

A Parallelization of Dijkstra's Shortest Path Algorithm

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Abstract. The single source shortest path (SSSP) problem lacks parallel solutions which are fast and simultaneously work-efficient. We propose simple criteria which divide Dijkstra's sequential SSSP algorithm into a number of phases, such that the operations within a phase can be done in parallel. We give a PRAM algorithm based on these criteria and analyze its performance on random digraphs with random edge weights uniformly distributed in $[0, 1]$. We use the $\mathcal{G}(n, d/n)$ model: the graph consists of n nodes and each edge is chosen with probability d/n . Our PRAM algorithm needs $\mathcal{O}(n^{1/3} \log n)$ time and $\mathcal{O}(n \log n + dn)$ work with high probability (whp). We also give extensions to external memory computation. Simulations show the applicability of our approach even on non-random graphs.

1 Introduction

Computing shortest paths is an important combinatorial optimization problem with numerous applications. Let $G = (V, E)$ be a directed graph, $|E| = m$, $|V| = n$, let s be a distinguished vertex of the graph, and c be a function assigning a non-negative real-valued *weight* to each edge of G . The *single source shortest path problem* (SSSP) is that of computing, for each vertex v reachable from s , the weight $\text{dist}(v)$ of a minimum-weight path from s to v ; the weight of a path is the sum of the weights of its edges.

The theoretically most efficient sequential algorithm on digraphs with non-negative edge weights is Dijkstra's algorithm [8]. Using Fibonacci heaps its running time is $\mathcal{O}(n \log n + m)$ ¹. Dijkstra's algorithm maintains a partition of V into *settled*, *queued* and *unreached* nodes and for each node v a *tentative distance* $\text{tent}(v)$; $\text{tent}(v)$ is always the weight of some path from s to v and hence an upper bound on $\text{dist}(v)$. For unreached nodes, $\text{tent}(v) = \infty$. Initially, s is queued, $\text{tent}(s) = 0$, and all other nodes are unreached. In each iteration, the queued node v with smallest tentative distance is selected and declared settled and all edges (v, w) are relaxed, i.e., $\text{tent}(w)$ is set to $\min\{\text{tent}(w), \text{tent}(v) + c(v, w)\}$.

¹ There is also an $\mathcal{O}(n + m)$ time algorithm for undirected graphs [20], but it requires the RAM model instead of the comparison model which is used in this work.

If w was unreached, it is now queued. It is well known that $\text{tent}(v) = \text{dist}(v)$, when v is selected from the queue.

The queue may contain more than one node v with $\text{tent}(v) = \text{dist}(v)$. All such nodes could be removed simultaneously, the problem is to identify them. In Sect. 2 we give simple sufficient criteria for a queued node v to satisfy $\text{tent}(v) = \text{dist}(v)$. We remove all nodes satisfying the criteria simultaneously.

Although there exist worst-case inputs needing $\Theta(n)$ phases, our approach yields considerable parallelism on random directed graphs: We use the random graph model $\mathcal{G}(n, d/n)$, i.e., there are n nodes and each theoretically possible edge is included into the graph with probability d/n . Furthermore, we assume random edge weights uniformly distributed in $[0, 1]$: In Sect. 3 we show that the number of phases is $\mathcal{O}(\sqrt{n})$ using a simple criterion, and $\mathcal{O}(n^{1/3})$ for a more refined criterion with high probability (whp)².

Sect. 4 presents an adaption of the phase driven approach to the CRCW PRAM model which allows p processors (PUs) concurrent read/write access to a shared memory in unit cost (e.g. [13]). We propose an algorithm for random graphs with random edge weights that runs in $\mathcal{O}(n^{1/3} \log n)$ time whp. The work, i.e., the product of its running time and the number of processors, is bounded by $\mathcal{O}(n \log n + dn)$ whp.

In Sect. 5 we adapt the basic idea to external memory computation (I/O model [22]) where one assumes large data structures to reside on D disks. In each I/O operation, D blocks from distinct disks, each of size B , can be accessed in parallel. We derive an algorithm which needs $\mathcal{O}(\frac{n}{D} + \frac{dn}{DB} \log_{S/B} \frac{dn}{DB})$ I/Os on random graphs whp and can use up to $D = \mathcal{O}(\min\{n^{2/3}/\log n, \frac{S}{B}\})$ independent disks. S denotes the size of the internal memory.

