Algorithmen II

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Web:

`http://algo2.iti.kit.edu/AlgorithmenII_WS18.php`

5 Maximum Flows and Matchings

[mit Kurt Mehlhorn, Rob van Stee]

Folien auf Englisch

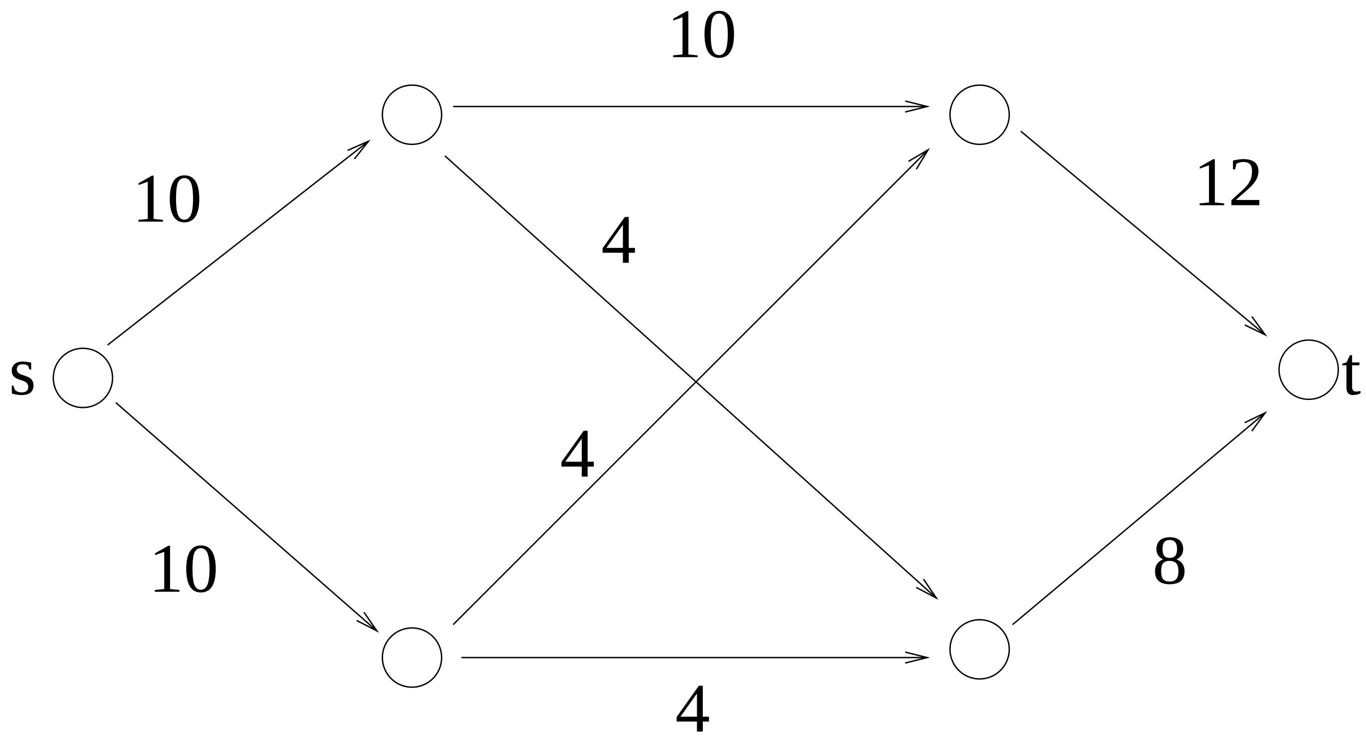
Literatur:

[Mehlhorn / Näher, The LEDA Platform of Combinatorial and Geometric Computing, Cambridge University Press, 1999]
`http://www.mpi-inf.mpg.de/~mehlhorn/ftp/LEDAbook/Graph_alg.ps`

[Ahuja, Magnanti, Orlin, Network Flows, Prentice Hall, 1993]

