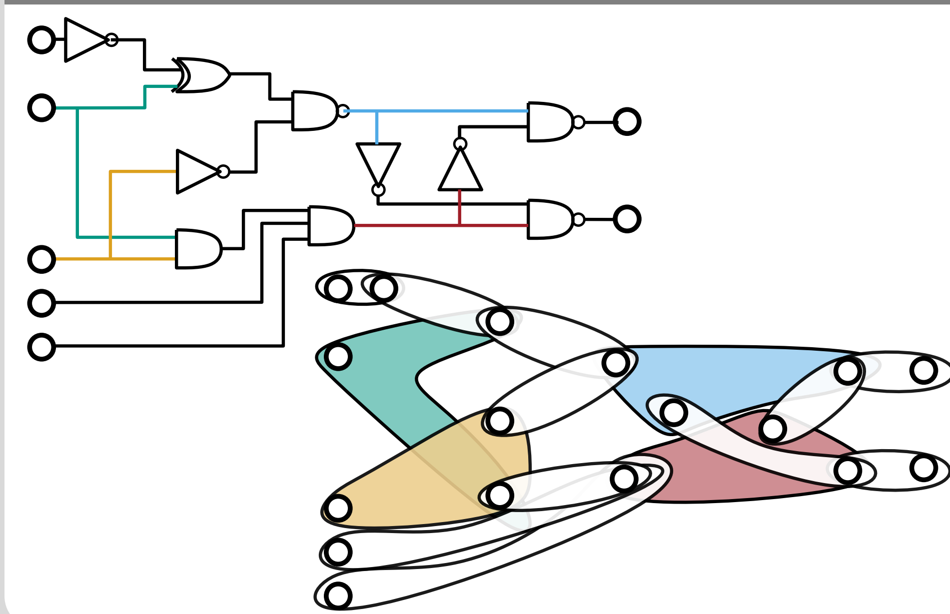
Karlsruhe Institute of Technology

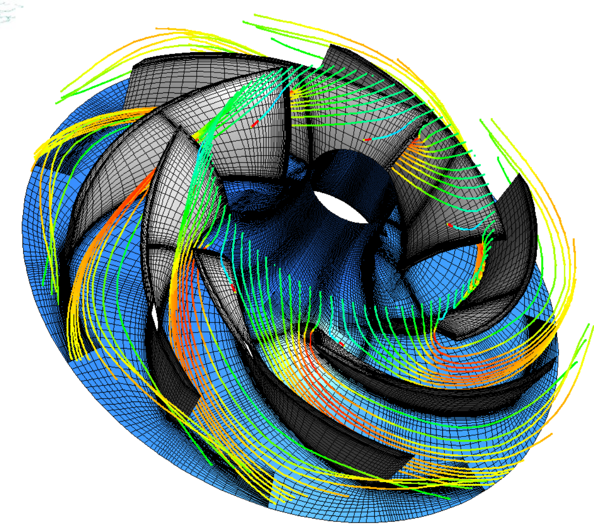
High-Quality (Hyper)Graph Partitioning

ICIAM · Combinatorial Scientific Computing · July 15th, 2019

Y. Akhremtsev, T. Heuer, P. Sanders, S. Schlag, C. Schulz, D. Seemaier, D. Strash

INSTITUTE OF THEORETICAL INFORMATICS · ALGORITHMICS GROUP





Research Areas in Peter Sanders' Group

Graph Partitioning

Shared-Memory Data Structures

Route Planning

Text Indexing

Hypergraph Partitioning

SAT Solving

Parallel Sorting

Communication-Efficient Algorithms

Graph Generators

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This Talk: Hypergraph & Graph Partitioning

Graph Partitioning

Shared-Memory
Data Structures

Route Planning

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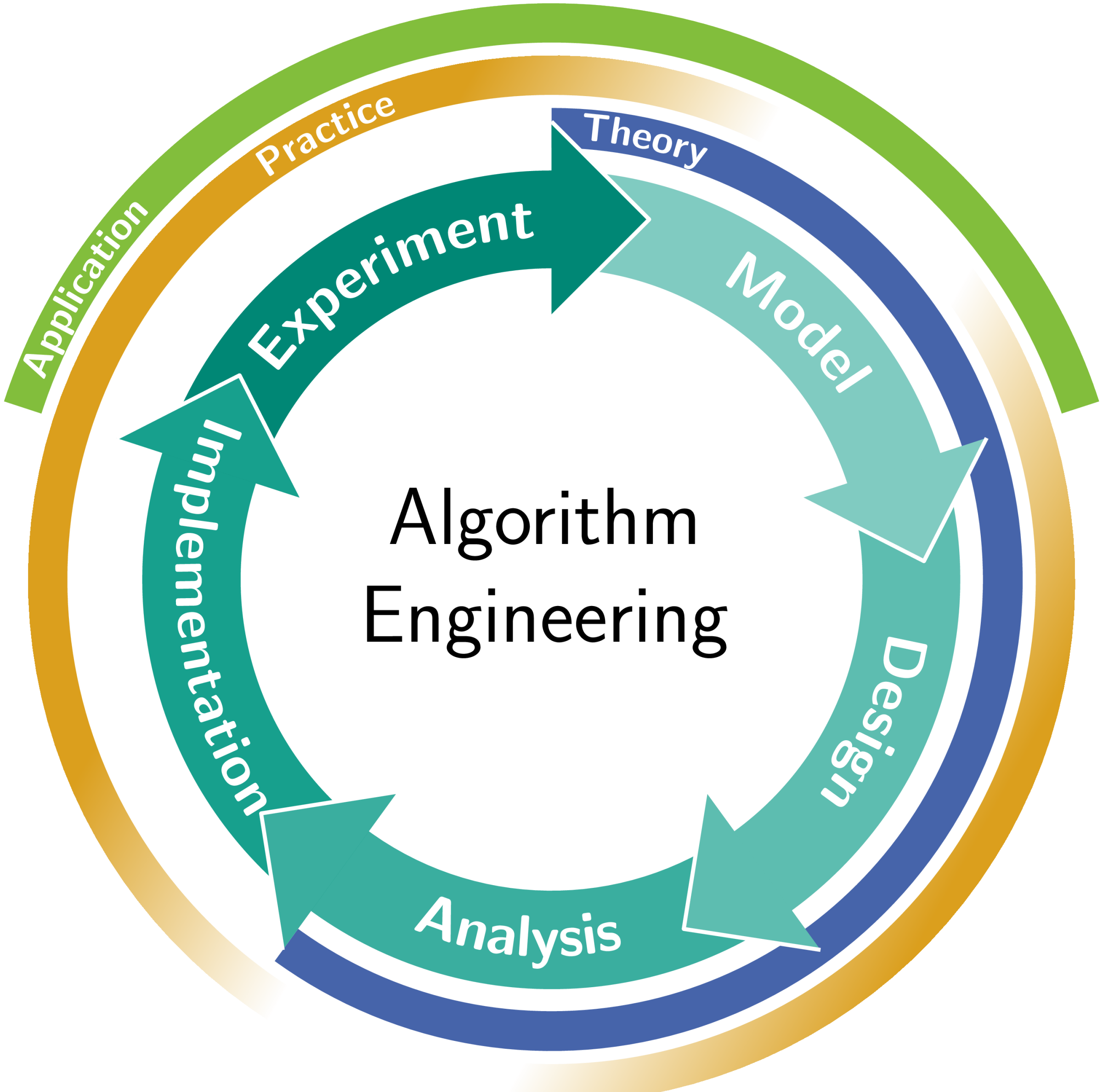
Research Methodology



Graphs and Hypergraphs

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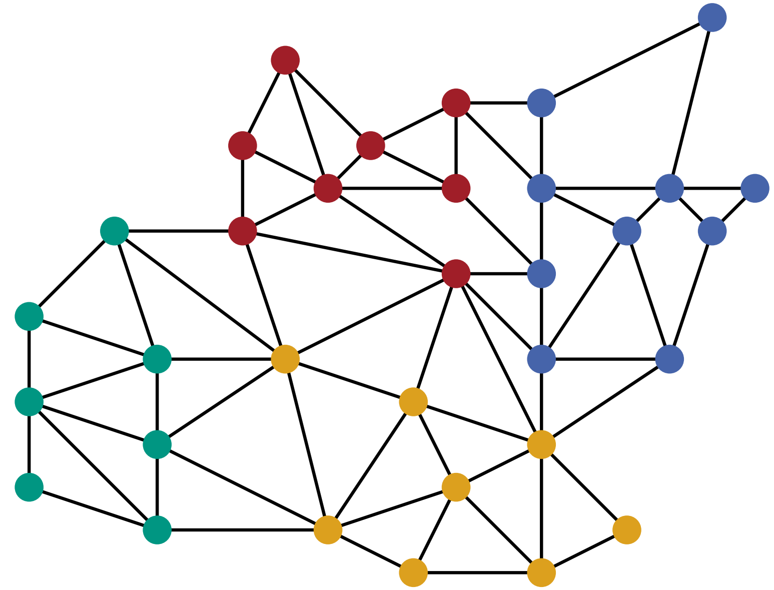
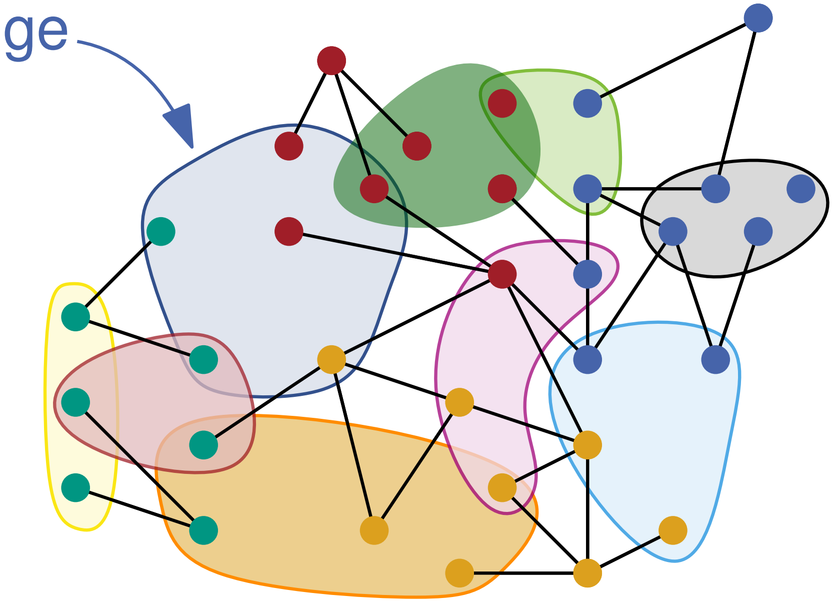
Graph $G = (V, E)$

vertices → V, edges → E



- Models relationships between objects
- Dyadic ($\mathbf{2}$-ary) relationships

hyperedge



Hypergraph $H = (V, E)$

- Generalization of a graph
  $\Rightarrow$ hyperedges connect ≥ 2 nodes
- Arbitrary ($\mathbf{d}$-ary) relationships
- Edge set $E \subseteq \mathcal{P}(V) \setminus \emptyset$

ε-Balanced Hypergraph Partitioning (HGP)

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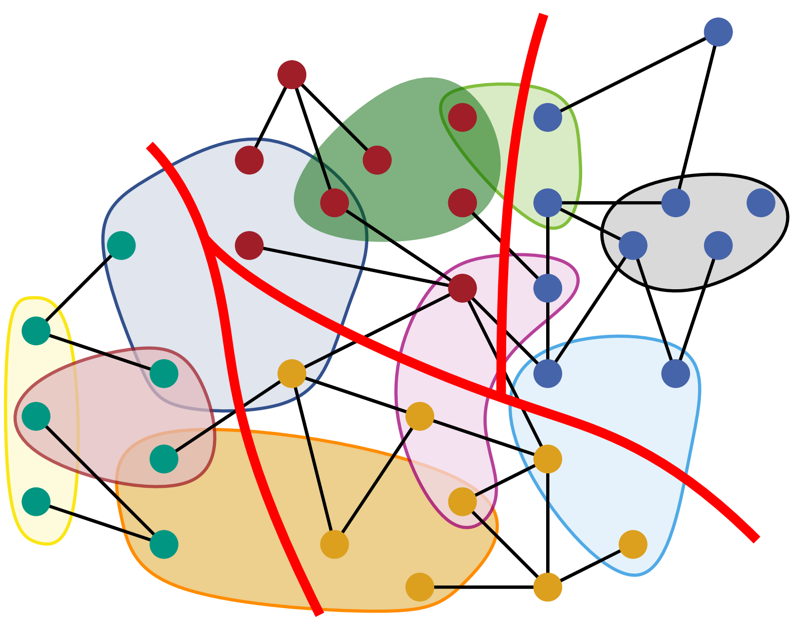
Partition hypergraph $H = (V, E, c : V \to \mathbb{R}_{>0}, \omega : E \to \mathbb{R}_{>0})$ into **k** disjoint blocks $\Pi = \{V_1, \ldots, V_k\}$ such that

- Blocks V_i are roughly equal-sized:

$$c(V_i) \leq (1 + \varepsilon) \left\lceil \frac{c(V)}{k} \right\rceil$$

- Objective function on hyperedges is minimized



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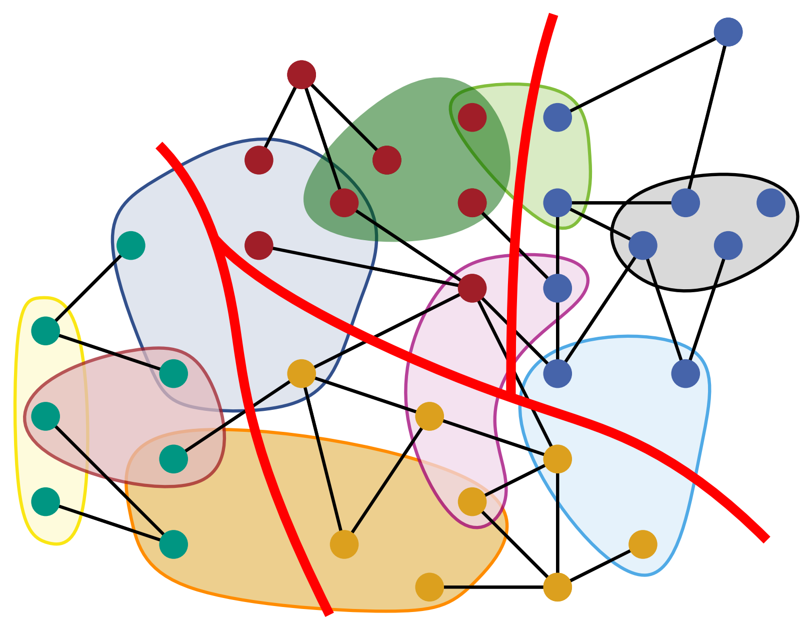

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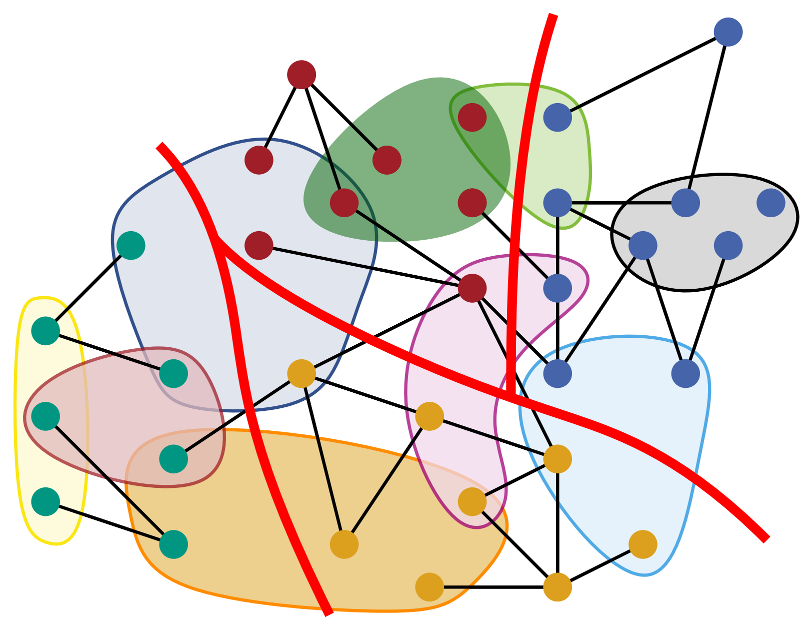
Common HGP Objectives:

- Cut-Net: $\sum_{e \in \text{Cut}} \omega(e)$



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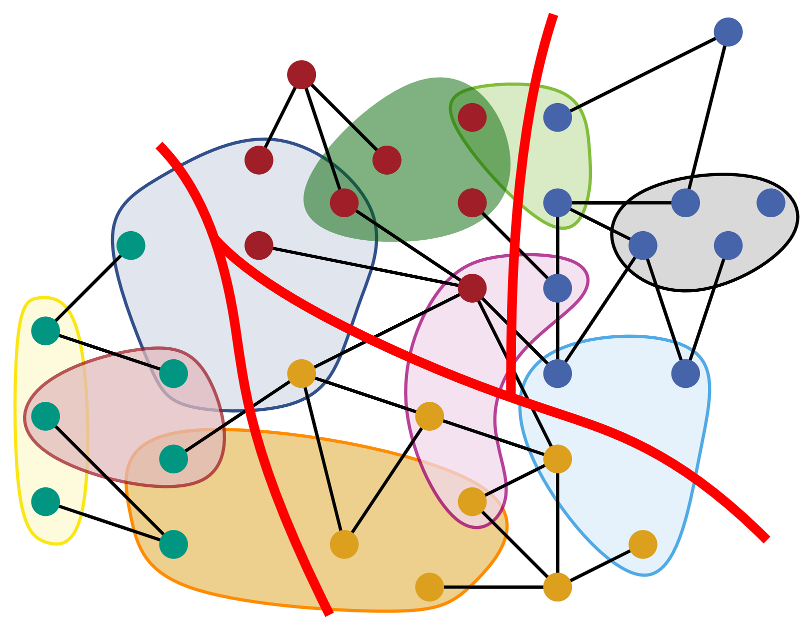
- Cut-Net: $\sum_{e \in \mathrm{Cut}} \omega(e)$
- Connectivity: $\sum_{e \in \mathrm{cut}} (\lambda - 1)\, \omega(e)$