In Sect. 6 we report on simulations concerning the number of phases needed for both random graphs and real world data. Finally, Sect. 7 summarizes the results and sketches some open problems and future improvements.

Previous Work

PRAM algorithms: There is no parallel $\mathcal{O}(n \log n + m)$ work PRAM algorithm with sublinear running time for general digraphs with non-negative edge weights. The best $\mathcal{O}(n \log n + m)$ work solution [9] has running time $\mathcal{O}(n \log n)$. All known algorithms with polylogarithmic execution time are work-inefficient. ($\mathcal{O}(\log^2 n)$ time and $\mathcal{O}(n^3 (\log \log n / \log n)^{1/3})$ work for the algorithm in [11].) An $\mathcal{O}(n)$ time algorithm requiring $\mathcal{O}((n + m) \log n)$ work was presented in [3].

For special classes of graphs, like planar digraphs [21] or graphs with separator decomposition [6], more efficient algorithms are known. Randomization was used in order to find approximate solutions [5]. Random graphs with *unit* weight edges are considered in [4]. The solution is restricted to dense graphs ($d = \Theta(n)$) or edge probability $d = \Theta(\log^k n/n)$ ($k > 1$). In the latter case $\mathcal{O}(n \log^{k+1} n)$ work is needed. Properties of shortest paths in complete graphs ($d = n$) with

² Throughout this paper “whp” stands for “with high probability” in the sense that the probability for some event is at least $1 - n^{-\beta}$ for a constant $\beta > 0$.

random edge weights are investigated in [10,12]. In contrast to all previous work on random graphs, we are most interested in the case of small, even constant d . **External Memory:** The best result on SSSP was published in [16]. This algorithm requires $\mathcal{O}(n + \frac{m}{DB} \log_2 \frac{m}{B})$ I/Os. The solution is only suitable for small n because it needs $\Theta(n)$ I/Os.

2 Running Dijkstra's Algorithm in Phases

We give several criteria for dividing the execution of Dijkstra's algorithm into phases. In the first variant (OUT-version) we compute a threshold defined via the weights of the *outgoing* edges: let $L = \min\{\text{tent}(u) + c(u, z) : u \text{ is queued and } (u, z) \in E\}$ and remove all nodes v from the queue which satisfy $\text{tent}(v) \leq L$. Note that when v is removed from the queue then $\text{dist}(v) = \text{tent}(v)$. The threshold for the OUT-criterion can either be computed via a second priority queue for $o(v) = \text{tent}(v) + \min\{c(v, u) : (v, u) \in E\}$ or even on the fly while removing nodes.

The second variant, the IN-version, is defined via the *incoming* edges: let $M = \min\{\text{tent}(u) : u \text{ is queued}\}$ and $i(v) = \text{tent}(v) - \min\{c(u, v) : (u, v) \in E\}$ for any queued vertex v . Then v can be safely removed from the queue if $i(v) \leq M$. Removable nodes of the IN-type can be found efficiently by using an additional priority queue for $i(\cdot)$.

Finally, the INOUT-version applies both criteria in conjunction.

3 The Number of Phases for Random Graphs

In this section we first investigate the number of delete-phases for the OUT-variant of Dijkstra's algorithm on random graphs. Then we sketch how to extend the analysis to the INOUT-approach. We start with mapping the OUT-approach to the analysis of the reachability problem as provided in [14] and [1, Sect. 10.5] and give lower bounds on the probability that many nodes can be removed from the queue during a phase.

Theorem 1. OUT-approach. *Given a random graph from $\mathcal{G}(n, d/n)$ with edge labels uniformly distributed in $[0, 1]$, the SSSP problem can be solved using $r = \mathcal{O}(\sqrt{n})$ delete-phases with high probability.*

We review some facts of the reachability problem using the notation of [1].

The following procedure determines all nodes reachable from a given node s in a random graph G from $\mathcal{G}(n, d/n)$. Nodes will be neutral, active, or dead. Initially, s is active and all other nodes are neutral, let time $t = 0$, and $Y_0 = 1$ the number of active nodes. In every time unit we select an arbitrary active node v and check all theoretically possible edges (v, w) , w neutral, for membership in G . If $(v, w) \in E$, w is made active, otherwise it stays neutral. After having treated all neutral w in that way, we declare v dead, and let Y_t equal the new number of active nodes. The process terminates when there are no active nodes.