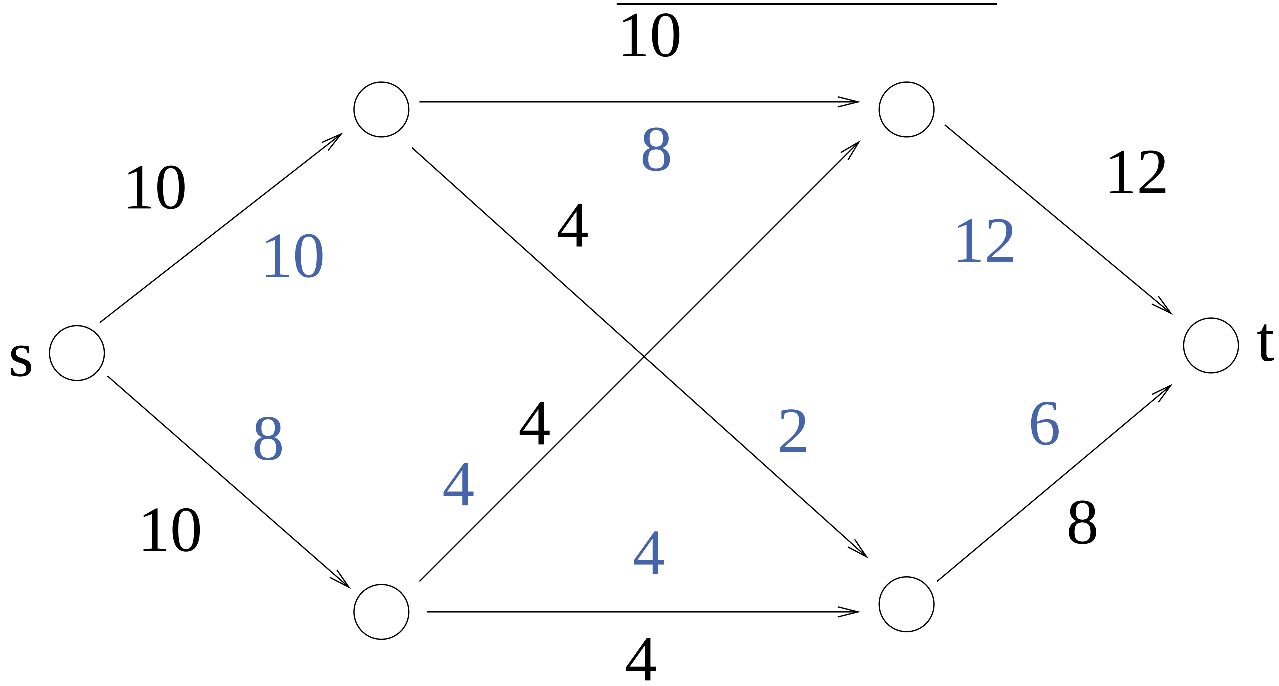
Definitions: Network

- Network = directed weighted graph with source node s and sink node t
- s has no incoming edges, t has no outgoing edges
- Weight c_e of an edge e = capacity of e (nonnegative!)



Definitions: Flows

- Flow = function f_e on the edges, $0 \leq f_e \leq c_e \forall e$
  $\forall v \in V \setminus \{s,t\}$: total incoming flow = total outgoing flow
- Value of a flow $\mathbf{val}(f) =$ total outgoing flow from $s =$ total flow going into t
- Goal: find a flow with maximum value