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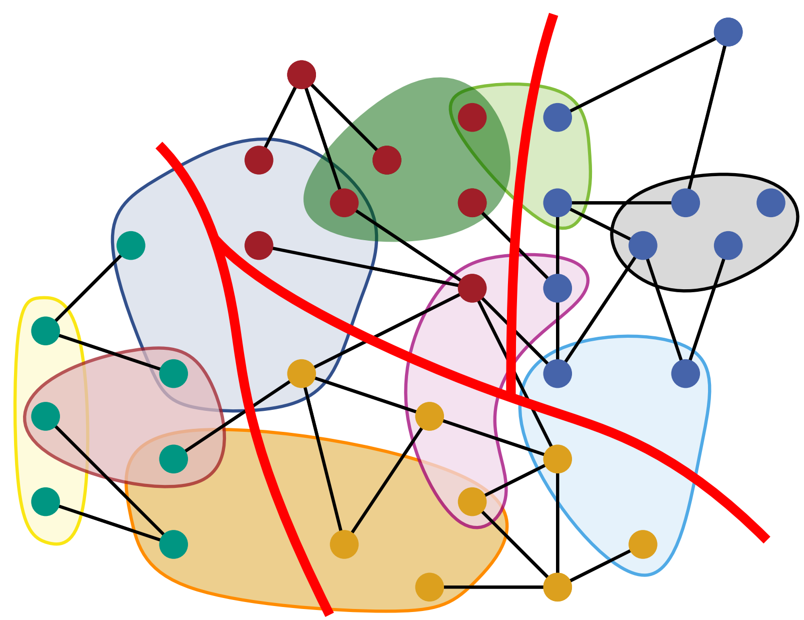
 # blocks connected by *e*



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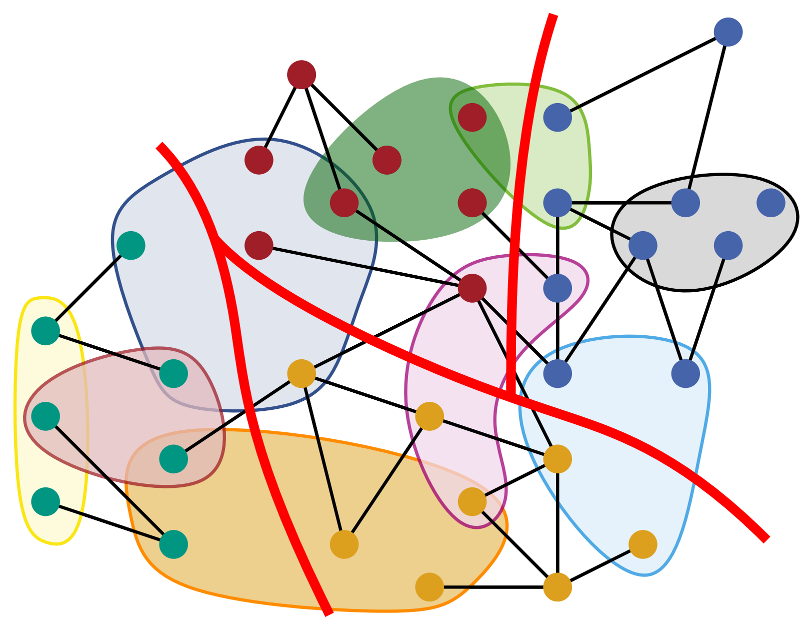
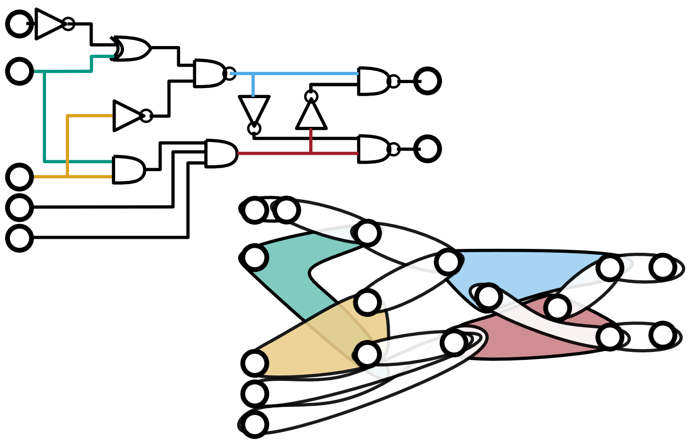
$\Rightarrow$ Both revert to edge-cut for graphs



Applications



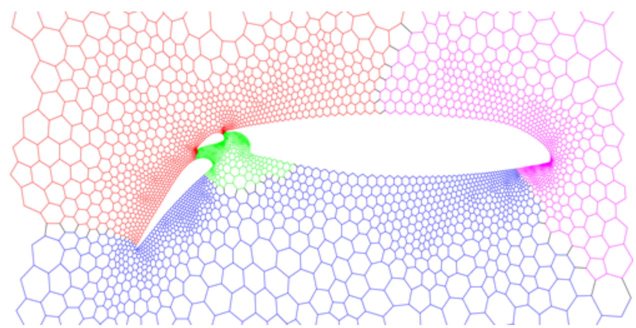
VLSI Design

Warehouse Optimization

[Martin Grandjean, via Wikimedia Commons]

Complex Networks

Route Planning

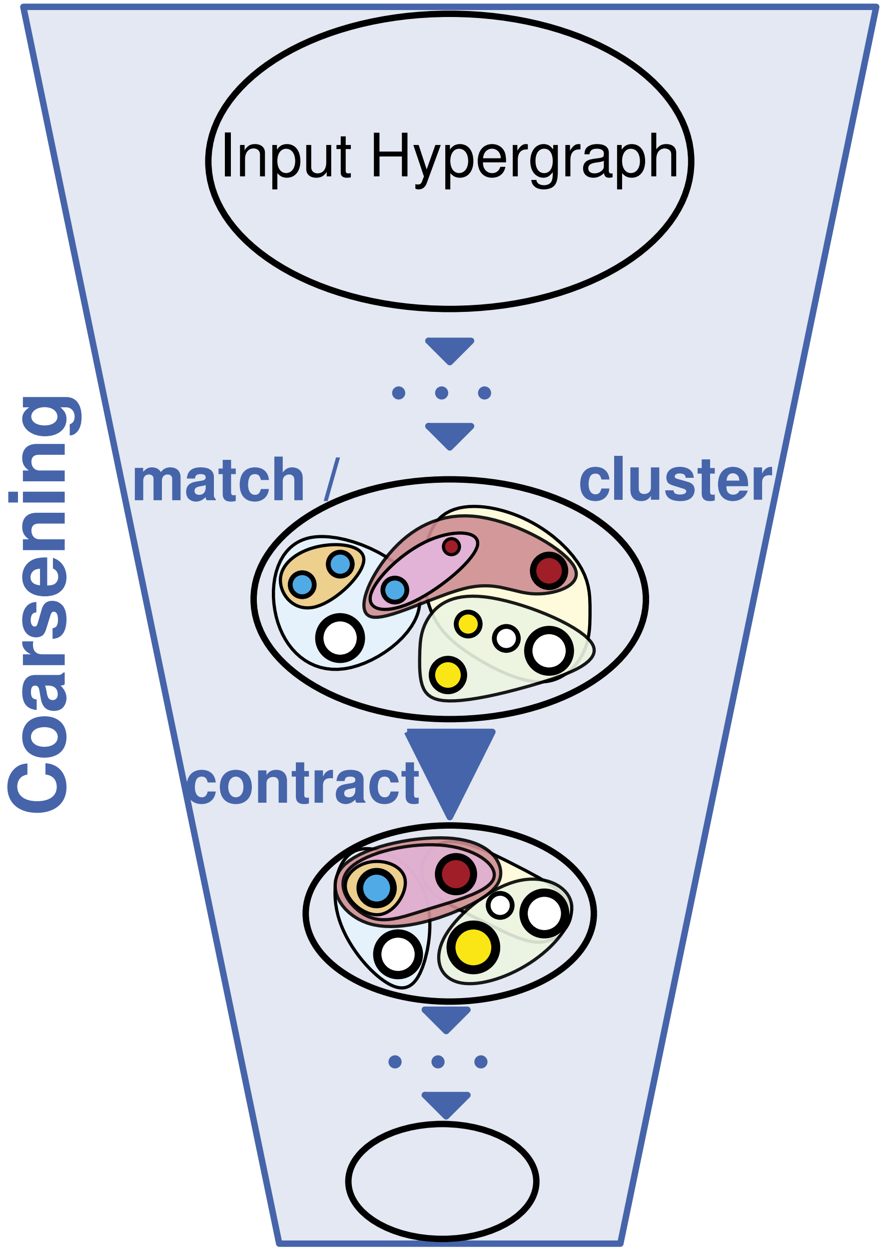


$$\mathbb{R}^{n \times n} \ni Ax = b \in \mathbb{R}^n$$

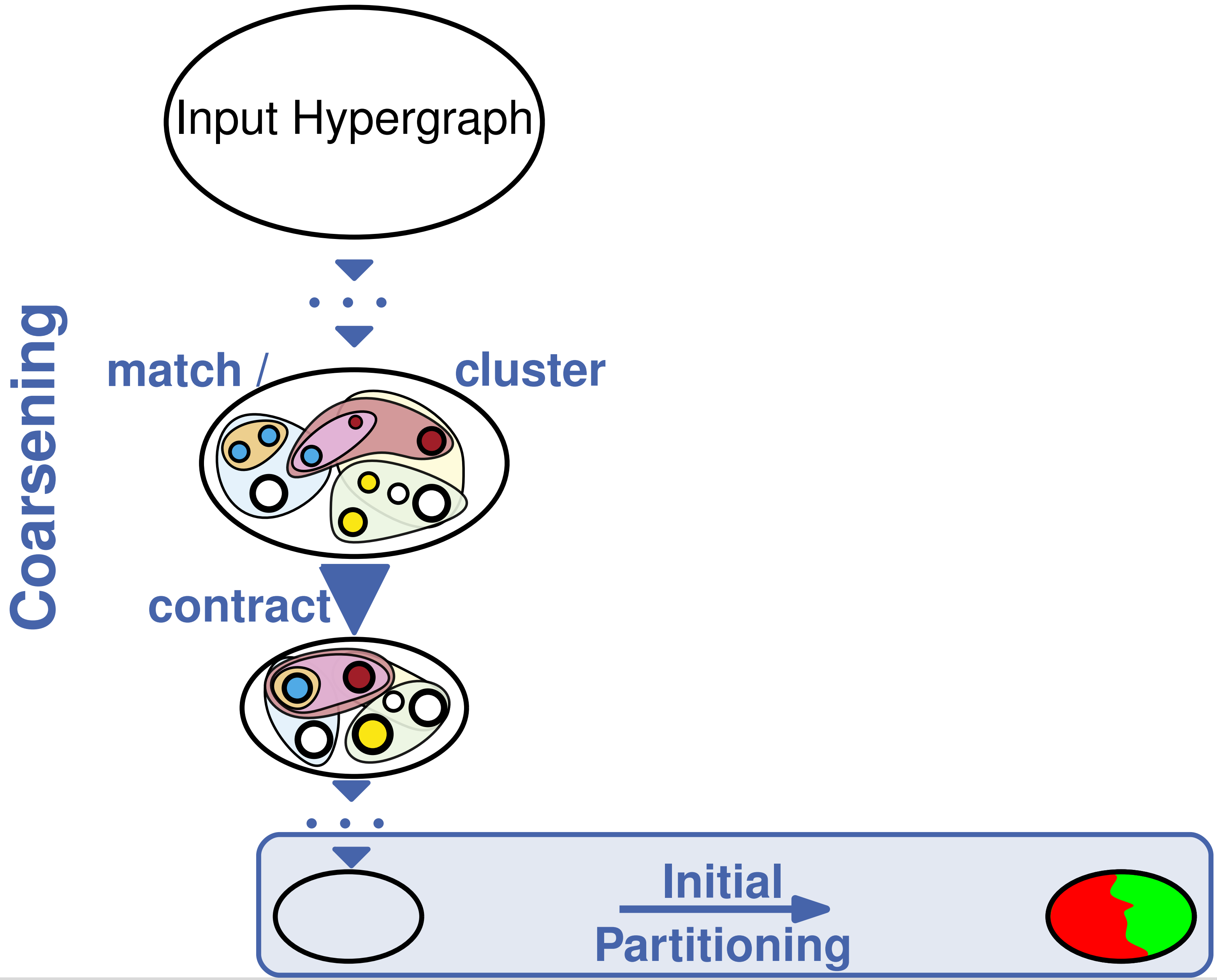
Scientific Computing

High-Quality Hypergraph Partitioning

Successful Heuristic: Multilevel Paradigm



Successful Heuristic: Multilevel Paradigm



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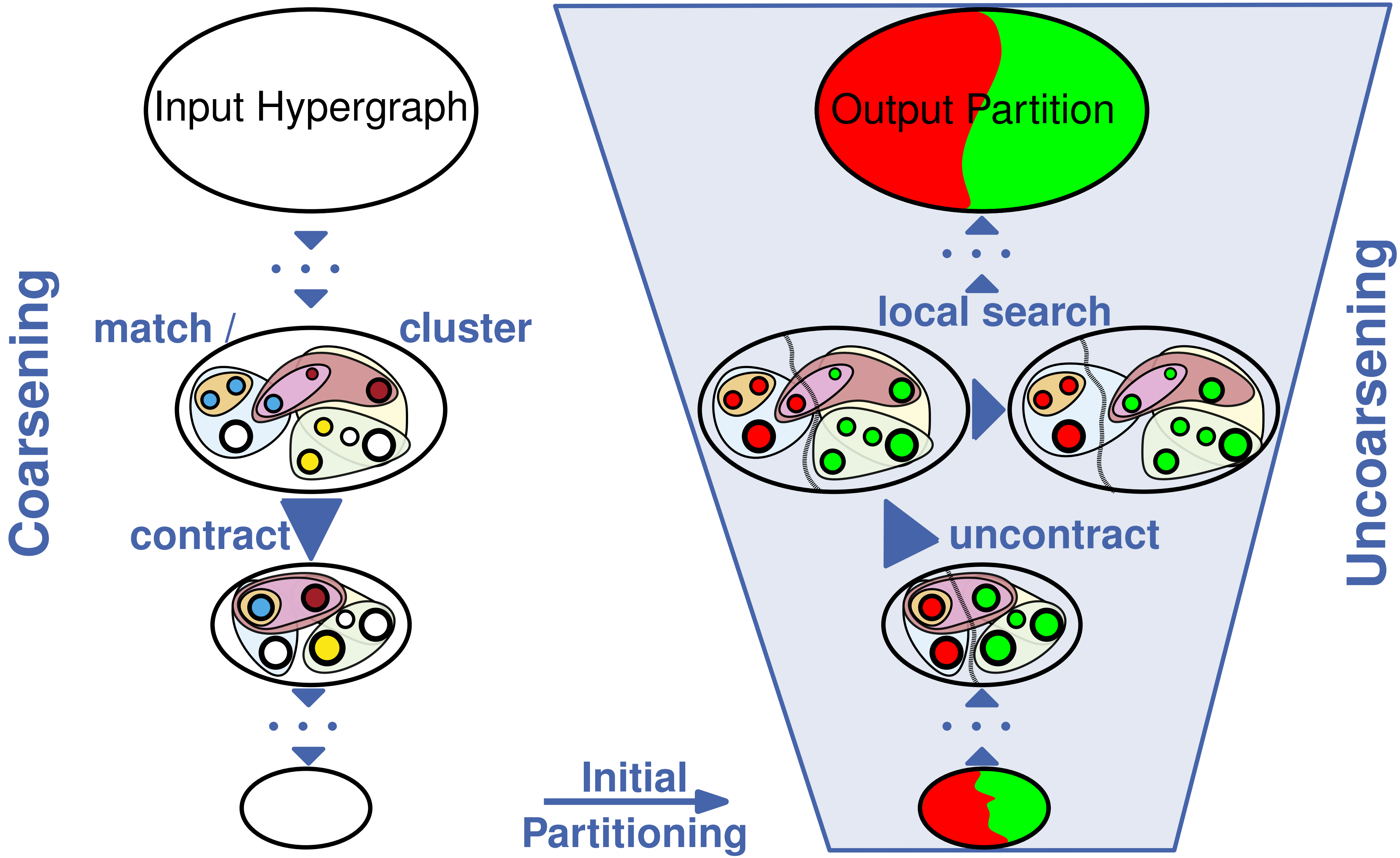
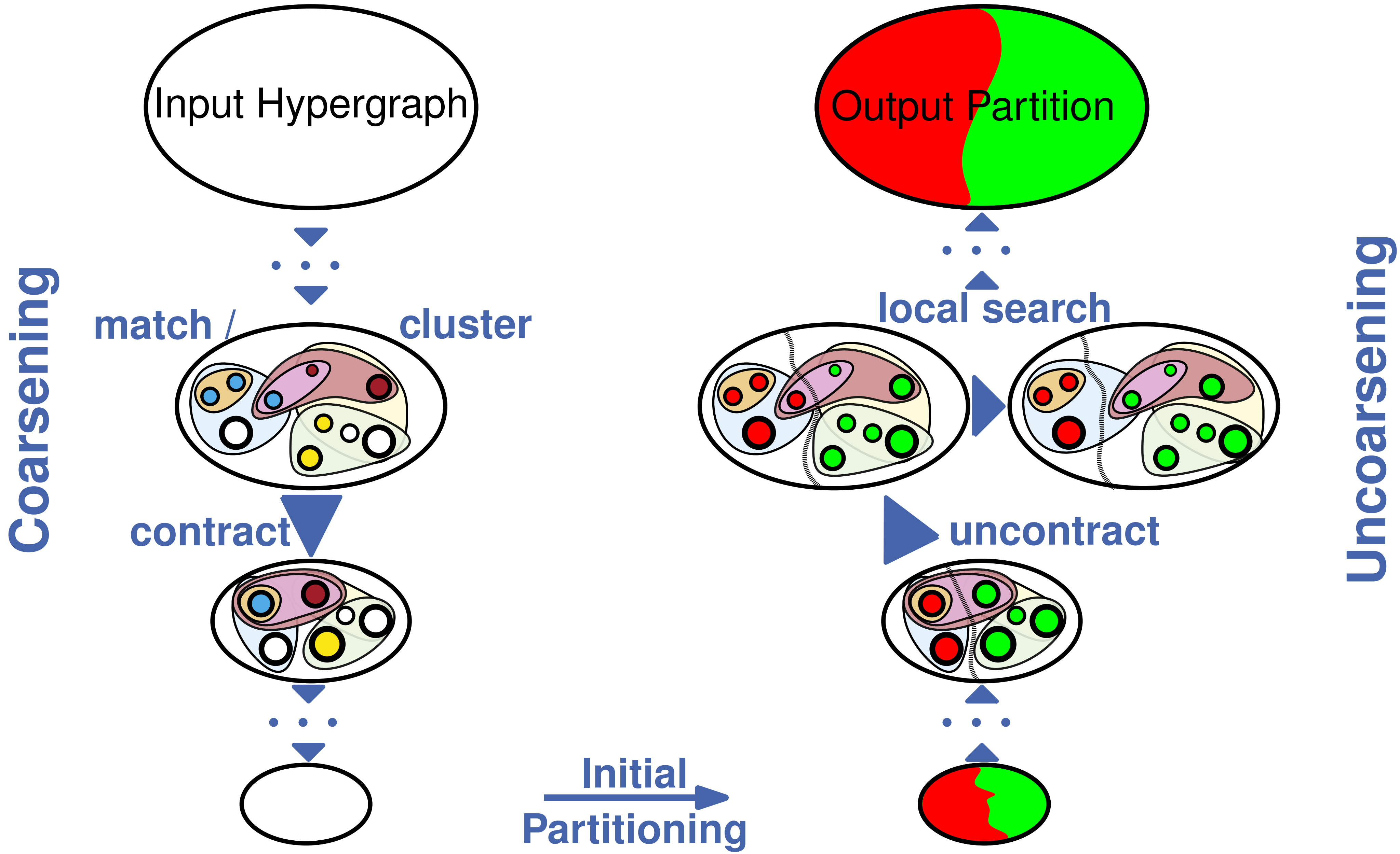


Why Yet Another Multilevel Algorithm?