The connection with the OUT-variant of Dijkstra's algorithm is easy: The distance labels determine the order in which queued vertices are considered and declared dead, and time is partitioned into intervals (=phases): If a phase of the OUT-variant removes k nodes this means that the time t increases by k .

Let Z_t be the number of nodes w that are reached for the first time at time t . Then $Y_0 = 1$, $Y_t = Y_{t-1} + Z_t - 1$ and $Z_t \sim B[n - (t-1) - Y_{t-1}, d/n]$ where $B[n, q]$ denotes the binomial distribution for n trials and success probability q .

Let T be the least t for which $Y_t = 0$. Then T is the number of nodes that are reachable from s . The recursive definition of Y_t is continued for all t , $0 \leq t \leq n$. We have $Y_t \sim B[n-1, 1 - (1-d/n)^t] + 1 - t$.

It is shown in [1] that the number of nodes reachable from s is either very small (less than $\mathcal{O}(\log n)$) or concentrates around $T_0 = \alpha_0 n$, where $0 < \alpha_0 < 1$, and $\alpha_0 = 1 - e^{-d\alpha_0}$. Only the case $T \approx T_0$ requires analysis; if $T = \mathcal{O}(\log n)$ the number of phases is certainly small. Chernoff bounds yield:

Lemma 1. *Except for small t ($t \leq \sqrt{n}$) and large t ($t \geq T_0 - n^{1/2+\epsilon}$) Y_t is $(1 \pm o(1/n^2))\mathbf{E}[Y_t]$ with high probability.*

The *yield* of a phase in the OUT-variant is the number of nodes that are removed in a phase. We call a phase starting at time t *profitable* if its yield is $\Omega(\sqrt{Y_t/d})$ and *highly profitable* if its yield is $\Omega(\sqrt{(Y_{t/2} - t/2)t/n})$ and show:

Lemma 2. *A phase is profitable with probability at least $1/8$. A phase starting at time t with $\frac{n \ln d}{d} \leq t \leq \alpha_0 n - n/d$ is highly profitable with probability at least $1/8$.*

Theorem 1 follows fairly easily from lemmas 1 and 2: We call a phase with starting time t *early extreme* if $t \leq \sqrt{n}$, *early intermediate* if $\sqrt{n} < t \leq (n \ln d)/d$, *early central* if $(n \ln d)/d < t \leq n/2$, *late central* if $n/2 < t \leq \alpha_0 n - n/d$, *late intermediate* if $\alpha_0 n - n/d < t \leq \alpha_0 n - n^{1/2+\epsilon}$, and *late extreme* if $\alpha_0 n - n^{1/2+\epsilon} < t$, and show that there are only $\mathcal{O}(\sqrt{n})$ phases of each kind with high probability. Consider, for example, the late intermediate phases. A profitable late intermediate phase starting at time t has yield $\Omega(\sqrt{Y_t/d}) = \Omega(\sqrt{\mathbf{E}[Y_t]/d}) = \Omega(\sqrt{(\alpha_0 n - t)/d})$, where the first equality holds with high probability by Lemma 1. Let $t' := \alpha_0 n - t$. The number of profitable phases with $2^i \leq t' < 2^{i+1}$ is therefore $\mathcal{O}(\sqrt{2^i d})$ and the number of profitable phases with $\alpha_0 n - n/d \leq t = \alpha_0 n - t'$ is therefore $\sum_{i \leq \log(n/d)} \mathcal{O}(\sqrt{2^i d}) = \mathcal{O}(\sqrt{n})$. Since a phase is profitable with probability at least $1/8$, the number of phases is also $\mathcal{O}(\sqrt{n})$ with high probability. The number of early extreme phases is $\mathcal{O}(\sqrt{n})$ trivially. For the number of late extreme phases we argue as follows. We first show that $T \leq \alpha_0 n + n^{1/2+\epsilon}$ with high probability and then consider the first time t_1 , $t_1 \geq \alpha_0 n - n^{1/2+\epsilon}$, with $Y_{t_1} \leq n^{1/4}$. Lemma 1 implies that the number of late extreme phases starting before t_1 is $\mathcal{O}(\sqrt{n})$. If the number of phases starting after t_1 is \sqrt{n} or more, then $Z_{t_1} + Z_{t_1+1} + \dots + Z_{t_1+\sqrt{n}} \geq \sqrt{n} - n^{1/4} \geq \sqrt{n}/2$. The probability of this event is bounded by $\mathbf{P}[B[n^{1/2}(n - (n - n^{1/2+\epsilon})), d/n] \geq \sqrt{n}/2]$, which is exponentially small.