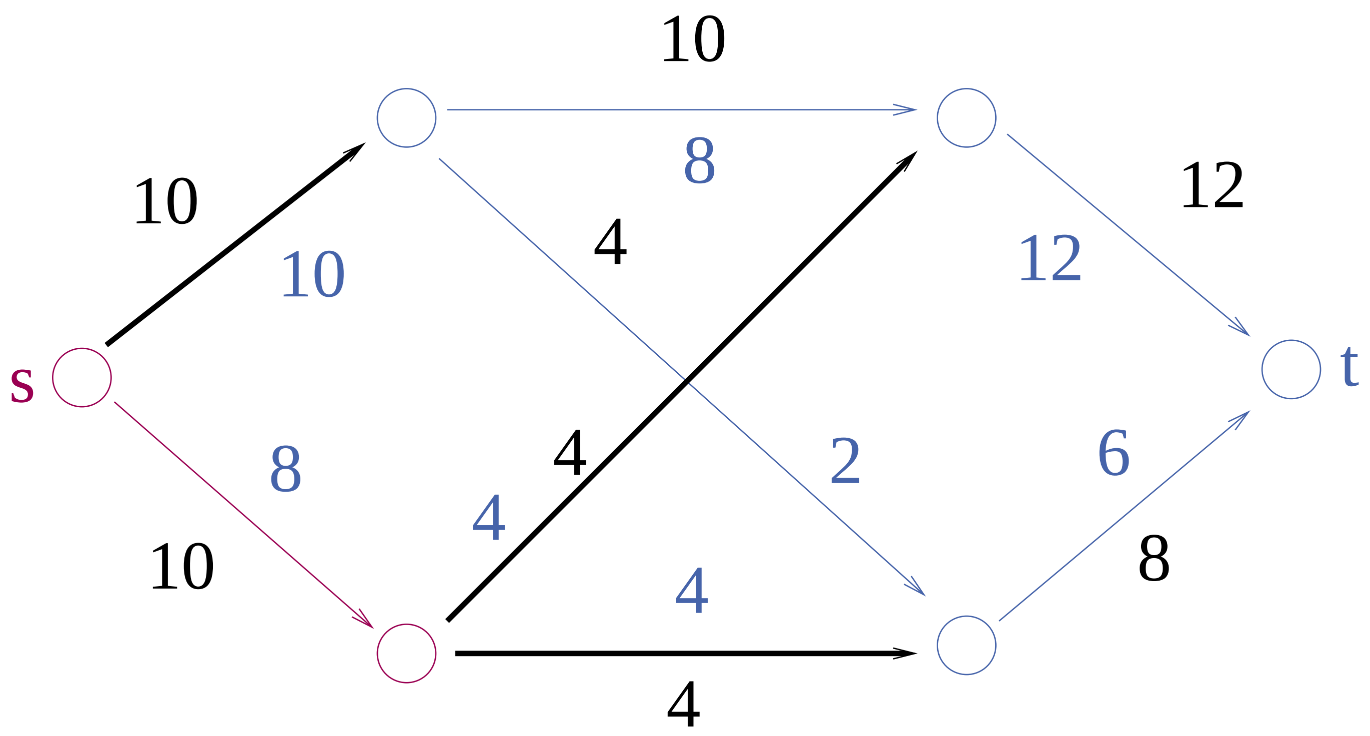


Definitions: (Minimum) s-t Cuts

An s-t cut is partition of V into S and T with $s \in S$ and $t \in T$.

The capacity of this cut is:

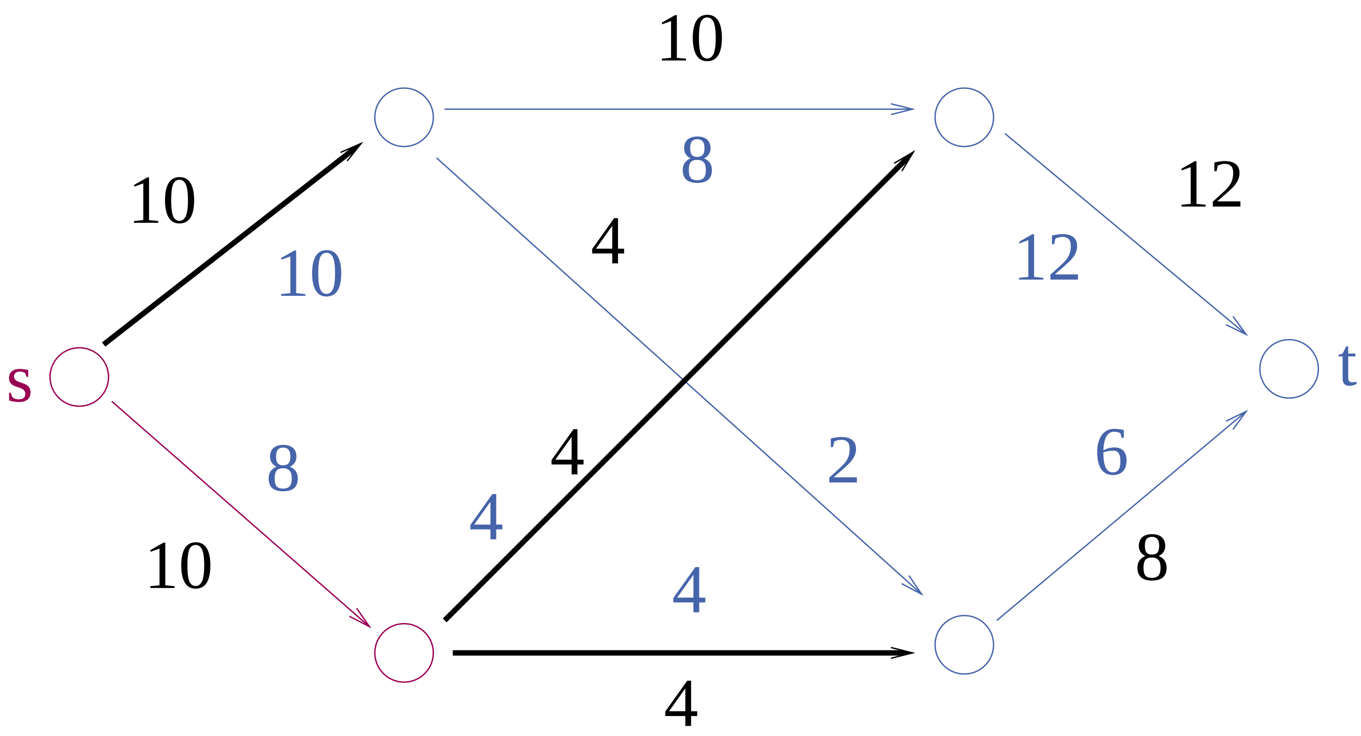
$$\sum \{c_{(u,v)} : u \in S, v \in T\}$$