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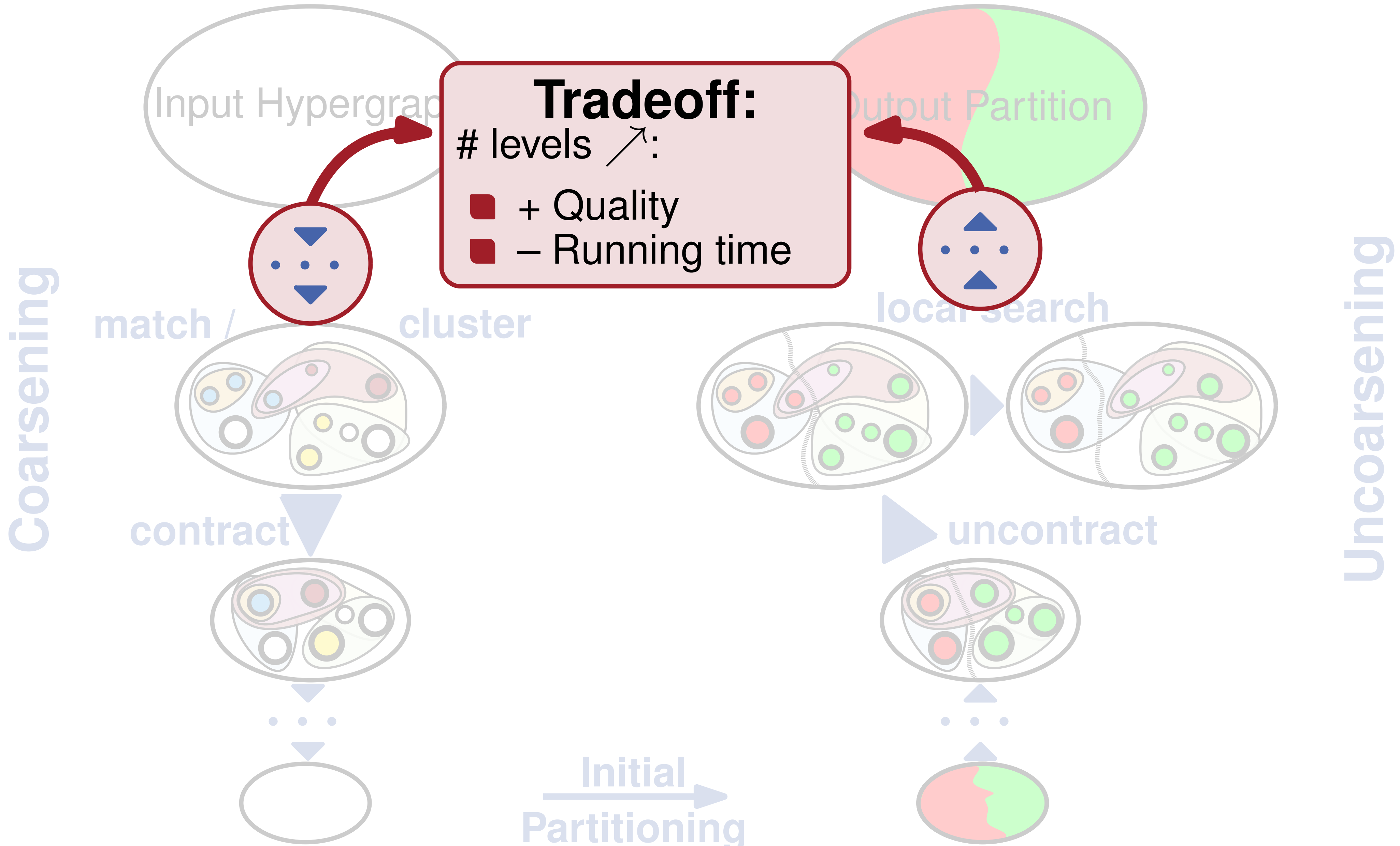

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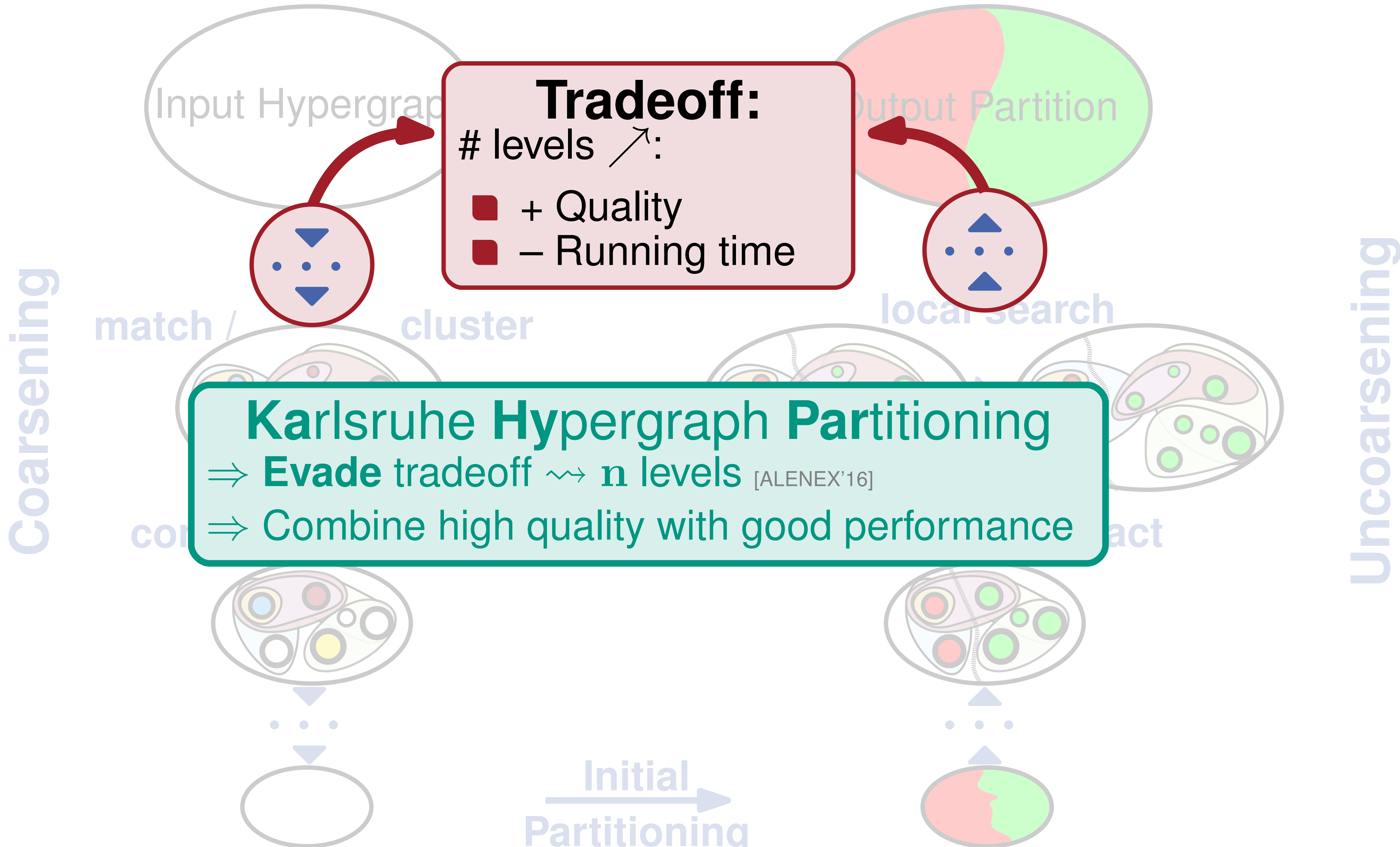

Tradeoff:

\# levels $\nearrow$:

- + Quality
- – Running time

Karlsruhe **Hy**pergraph **Par**titioning

$\Rightarrow$ **Evade** tradeoff $\rightsquigarrow$ $\mathbf{n}$ levels [ALENEX'16]

$\Rightarrow$ Combine high quality with good performance

KaHyPar: Novel Algorithmic Ingredients

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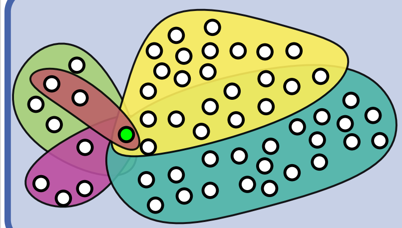
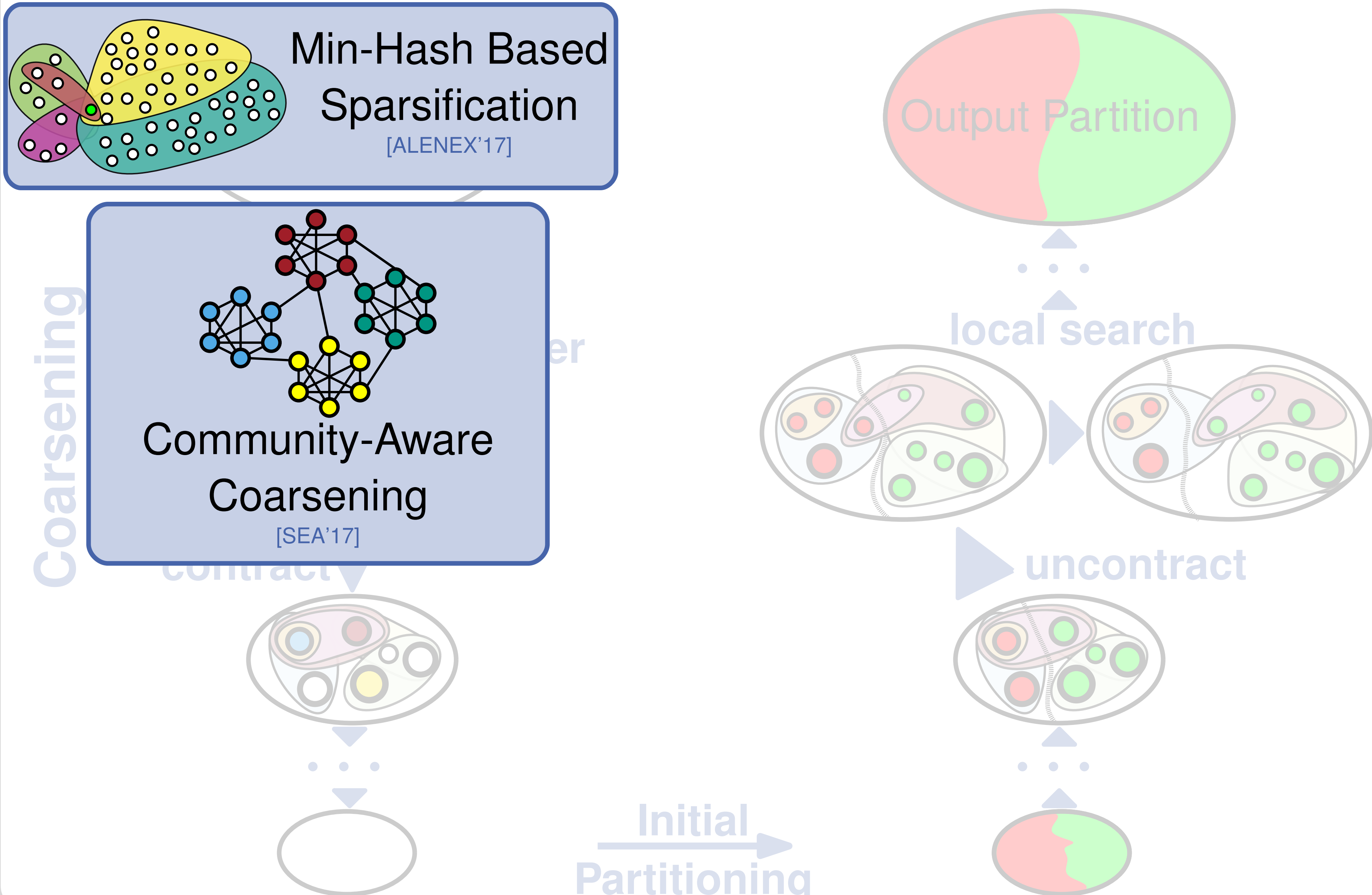


Min-Hash Based Sparsification

[ALENEX'17]

KaHyPar: Novel Algorithmic Ingredients



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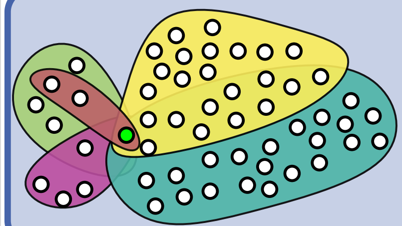
KaHyPar: Novel Algorithmic Ingredients



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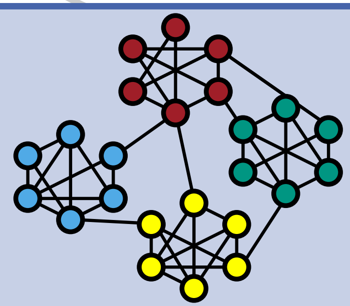
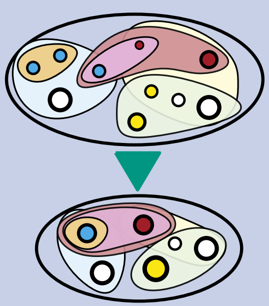


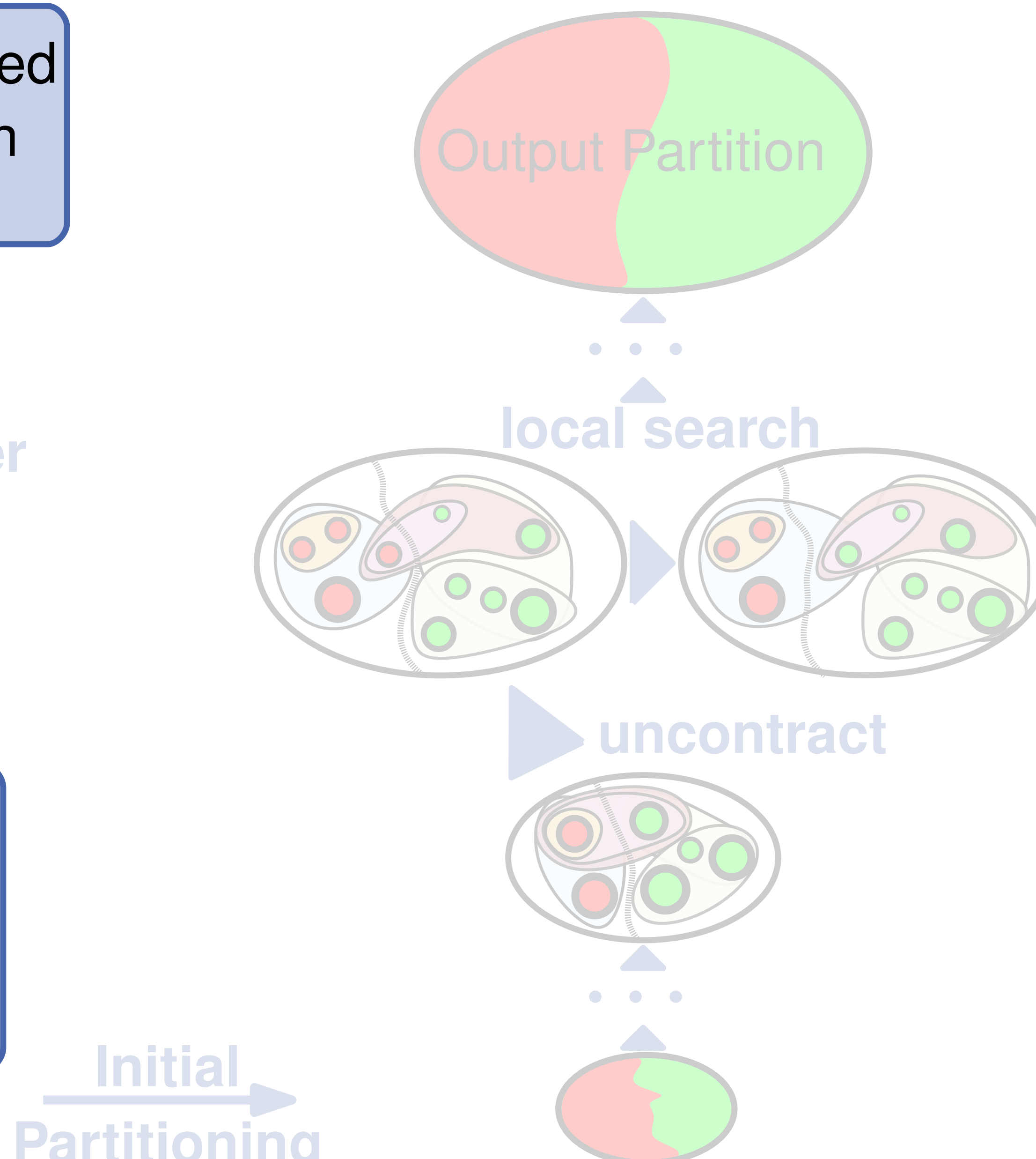
Community-Aware Coarsening

[SEA'17]