The idea for the proof of Lemma 2 is as follows. Let v_1, v_2, \dots, v_q , $q = Y_t$, be the queued nodes in order of increasing tentative distances, and let L' be the value of L in the previous phase. The distance labels $\text{tent}(v_i)$ are random variables in $[L', L' + 1]$. We show that their values are independent and their distributions are biased towards smaller values (since $\text{tent}(v_i) = \min\{\text{dist}(v) + c(v, v_i), v \text{ settled and } (v, v_i) \in E\}$, $\text{dist}(v) \leq L'$, $c(v, v_i)$ uniform in $[0, 1]$). The value of $\text{tent}(v_r)$ is therefore less than r/q with constant probability for arbitrary r , $1 \leq r \leq q$. The number of edges out of v_1, \dots, v_r is $r(d/n)n = rd$ on the average and not much more with constant probability. The shortest of these edges has length about $\frac{1}{rd}$. We remove v_1, \dots, v_r from the queue if $\text{tent}(v_r)$ is smaller than the length of the shortest edge out of v_1, \dots, v_r . This is the case (with constant probability) if $r/q \leq \frac{1}{rd}$ or $r \leq \sqrt{q/d}$.

For the phases starting at time t with $(n \ln d)/d \leq t \leq \alpha_0 n - n/d$ we refine the argument as follows. We call a node queued at time t *old* if it was already queued before time $t/2$ and show that the number of old queued nodes at time t is at least $Y_{t/2} - t/2$. Each old queued node has an expected indegree from settled nodes of at least $\frac{t}{2} \frac{d}{n}$. We use this fact to deduce that $\text{tent}(v_r)$ is less than $r / (\frac{td}{2n}(Y_{t/2} - t/2))$ with constant probability and then proceed as above.

INOUT Approach. If both IN- and OUT-criterion are applied together, the tentative distance labels of queued nodes may spread over a range as large as $[L', L' + 2)$, while the edge weights are only in $[0, 1]$. In order to reuse the analysis of the OUT-part we analyze a slightly slower version which alternates the two criteria in the following way:

I-Step: Let q be the current queue size. Apply the IN-criterion to the $g(q)$ nodes with smallest tentative distances where g is a function we are free to choose³. Let L be the largest distance of any removed node. Switch to *O-Step*.

O-Step: Repeatedly apply the OUT-criterion until no tentative distance is smaller than L . Then switch back to *I-Step*.

The function $g()$ is chosen in such a way that there is both a constant probability for a large yield in an *I-Step* and the expected number of subsequent *O-Steps* is constant. The function $g()$ is chosen dependent of the current phase type. For example, during late intermediate phases we take $g(q) = cq^{2/3}/d^{1/3}$ for some constant c . A super-phase consisting of an *I-Step* and series of *O-Steps* is now profitable if at most a constant number of *O-Steps* is needed and if its total yield is $\Omega(Y_t^{2/3}/d^{1/3})$, highly profitable if its yield is $\Omega((Y_{t/2} - t/2)^{2/3}/(n/t)^{1/3})$. Then one has to show again that a super-phase is (highly) profitable with constant probability.

Theorem 2. INOUT-approach. *Given a random graph from $\mathcal{G}(n, d/n)$ with edge labels uniformly distributed in $[0, 1]$, the SSSP problem can be solved using $r = \mathcal{O}(n^{1/3})$ delete-phases with high probability.*

³ Note that the implementation does not need to know this function since it uses the faster combined criterion.

4 Parallelization

We now show how the sequential OUT-variant of Sect. 2 can be efficiently implemented on an arbitrary-write CRCW PRAM for random graphs from $\mathcal{G}(n, d/n)$ and random edge weights. The actual number of edges is $m = \Theta(dn)$ whp.