Duality Between Flows and Cuts

Theorem: [Elias/Feinstein/Shannon, Ford/Fulkerson 1956]

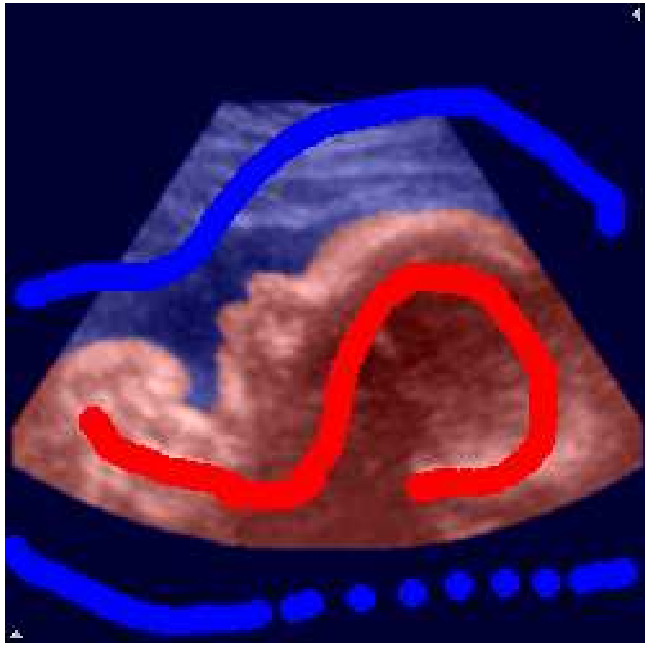
Value of an s-t max-flow $=$ minimum capacity of an s-t cut.