Fast *n*-Level Coarsening

[ALENEX'16, ALENEX'17]



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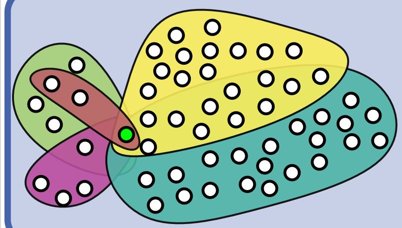

Min-Hash Based Sparsification
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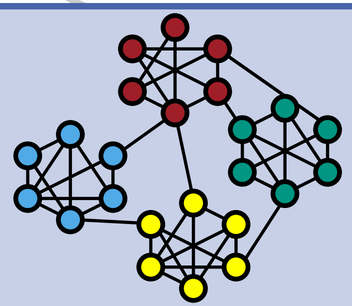

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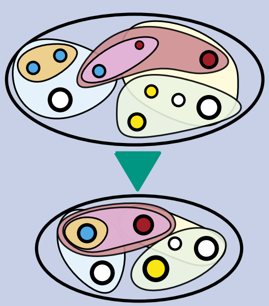

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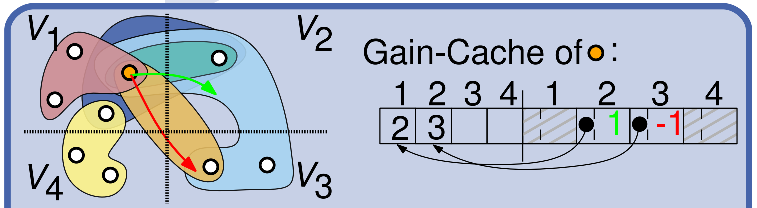

Engineered *k*-way FM
[ALENEX'17]

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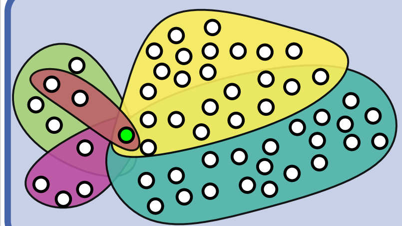
KaHyPar: Novel Algorithmic Ingredients



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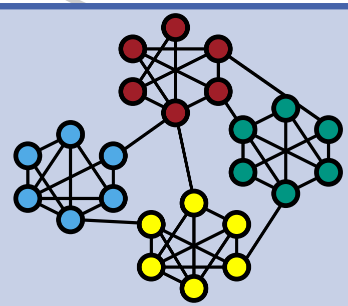

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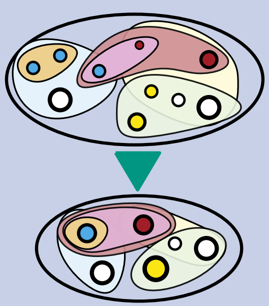

Fast n-Level Coarsening
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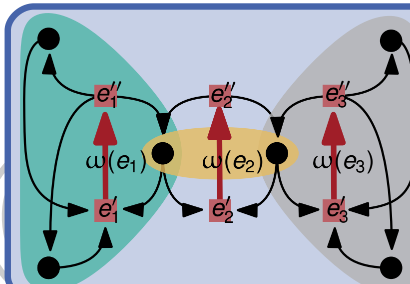
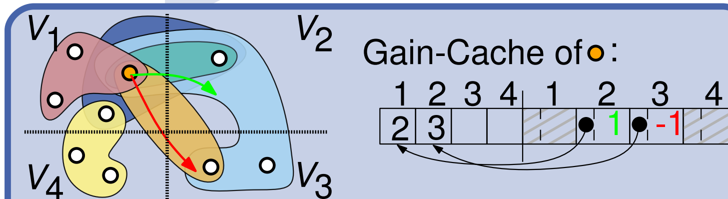


Max-Flow Min-Cut Refinement
[SEA'18, JEA'19]



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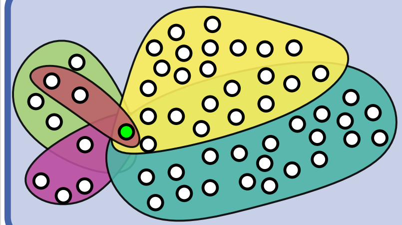
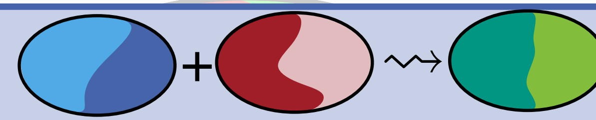
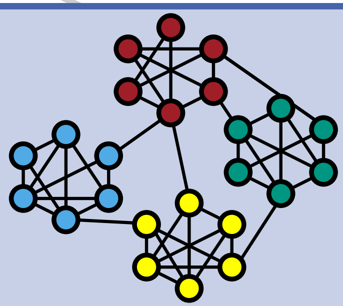


Min-Hash Based Sparsification
[ALENEX'17]



Memetic Multilevel Algorithm
[GECCO'18]



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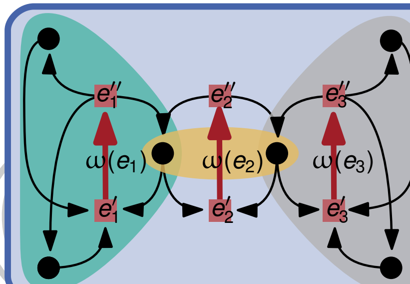

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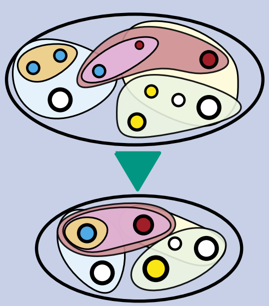

Fast *n*-Level Coarsening
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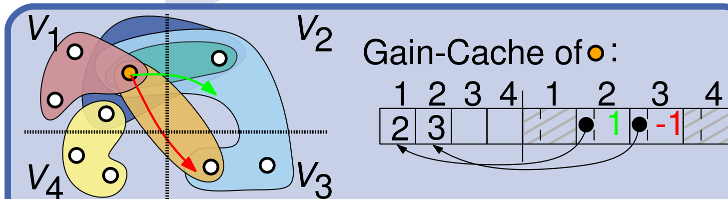

Engineered *k*-way FM
[ALENEX'17]

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Experiments – Benchmark Setup

- System: 1 core of 2 Intel Xeon E5-2670 @ 2.6 Ghz, 64 GB RAM
- # Hypergraphs: [publicly available]
  - SuiteSparse Matrix Collection 184
  - SAT Competition 2014 (3 representations) 92·3
  - ISPD98 & DAC2012 VLSI Circuits 28
- $k \in \{2, 4, 8, 16, 32, 64, 128\}$ with imbalance: $\varepsilon = 3\%$
- Comparing KaHyPar with:
  - hMetis-R & hMetis-K
  - PaToH-Default & PaToH-Quality
  - HYPE
  - Zoltan-AlgD

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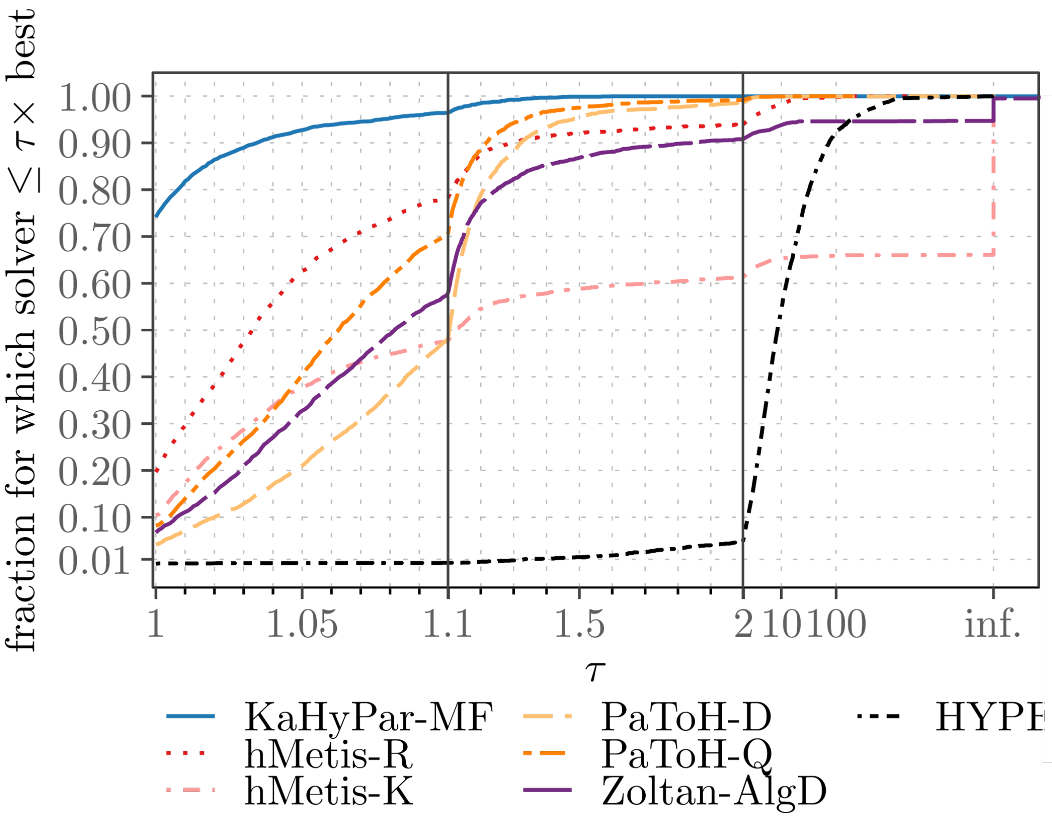
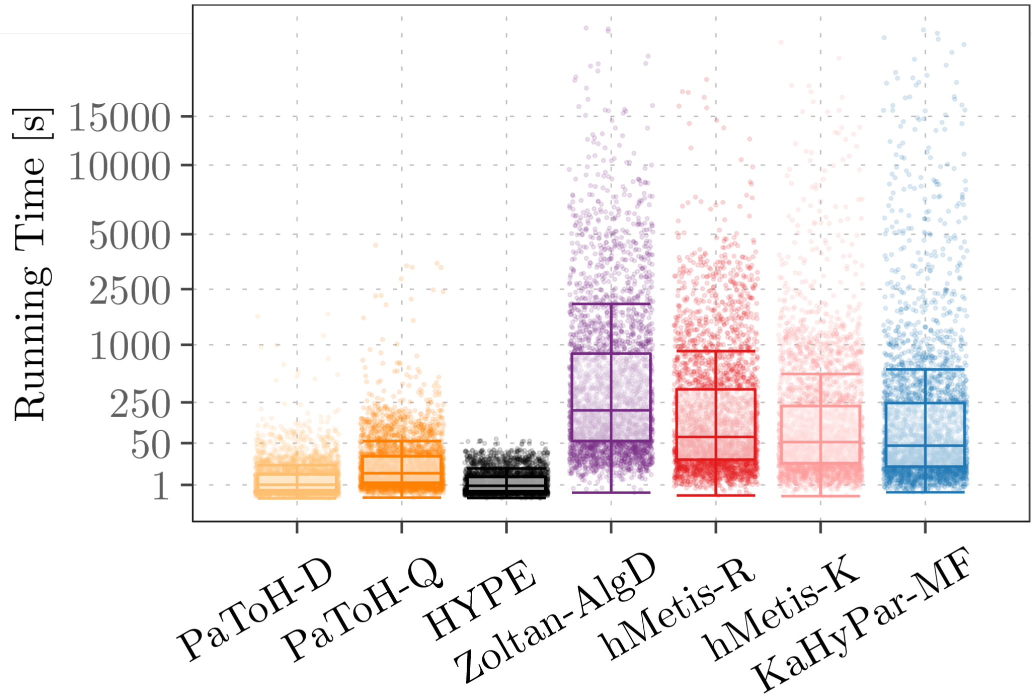
Experiments: Connectivity Optimization





⇒ Similar results for cut-net optimization

Parallel Shared-Memory Graph Partitioning

Parallel GP: Coarsening [EuroPar'18]

Algorithm: Parallel label propagation [SM'16] with improved load balancing

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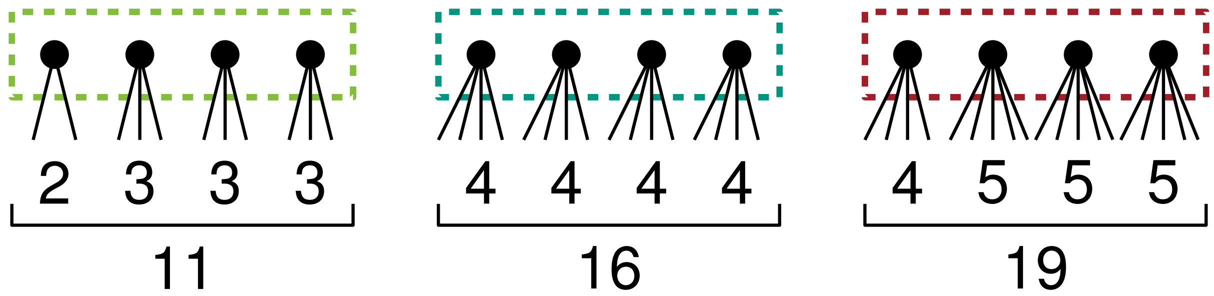
Parallel GP: Coarsening [EuroPar'18]

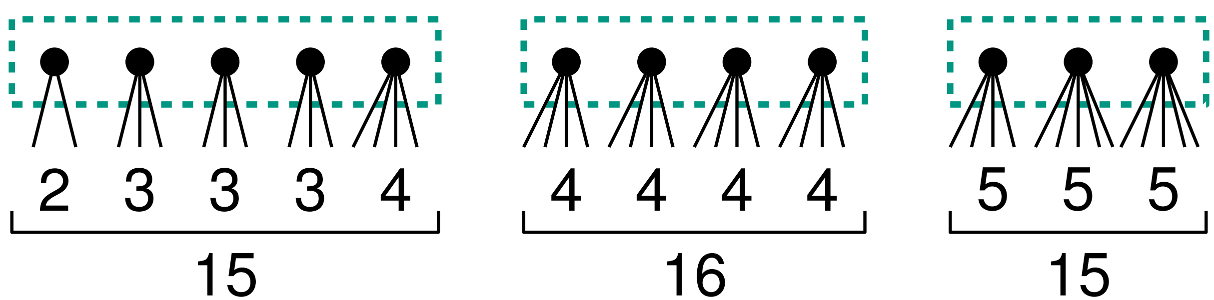
Algorithm: Parallel label propagation [SM'16] with improved load balancing

Problem: Vertex-based distribution ⇒ bad load-balance



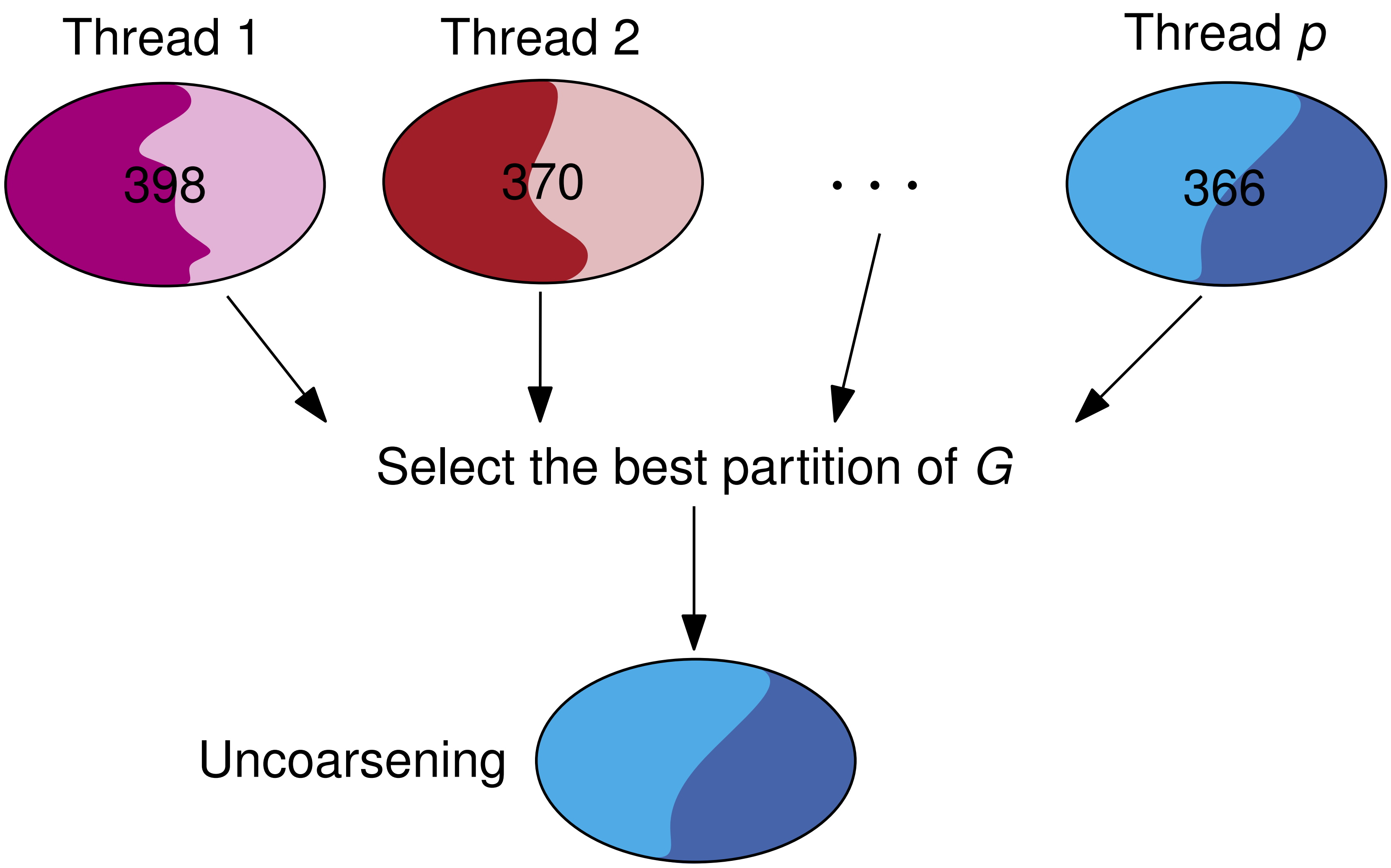
Solution: Edge-based distribution

- Packets P: $\sqrt{|E|} \leq \sum_{v \in P} d(v) \leq \sqrt{|E|} + \Delta$, $\Delta = \max_{v \in V} d(v)$