The algorithm keeps a global array $\text{tent}(\cdot)$ for all tentative distance values. Each processor P_i , $0 \leq i < p$ is responsible for two sequential priority queues: Q_i and Q_i^* . Each pair (Q_i, Q_i^*) only deals with a subset of nodes, the distribution is made randomly and stored in a global array $\text{ind}()$. Furthermore, each PU maintains a buffer array for incoming requests.

The queues Q_i handle tentative node distances for the nodes they are responsible for, the key of a node $v \in Q_i^*$ is given by $\text{tent}(v) + \delta_o(v)$ where $\delta_o(v) := \min\{c(v, w) : (v, w) \in E\}$; $\delta_o(v)$ is recomputed once and for all upon initialization. The Q_i^* queues are used to efficiently derive the criterion of the OUT-version indicating whether a node can be deleted in a phase. The queues are implemented as relaxed heaps [9] because they provide worst-case running times: `findMin`, `insert` and `decreaseKey` are performed in $\mathcal{O}(1)$ time and `delete/deleteMin` in $\mathcal{O}(\log q)$ time where q denotes the local queue size.

Let r be the number of delete-phases which are needed, e.g. for the OUT-variant $r = \mathcal{O}(\sqrt{n})$ whp. For the analysis we fix the number of processors as $p = \max\{\frac{n}{r \log n}, \frac{dn}{r \log^2 n}\}$; so from now on a time bound T implies a work bound pT .

The algorithm works similar to Dijkstra's algorithm: The queues start with only s in $Q_{\text{ind}(s)}$ and $Q_{\text{ind}(s)}^*$ and all other local queues empty. This and the initialization of other arrays and buffers ($\text{ind}()$, outgoing edges, ...) can be done in time $\mathcal{O}((n+m)/p) = \mathcal{O}(r \log^2 n)$ whp, even if the input uses an adjacency-list representation.

While any queue is nonempty the algorithm performs a phase consisting of five steps. These steps are now further explicated together with the most interesting part of their analysis, namely for the case that at most n/r nodes are deleted in this phase.

Step 1 finds the global minimum L of all elements in all Q_i^* and can clearly be performed in $\mathcal{O}(\log p) \leq \mathcal{O}(\log n)$ time.

In **Step 2** each PU i removes the nodes with $\text{tent}(v) \leq L$ from Q_i and Q_i^* . Let \check{R} denote the union of all these sets of deleted nodes. Our index distribution ensures that no PU has to deal with more than $\mathcal{O}(\log p + |\check{R}|/p)$ `deleteMins` whp. A single `deleteMin` or `delete` operation takes $\mathcal{O}(\log n)$ time, thus due to $|\check{R}| \leq n/r$ and $p = \max\{\frac{n}{r \log n}, \frac{dn}{r \log^2 n}\}$ Step 2 can be performed in $\mathcal{O}(\log^2 n)$ time whp.

In **Step 3** all PUs cooperate to generate a set $\text{Req} := \{(w, \text{tent}(v) + c((v, w))) : v \in \check{R} \text{ and } (v, w) \in E\}$ of *requests*. By compacting \check{R} and using prefix sums to schedule the PUs this task can be perfectly load balanced. Since $|\text{Req}| = \mathcal{O}(d|\check{R}| + \log n)$ whp for $|\check{R}| \leq n/r$, this step can be performed in time $\mathcal{O}(m/(rp) + \log n) = \mathcal{O}(\log^2 n)$ whp.

Step 4 permutes the requests such that (w, x) is put into a buffer array $B_{\text{ind}(w)}$. Altogether there are at most $\mathcal{O}(d|\check{R}|)$ requests whp that are spread over p buffers, thus, because of the random node distribution, each buffer gets $\mathcal{O}(\log n + d|\check{R}|/p) = \mathcal{O}(\log^2 n)$ requests whp (Chernoff bounds, $|\check{R}| \leq n/r$, $p = \max\{\frac{n}{r \log n}, \frac{dn}{r \log^2 n}\}$). The requests are placed by “randomized dart throwing” [18]. If each processor is responsible for the placement of a group of $\mathcal{O}(\log^2 n)$ requests (which may go to different buffers) Step 4 takes $\mathcal{O}(\log^2 n)$ time whp. The dart throwing progress is regularly monitored. In the unlikely case of stagnation (buffers are chosen too small), the buffer sizes are adapted.

Finally, in **Step 5** PU i scans buffer i and for each request (w, x) with $x < \text{tent}(w)$ it updates $\text{tent}(w)$ to x and calls `decreaseKey`(Q_i, w, x), `decreaseKey`($Q_i^*, w, x + \delta_o(w)$) (respectively `insert` for new nodes). Each operation can be executed in $\mathcal{O}(1)$ time, so for $|\check{R}| \leq n/r$ Step 5 needs time $\mathcal{O}(\log^2 n)$ whp.

Phases with $|\check{R}| > n/r$ show whp at least as balanced queue access patterns as those phases deleting less elements, thus time and work of a phase increase at most linearly. Let k_i denote the number of nodes removed in phase i . Then $\sum_{i \leq r} k_i \leq n$. The total time over all phases is $T = \mathcal{O}(\sum_{i \leq r} [k_i r/n] \log^2 n) = \mathcal{O}(r \log^2 n + (nr/n) \log^2 n) = \mathcal{O}(r \log^2 n)$ whp.

For $d > r \log^2 n$ more than n PUs can be used by dropping explicit queues: n global bits denote whether an element is “queued” or not and p/n PUs take care of each buffer area in order to cope with the increased number of requests. Alternatively, one can apply an initial filtering step because all but the $c \log n$ smallest edges per node, c some constant, can be ignored whp without changing the shortest paths [10, 12].

The INOUT-version is supported by p additional priority queues. Initialization of $\delta_i(v) := \min\{c(w, v) : (w, v) \in E\}$ involves collecting the weights of edges that are potentially distributed over $\Omega(d)$ adjacency-lists. For random graphs, the number of incoming edges of $k = \Omega(\log n)$ randomly selected nodes is $\mathcal{O}(dk)$ whp. Thus, we can use the randomized dart throwing to perform the initialization using $\mathcal{O}(dn)$ work whp.

Theorem 3. *If the number of delete-phases is bounded by r then the SSSP can be solved in $\mathcal{O}(r \log^2 n)$ time and $\mathcal{O}(n \log n + m)$ work whp. using $\max\{\frac{n}{r \log n}, \frac{m}{r \log^2 n}\}$ processors on a CRCW PRAM.*

The running time can be improved by a factor of $\mathcal{O}(\log n)$ if we choose an alternative implementation for the queues based on the parallel priority queue data structure from [19] which supports `insert` and `deleteMin` for $\mathcal{O}(p)$ elements in time $\mathcal{O}(\log n)$ using p PUs whp. In [7] we show how to augment this data structure so that `decreaseKey` and `delete` are also supported.

A queue is represented by three relaxed heaps: A main heap Q_1 , a buffer Q_0 for newly inserted elements plus the $\mathcal{O}(\log n)$ smallest ones and Q_d for elements whose key drops below a bound L' due to a `decreaseKey`. Deleted elements in Q_1 are only marked as deleted. More generally, `delete` and `deleteMin` are most of the time only performed on Q_0 and Q_d and only every $\mathcal{O}(\log n)$ phases

a function `cleanUp` is called which guarantees that Q_0 and Q_d do not grow too large. For an analysis we refer to [19, 7].

Corollary 1. *SSSP on random graphs with random edge weights uniformly distributed in $[0, 1]$ can be solved on a CRCW PRAM in $\mathcal{O}(n^{1/3} \log n)$ time and $\mathcal{O}(n \log n + m)$ work whp.*

The approach is relatively easy to adapt to distributed memory machines. The ind-array can be replaced by a hash-function and randomized dart throwing by routing. For random graphs, the PU scheduling for generating requests is unnecessary, if the number of PUs is decreased by a logarithmic factor.

The algorithm can also be adapted to a $\mathcal{O}(n^{1/3+\epsilon})$ time and $\mathcal{O}(n \log n + m)$ work EREW PRAM for an arbitrary small constant $\epsilon > 0$. Concurrent write accesses only occur during the randomized dart throwing. It can be replaced by $1/\epsilon$ reordering phases (essentially radix sorting), such that phase i groups all request for a subset of $p^{1-\epsilon i}$ queue pairs. Processors are rescheduled after each phase. After the last phase all requests to a certain queue pair are grouped together and can be handled sequentially.

5 Adaption to External Memory

The best previous external memory SSSP algorithm is due to [16]. It requires at least n I/Os and hence is unsuitable for large n . For our improved algorithm we use D to denote the number of disks and B to denote the block size. Let r be the number of delete-phases and assume for simplicity that each phase removes n/r elements from the queue.