Parallel GP: Initial Partitioning [EuroPar'18]

Parallel Initial Partitioning using KaHIP [SEA'14]

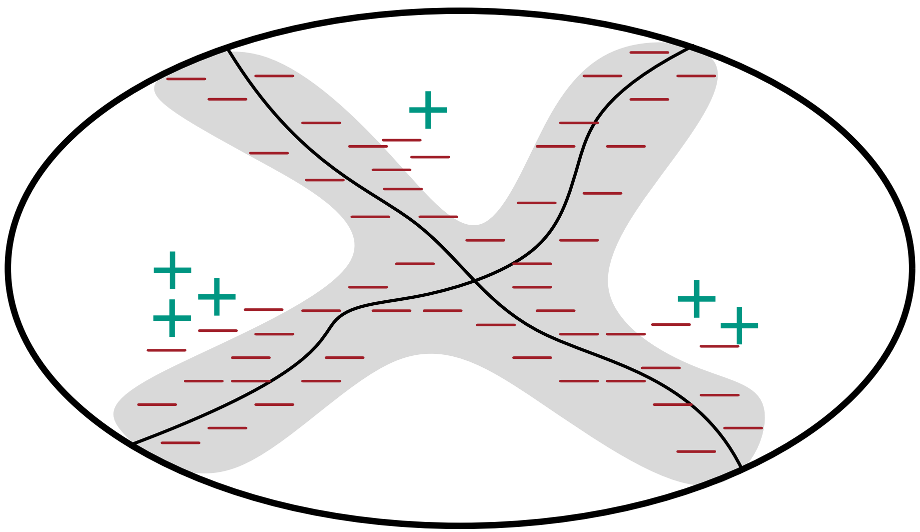


Parallel GP: Refinement [EuroPar'18]

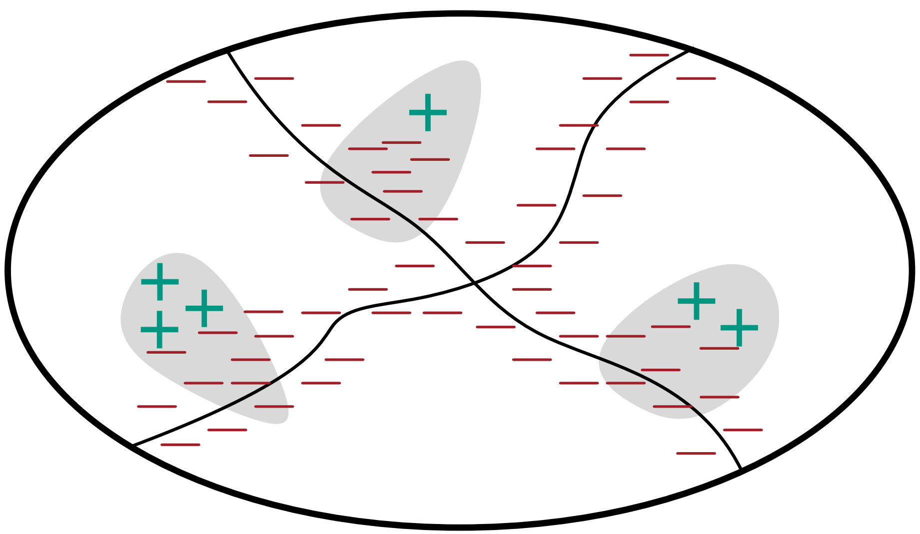
Algorithms:

- Parallel label propagation
- Parallel localized *k*-way local search
  - minimal coordination of searches
  - serialized execution of final moves

Traditional:

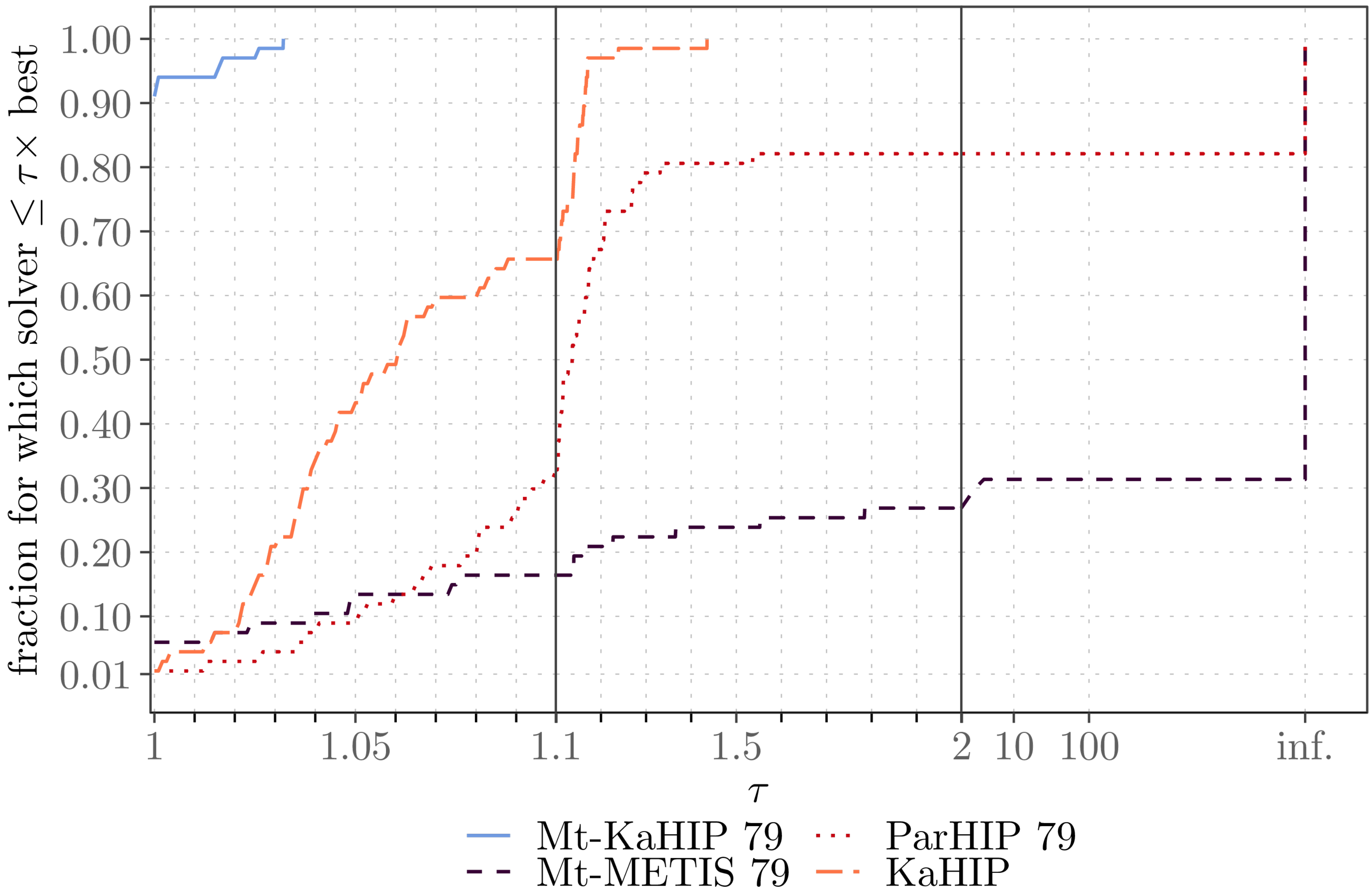


Localized:



Experiments: Solution Quality

38 Graphs with $k = \{16, 64\}$



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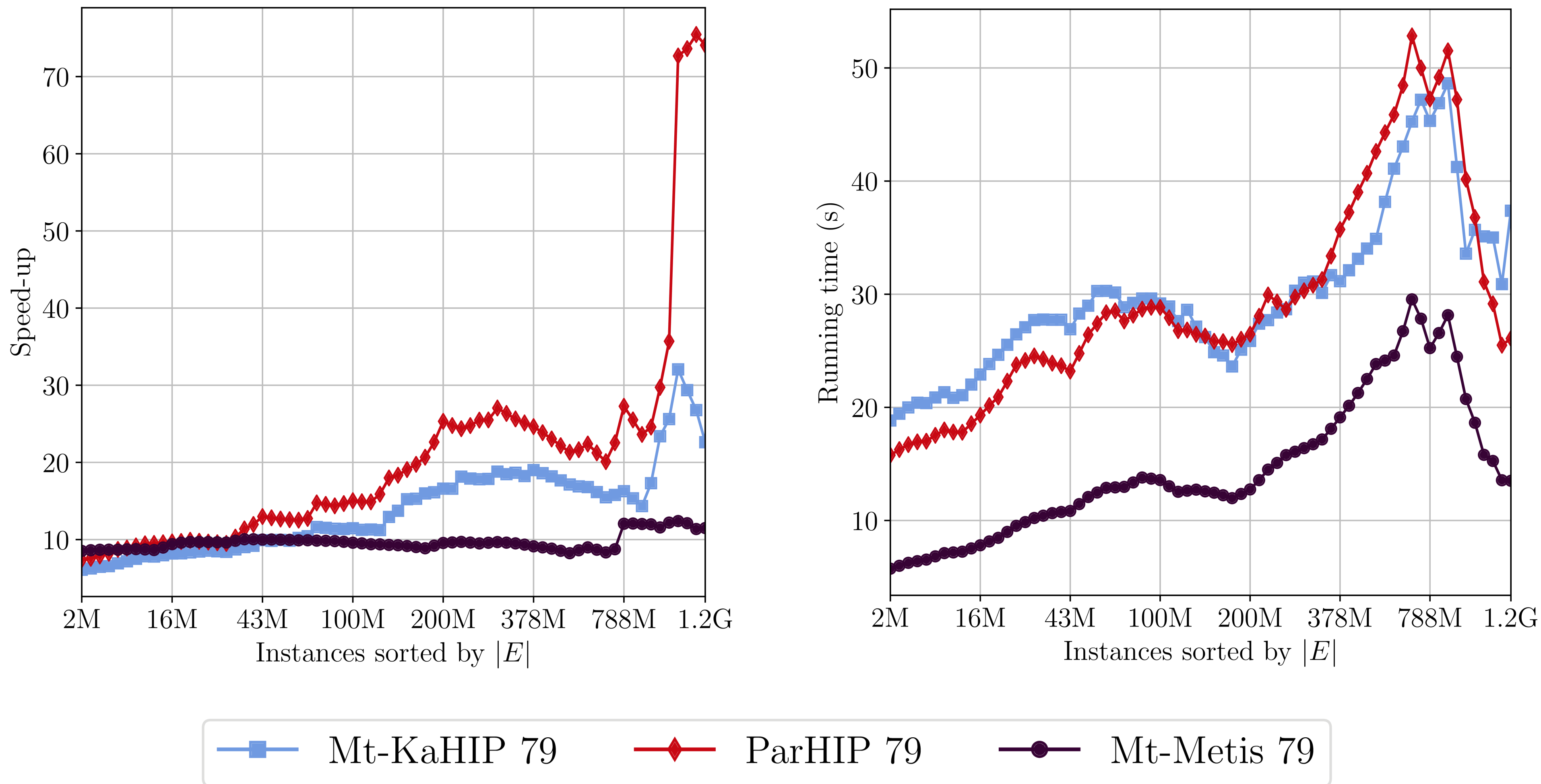
Experiments: Speedup & Running Time

4 Socket Machine with 79 Threads



Cumulative: $(x, y) \rightarrow$ speedup/ running time for graphs with $|E| \geq x = y$

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Scalable Edge Partitioning

The Edge Partitioning Problem

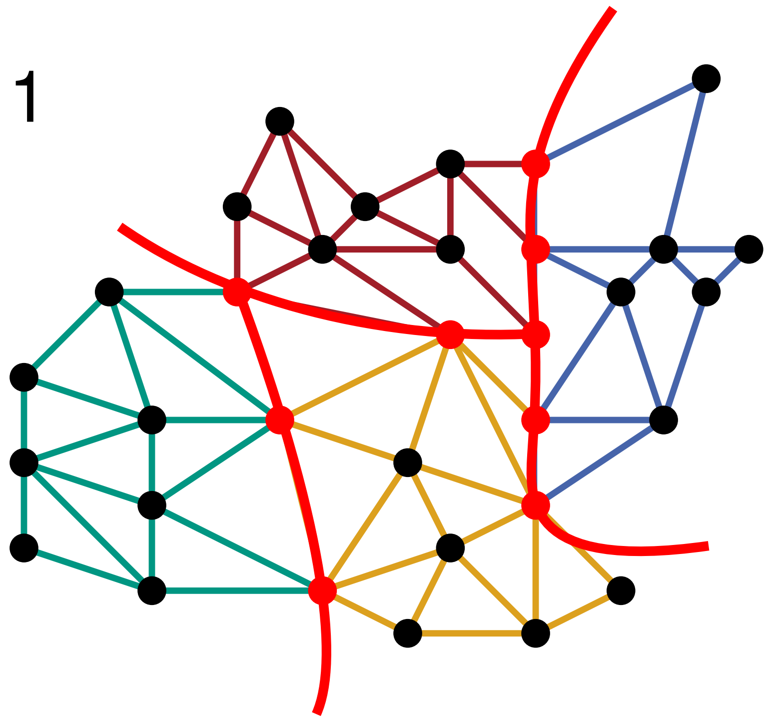
Partition edge set of graph $G = (V, E, c, \omega)$ into **k** disjoint blocks $\Pi = \{E_1, \dots, E_k\}$ such that

- Blocks E_i are roughly equal-sized:

$$\omega(E_i) \leq (1 + \varepsilon) \left\lceil \frac{\omega(E)}{k} \right\rceil$$

- minimize vertex cut:

$$\sum_{v \in V} |I(v)| - 1$$



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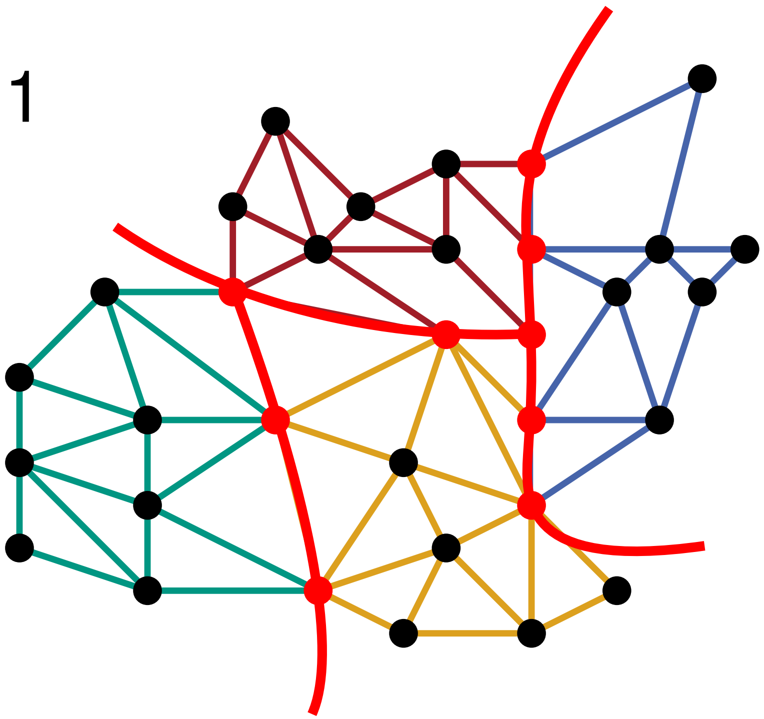
- Blocks E_i are roughly equal-sized:

$$\omega(E_i) \leq (1 + \varepsilon) \left\lceil \frac{\omega(E)}{k} \right\rceil$$

- minimize vertex cut:

$$\sum_{v \in V} |I(v)| - 1$$

 # blocks with edges incident to v



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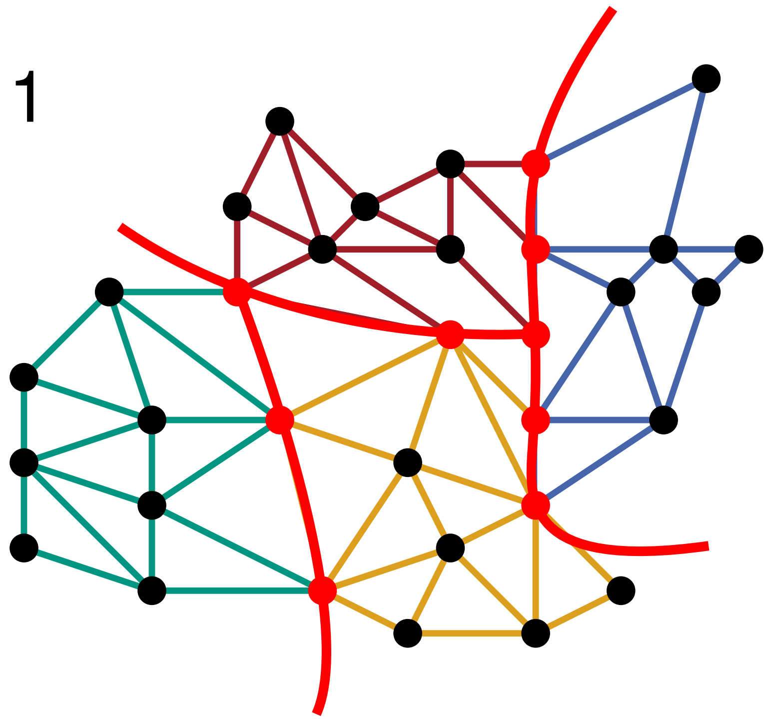

Motivation [Gonzalez et al.'12]:

- edge-centric distributed computations
- combat shortcomings of TLAV approaches
- duplicate node-centric computations

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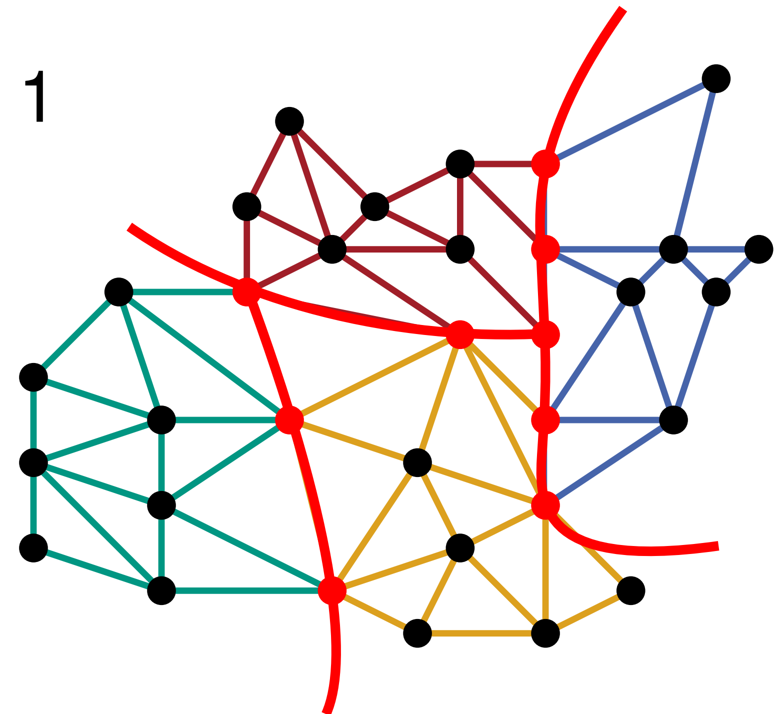
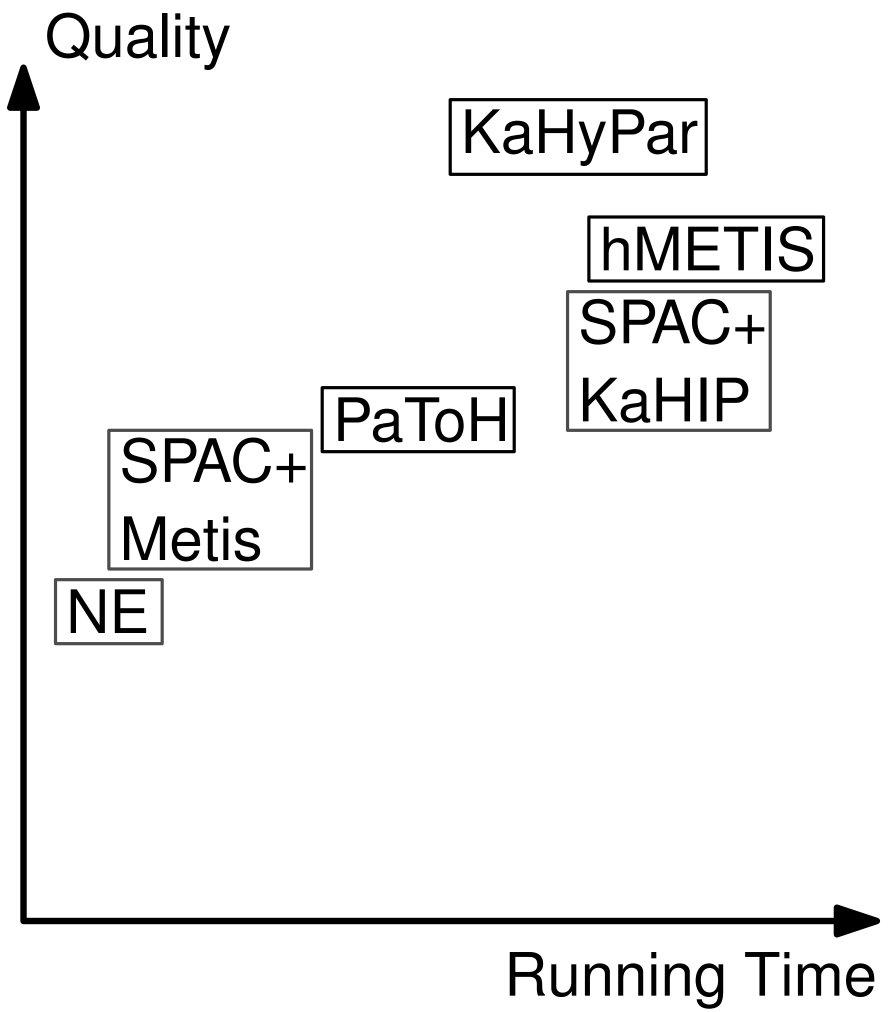
Think-Like-A-Vertex



Edge Partitioning Algorithms

Sequential



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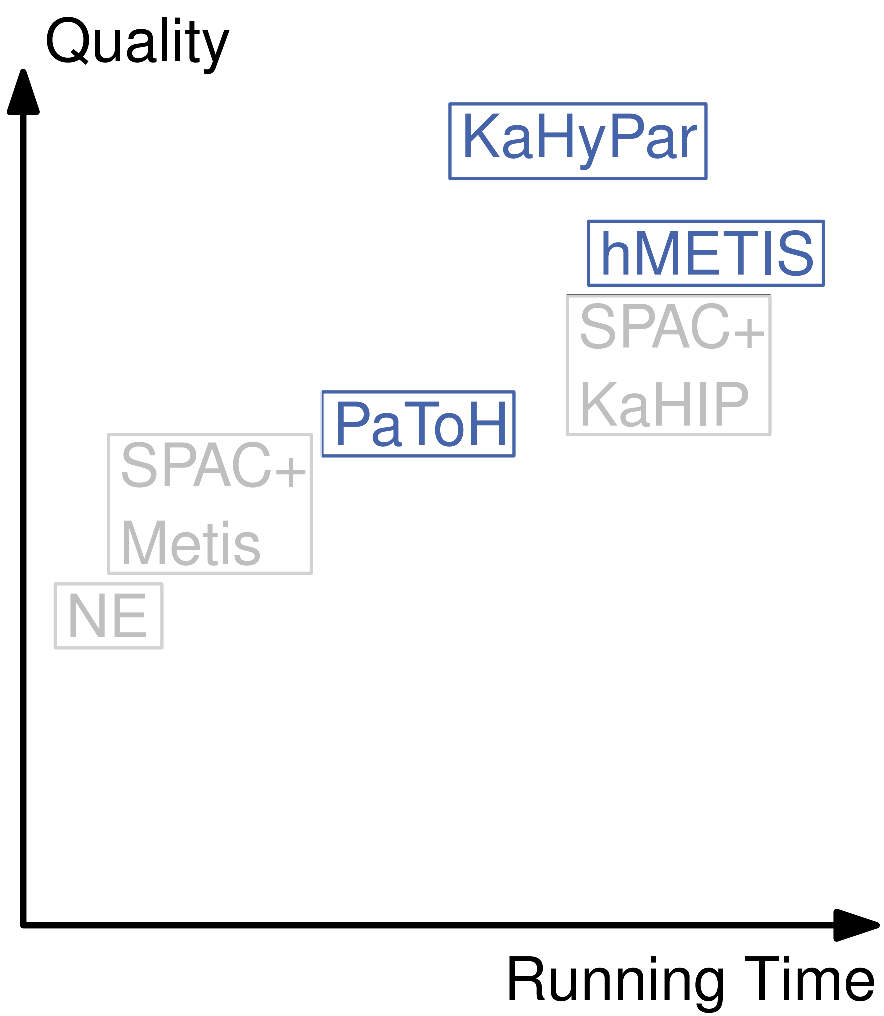
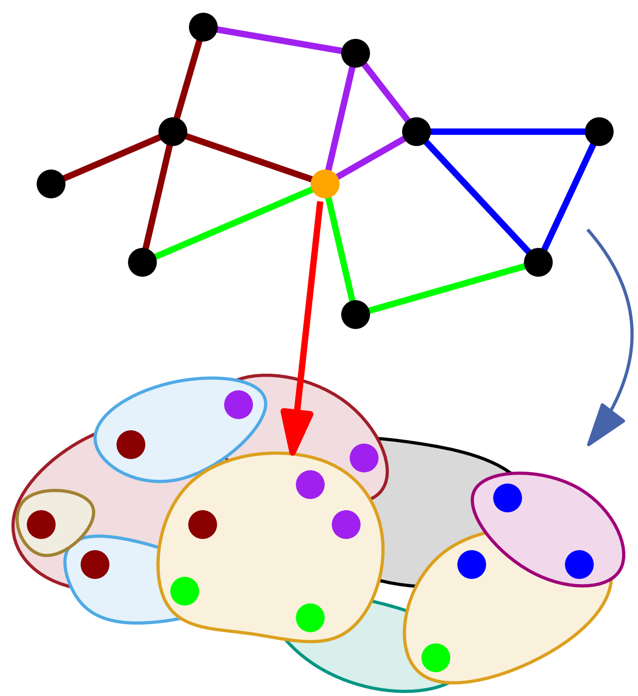


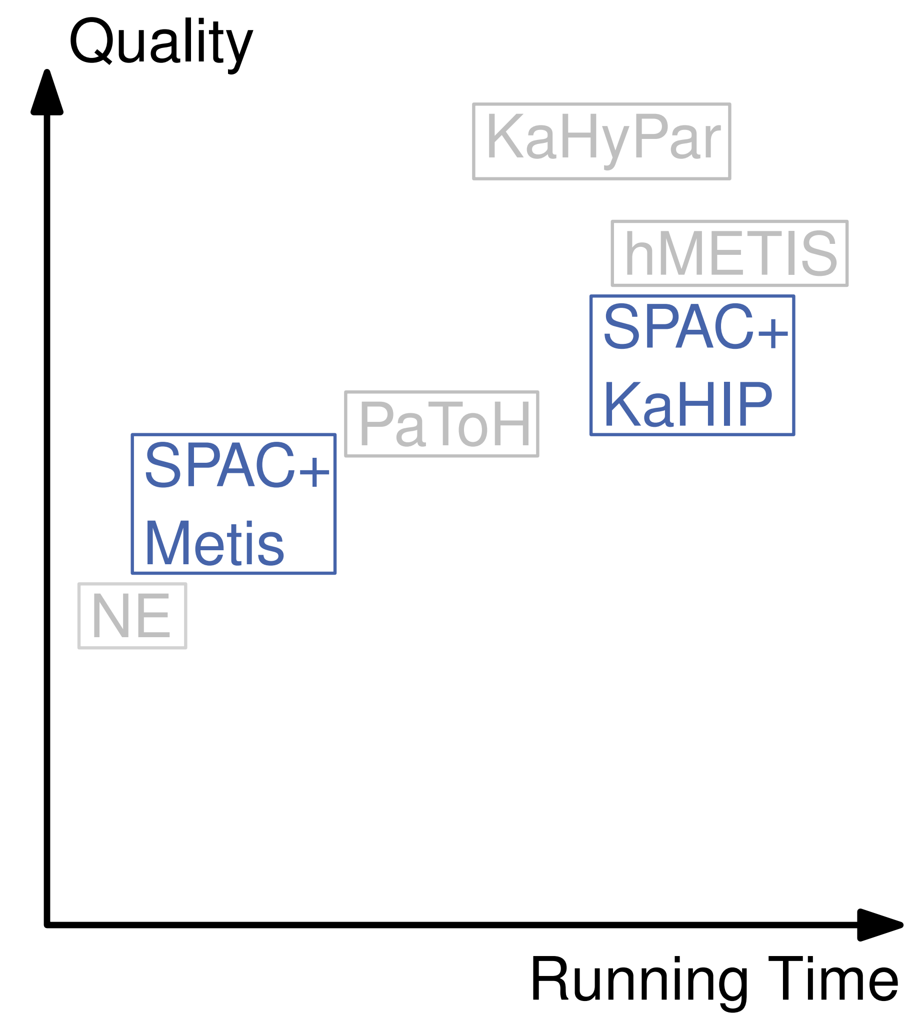
Hypergraph Model:

- Graph edge $\rightsquigarrow$ vertex
- Graph node $\rightsquigarrow$ hyperedge
- Optimize connectivity



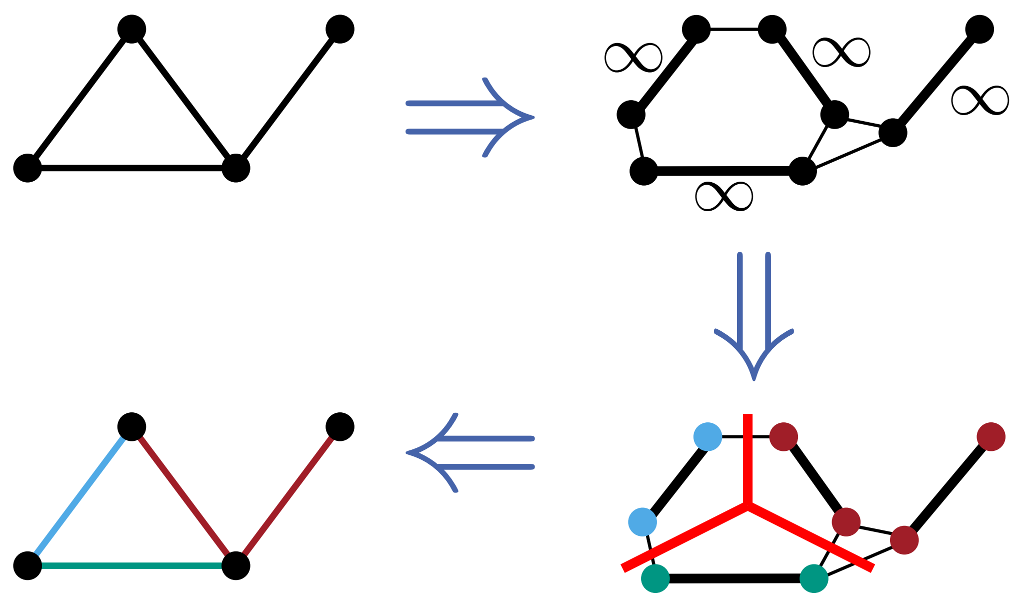
Edge Partitioning Algorithms

Sequential

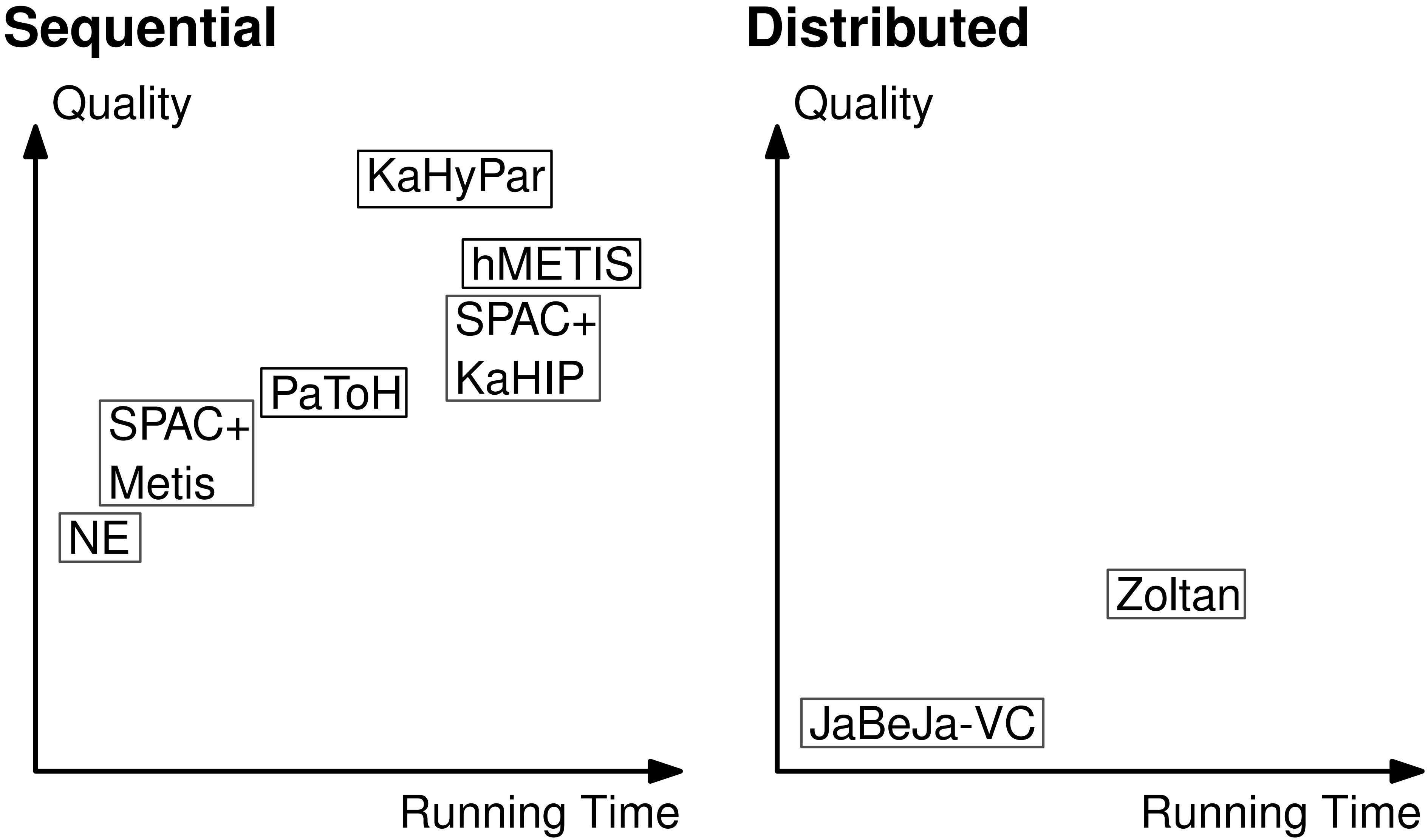


Split-And-Connect (SPAC) [Li et al.'17]:

- Build auxilary graph
- Use vertex partitioning algorithm



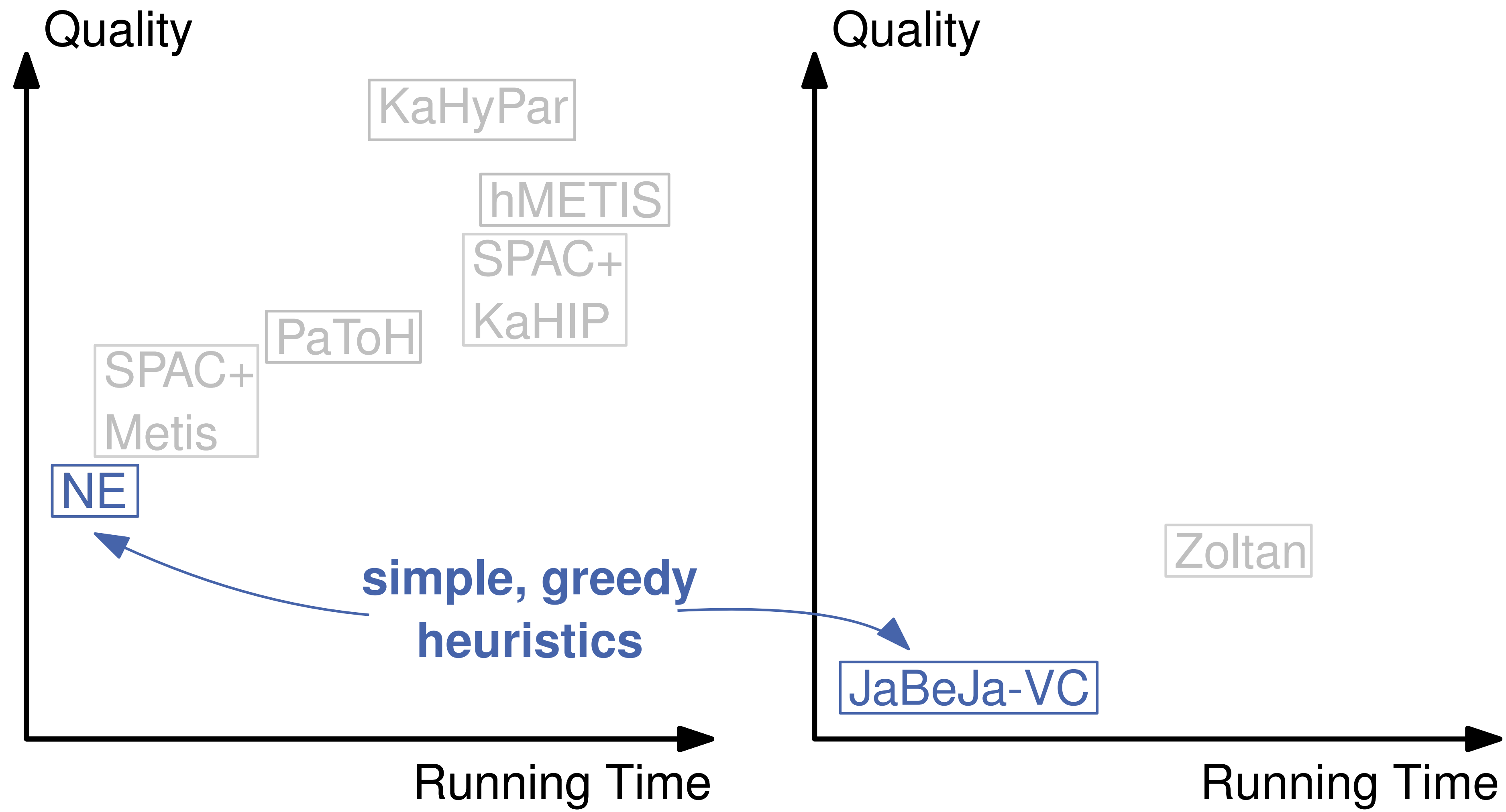
Edge Partitioning Algorithms



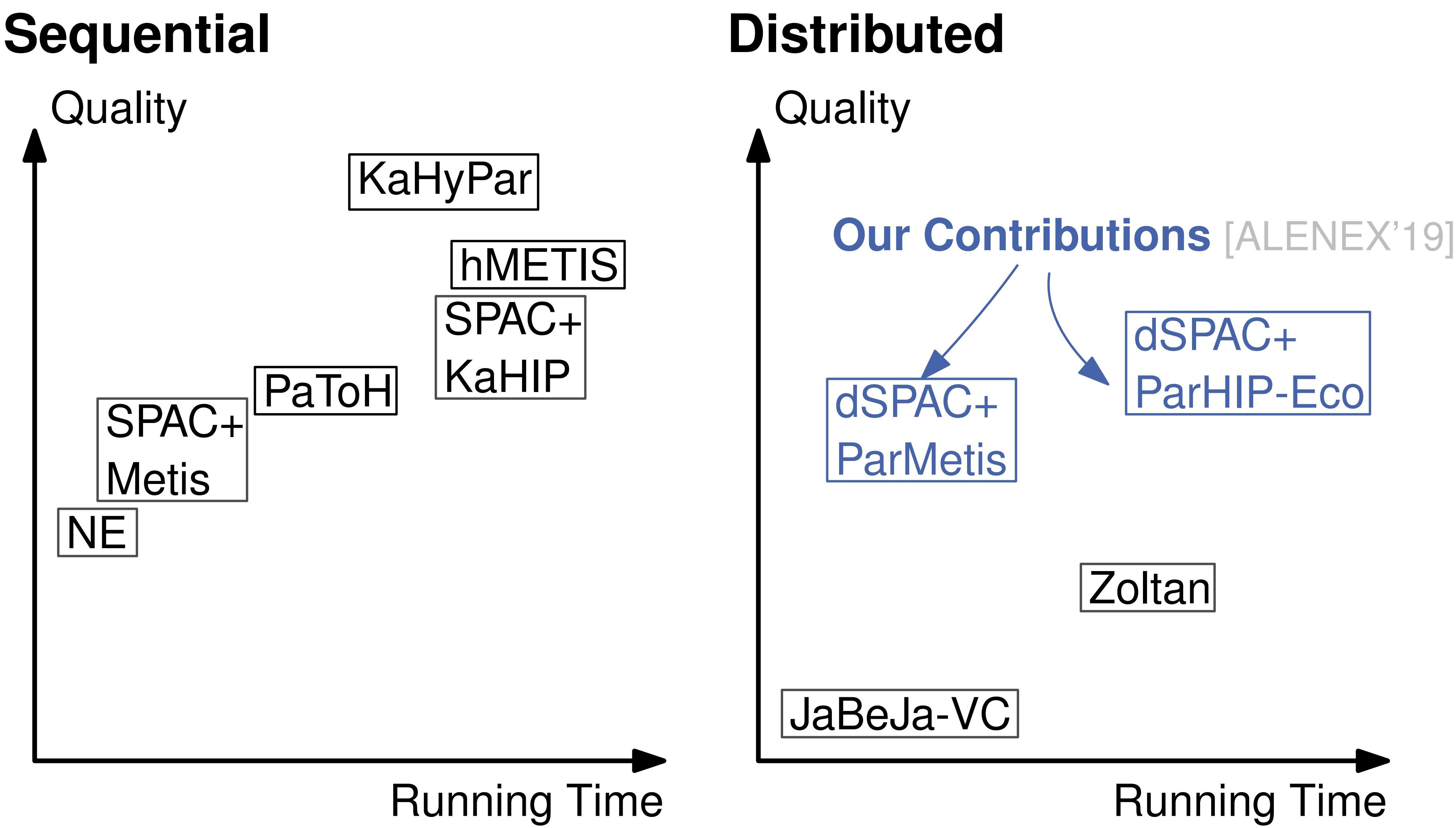
Edge Partitioning Algorithms

Sequential

Distributed



Edge Partitioning Algorithms



Experiments: Benchmark Setup

- Test suite: 70 graphs
  - Walshaw Graph Archive
  - Sparse Matrix-Vector Multiplication
  - Web & Social Graphs
  - Random Geometric Graphs
- $k \in \{2, 4, 8, 16, 32, 64, 128\}$
- Imbalance: $\epsilon = 3\%$
- Averages of 5 repetitions
- Sequential: 1 core
- Distributed: 32 * 20 cores

Competitors:

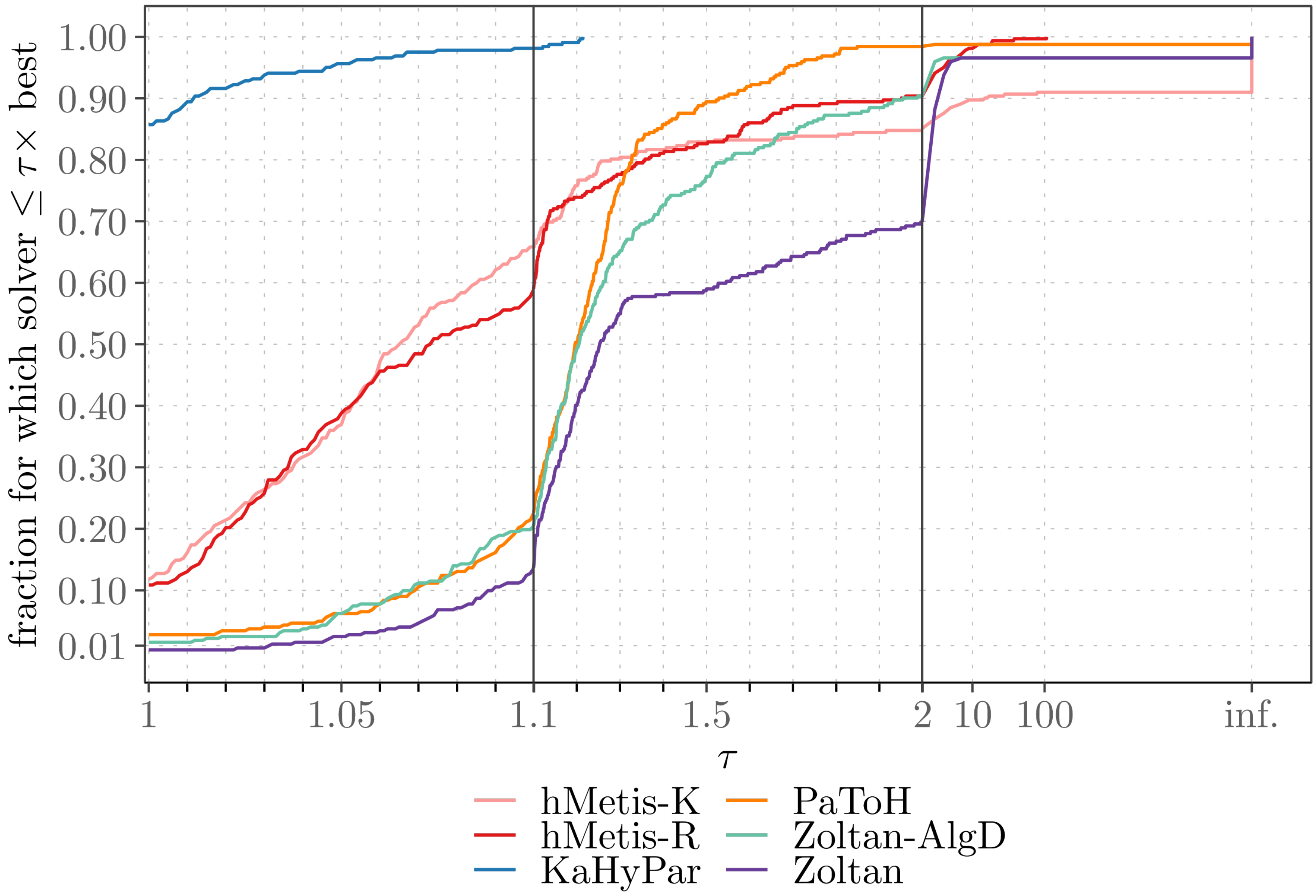
HGP's
- KaHyPar-MF
- PaToH
- Zoltan
- Zoltan-AlgD
- hMetis-{R, K}