Furthermore, we assume that $D \log D \leq n/r$ and that the internal memory, S , is large enough to hold *one* bit per node. It is indicated in [7] how to proceed if this reasonable assumption does not hold. We partition the adjacency-lists into blocks of size B and distribute the blocks randomly over the disks. All requests to adjacency-lists of a single phase are first collected in D buffers, in large phases they are possibly written to disk temporarily. At the end of a phase the requests are performed in parallel. If $D \log D \leq n/r$, the n/r adjacency-lists to be considered in a phase will distribute almost evenly over the disks whp, and hence the time spent in reading adjacency-lists is $\mathcal{O}(n/D + m/(DB))$ whp. We use a priority queue without `decreaseKey` operation (e.g. buffer trees [2]) and insert a node as often as it has incoming edges (each edge may give a different tentative distance). When a node is removed for the first time its bit is set. Later values for that node are ignored.

The total I/O complexity for this approach is given by $\mathcal{O}(\frac{n}{D} + \frac{m}{DB} \log_{S/B} \frac{m}{B})$ I/Os whp. The number of disks is restricted by $D = \mathcal{O}(\min\{\frac{n}{r \log n}, \frac{S}{B}\})$.

We note that it is useful to slightly modify the representation of the graph (provide each edge (v, w) with $\delta_o(w)$, the minimum weight of any edge out of w). This allows us to compute the L -value while deleting elements from the queue without the auxiliary queue Q^* . This online computing is possible because the nodes are deleted with increasing distances and the L -value initialized with

$\text{findMin}() + 1$ can only decrease. The preprocessing to adapt the graph takes $\mathcal{O}(\frac{n+m}{DB} \log_{S/B} \frac{m}{B})$ I/Os.

Theorem 4. *SSSP with r delete-phases can be solved in external memory using $\mathcal{O}(\frac{n}{D} + \frac{m}{DB} \log_{S/B} \frac{m}{B})$ I/Os whp if the number of disks is $D = \mathcal{O}(\min\{\frac{n}{r \log n}, \frac{S}{B}\})$ and S is large enough to hold one bit per node.*

6 Simulations

Simulations of the algorithm have greatly helped to identify the theoretical bounds to be proven. Furthermore, they give information about the involved constant factors.

For the OUT-variant on random graphs with random edge weights we found an average value of $2.5\sqrt{n}$ phases. The refined INOUT-variant needs about $6.0n^{1/3}$ phases on the average. A modification of the INOUT-approach which switches between the criteria as described in Sect. 2 takes about $8.5n^{1/3}$ phases.

We also ran tests on planar graphs taken from [15, GB_PLANE] where the nodes have coordinates uniformly distributed in a two-dimensional square and edge weights denote the Euclidean distance between respective nodes. The OUT-version finished in about $1.2n^{2/3}$ phases; taking random edge weights instead, about $1.7n^{2/3}$ phases sufficed on the average. The performance of the INOUT-version is less stable on these graphs; it seems to give only a constant factor improvement over the simpler OUT-variant.

Motivated from the promising results on planar graphs we tested our approach on real-world data: starting with a road-map of a town ($n = 10,000$) the tested graphs successively grew up to a large road-map of Southern Germany ($n = 157,457$). While repeatedly doubling the number of nodes, the average number of phases (for different starting points) only increased by a factor of about $1.63 \approx 2^{0.7}$; for $n = 157,457$ the simulation needed 6,647 phases.

7 Conclusions

We have shown how to subdivide Dijkstra's algorithm into delete phases and gave a simple CRCW PRAM algorithm for SSSP on random graphs with random edge weights which has sublinear running time and performs $\mathcal{O}(n \log n + m)$ work whp. Although the bounds only hold with high probability for random graphs, the approach shows good behavior on practically important real-world graph instances.

Future work can tackle the design and performance of more refined criteria for safe node deletions, in particular concerning non-random inputs.

Another promising approach is to relax the requirement of $\text{tent}(v) = \text{dist}(v)$ for deleted nodes. In [7, 17] we also analyze an algorithm which allows these two values to differ by an amount of Δ . While this approach yields more parallelism for random graphs, the safe criteria do not need tuning parameters and can better adapt to inhomogeneous distributions of edge weights over the graph.

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