- JaBeJa-VC
- NE

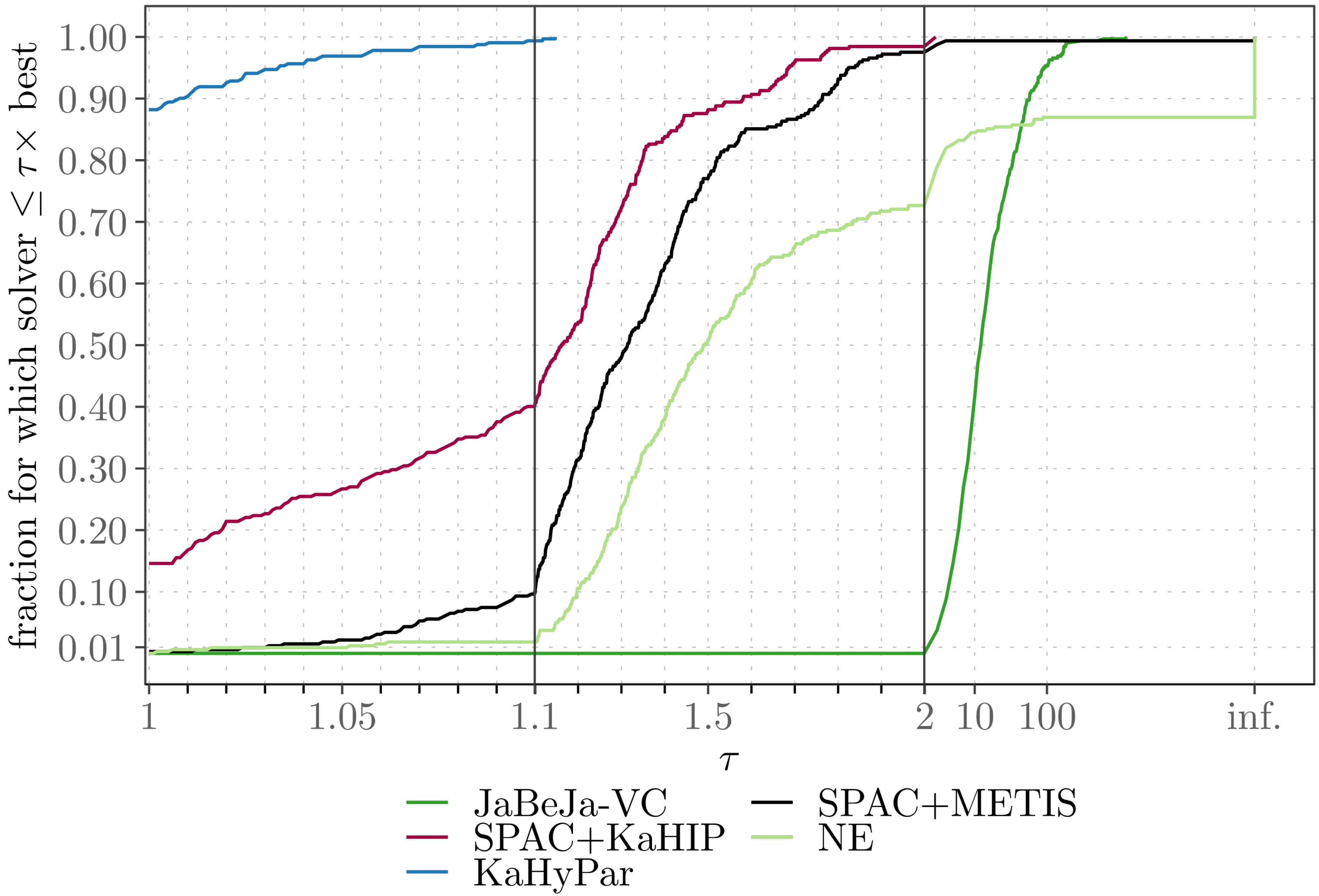
- SPAC + KaHIP
- SPAC + Metis

- dSPAC + ParHIP
- dSPAC + ParMetis

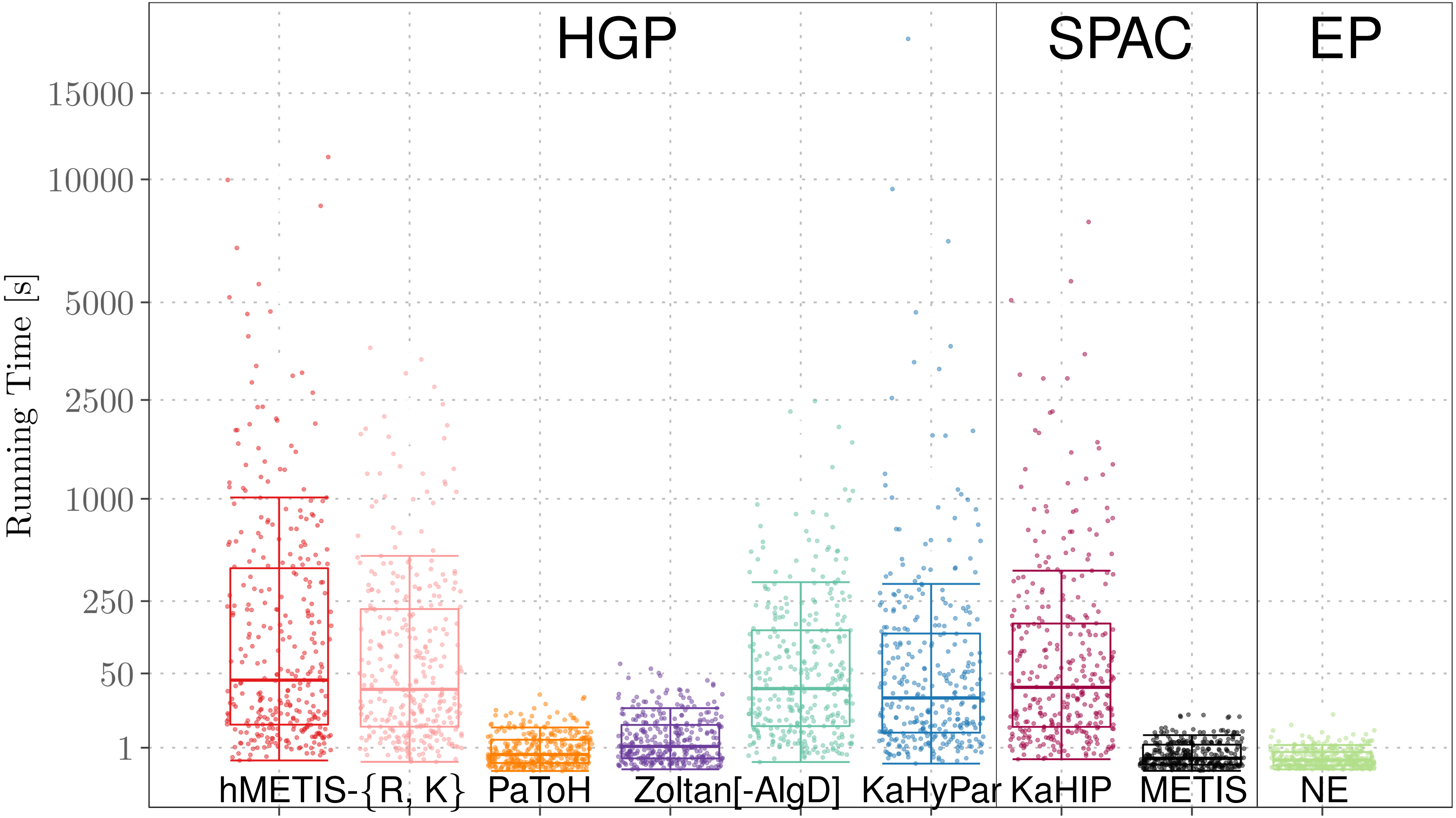
Experiments: Sequential HGP



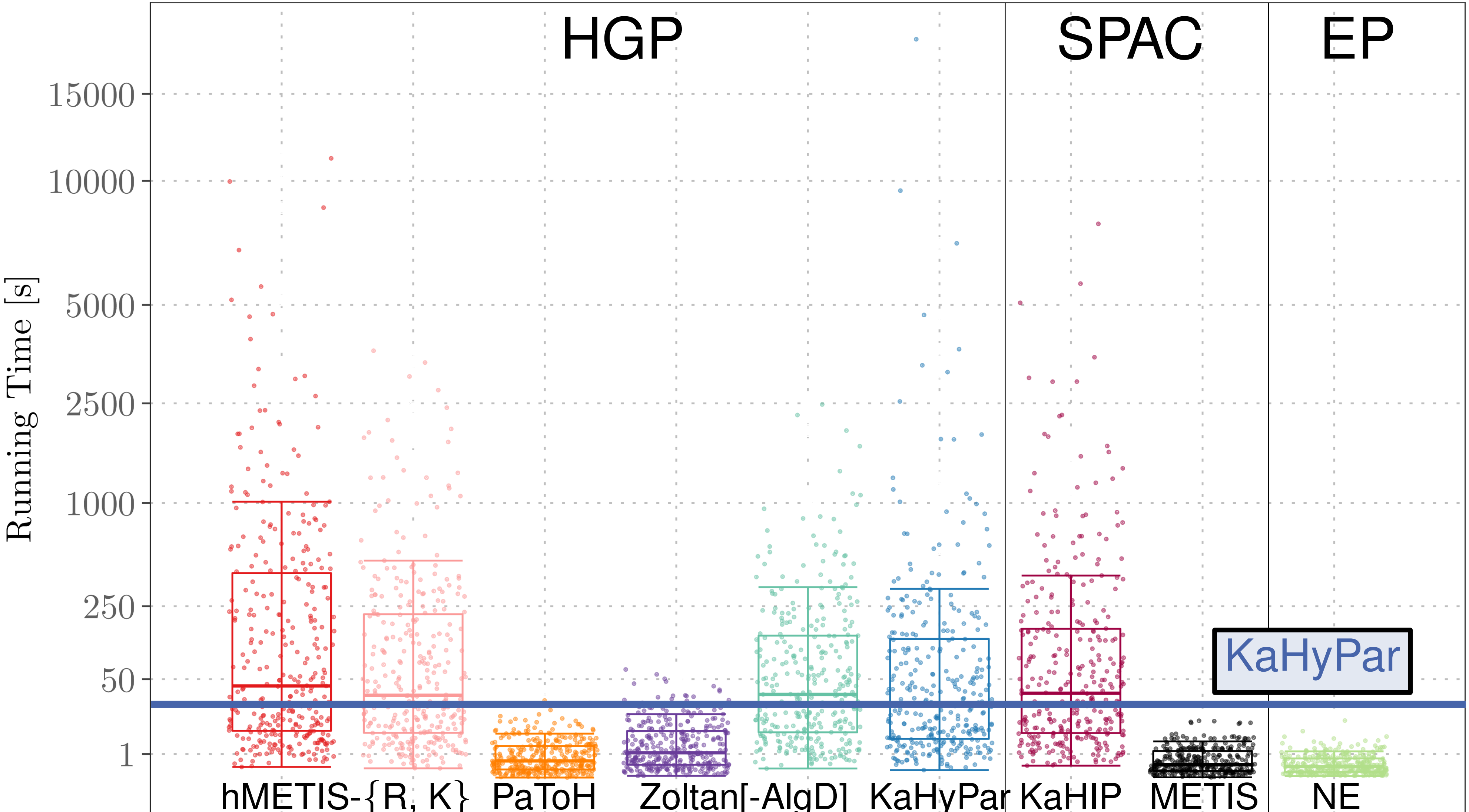
Experiments: Sequential HGP & SPAC+X



Experiments: Sequential Running Time



Experiments: Sequential Running Time



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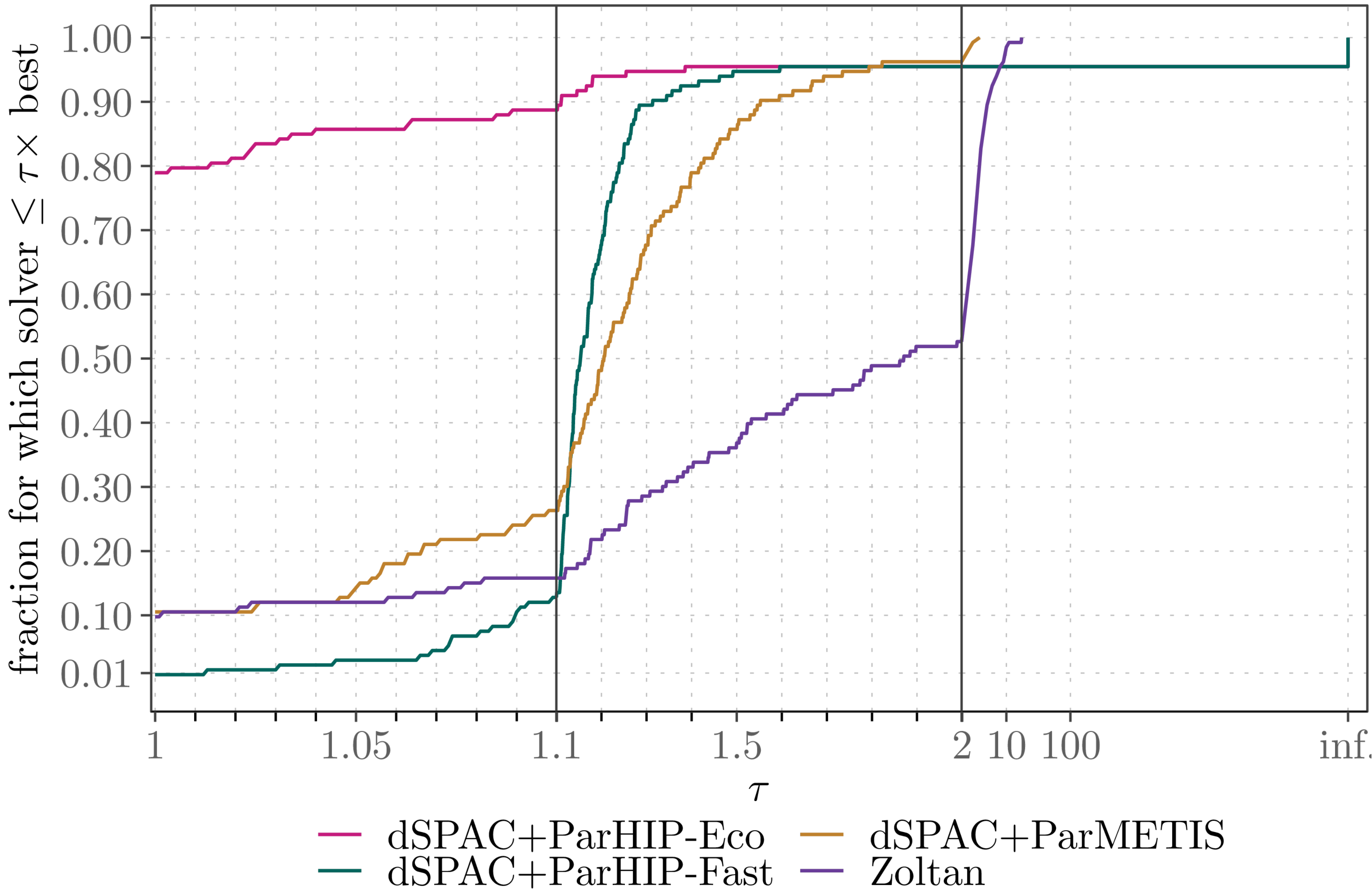
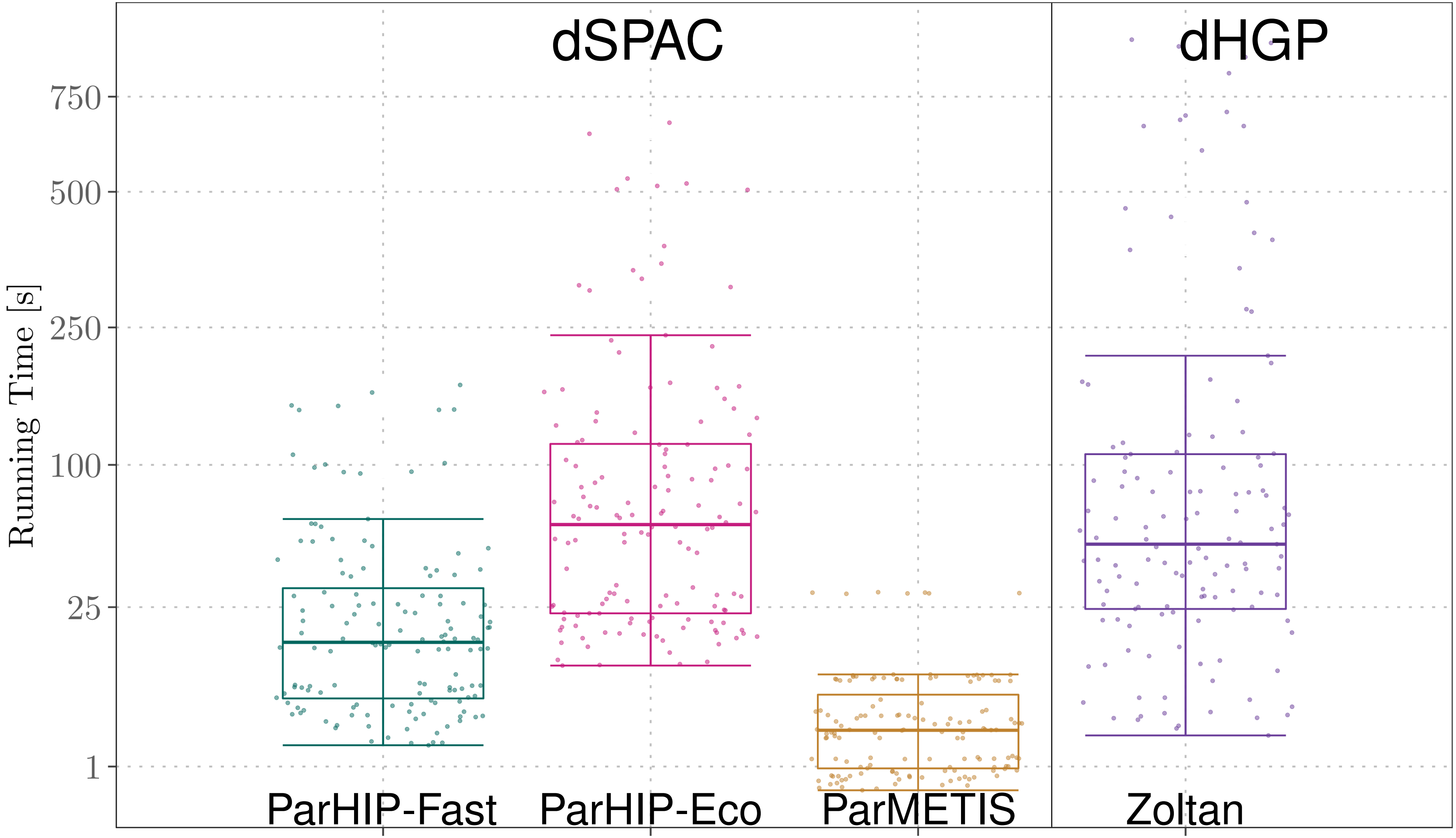
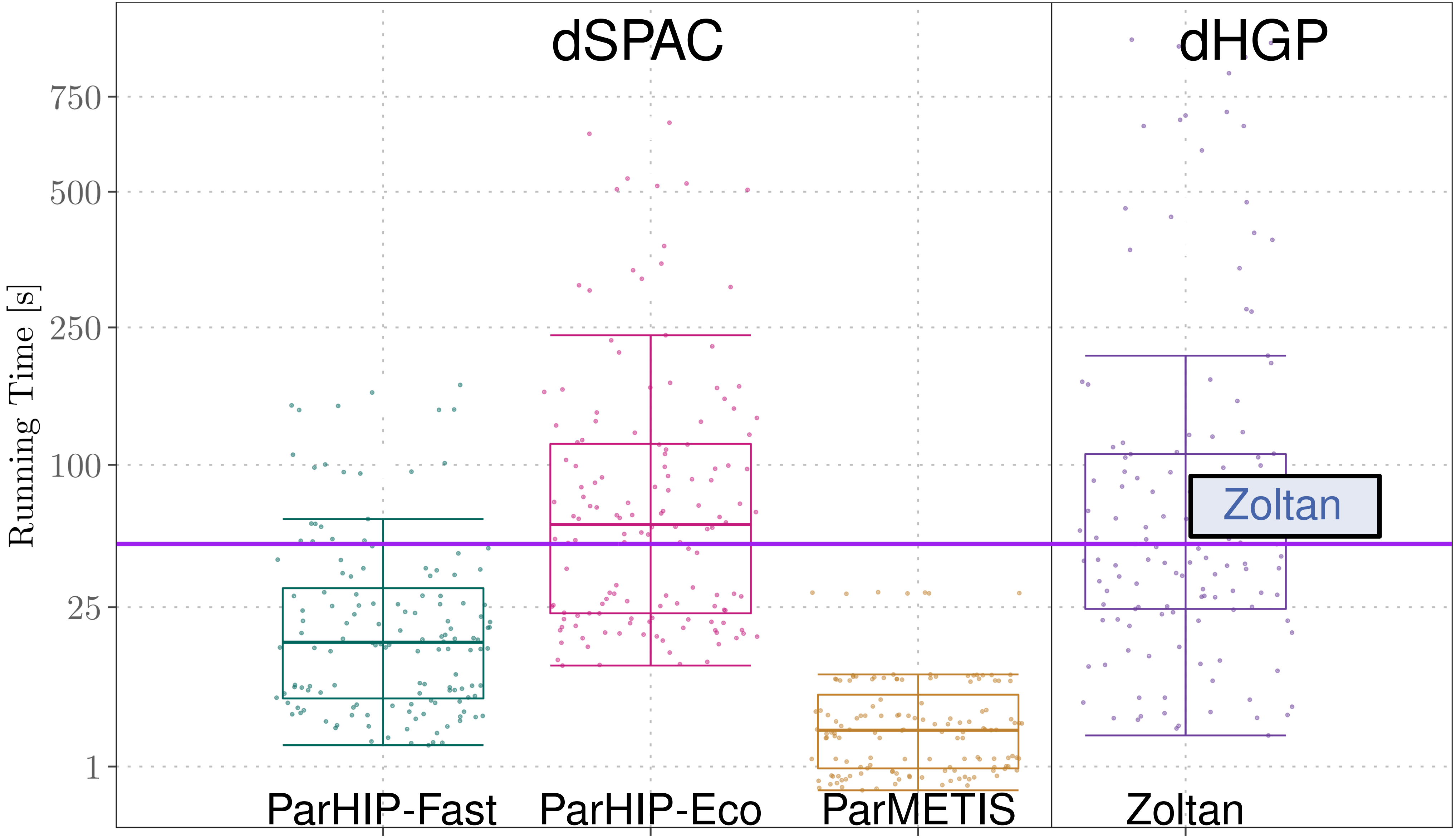
Experiments: Distributed HGP & dSPAC+X



Experiments: Distributed Running Time



Experiments: Distributed Running Time



Conclusion & Outlook

- **High-Quality Graph & Hypergraph Partitioning Frameworks:**
  - KaHIP– http://algo2.iti.kit.edu/kahip/
  - KaHyPar – http://www.kahypar.org
- **Future Work:**
  - Shared-Memory HGP
  - Distributed-Memory HGP
  - Shift focus towards fast (H)GP algorithms with reasonable quality
- **(Personal) Open Questions:**
  - What would benefit the CSC community?
  - What are ”difficult” instances?

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