Proof: later

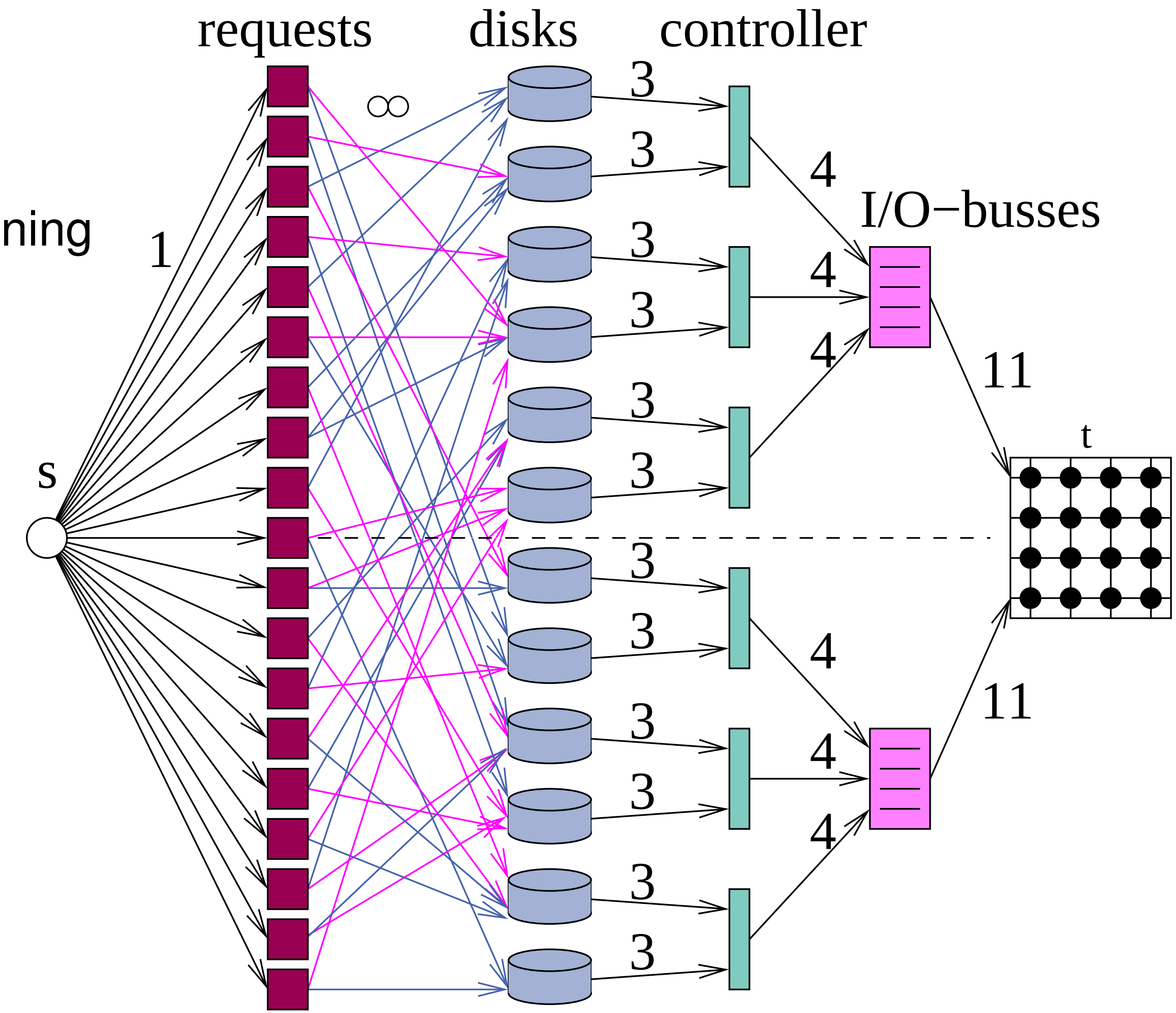
Applications

- Oil pipes
- Traffic flows on highways
- Image Processing `http://vision.csd.uwo.ca/maxflow-data`
  - segmentation
  - stereo processing
  - multiview reconstruction
  - surface fitting
- disk/machine/tanker scheduling
- matrix rounding
- ...



Applications in our Group

- multicasting using network coding
- balanced k partitioning
- disk scheduling



Option 1: linear programming

- Flow variables x_e for each edge e
- Flow on each edge is at most its capacity
- Incoming flow at each vertex = outgoing flow from this vertex
- Maximize outgoing flow from starting vertex

We can do better!

Algorithms 1956–now

Year	Author	Running time
1956	Ford-Fulkerson	$O(mnU)$
1969	Edmonds-Karp	$O(m^2 n)$
1970	Dinic	$O(mn^2)$
1973	Dinic-Gabow	$O(mn \log U)$
1974	Karzanov	$O(n^3)$
1977	Cherkassky	$O(n^2\sqrt{m})$
1980	Galil-Naamad	$O(mn \log^2 n)$
1983	Sleator-Tarjan	$O(mn \log n)$

$n =$ number of nodes

$m =$ number of arcs

$U =$ largest capacity

Year	Author	Running time
1986	Goldberg-Tarjan	$O(mn\log(n^2/m))$
1987	Ahuja-Orlin	$O(mn+n^2\log U)$
1987	Ahuja-Orlin-Tarjan	$O(mn\log(2+n\sqrt{\log U}/m))$
1990	Cheriyan-Hagerup-Mehlhorn	$O(n^3/\log n)$
1990	Alon	$O(mn+n^{8/3}\log n)$
1992	King-Rao-Tarjan	$O(mn+n^{2+\varepsilon})$
1993	Philipps-Westbrook	$O(mn\log n/\log\frac{m}{n}+n^2\log^{2+\varepsilon}n)$
1994	King-Rao-Tarjan	$O(mn\log n/\log\frac{m}{n\log n})$ if $m\geq 2n\log n$
1997	Goldberg-Rao	$O(\min\{m^{1/2},n^{2/3}\}m\log(n^2/m)\log U)$
2014	Lee-Sidford	$O(m\sqrt{n}\log^2 U)$

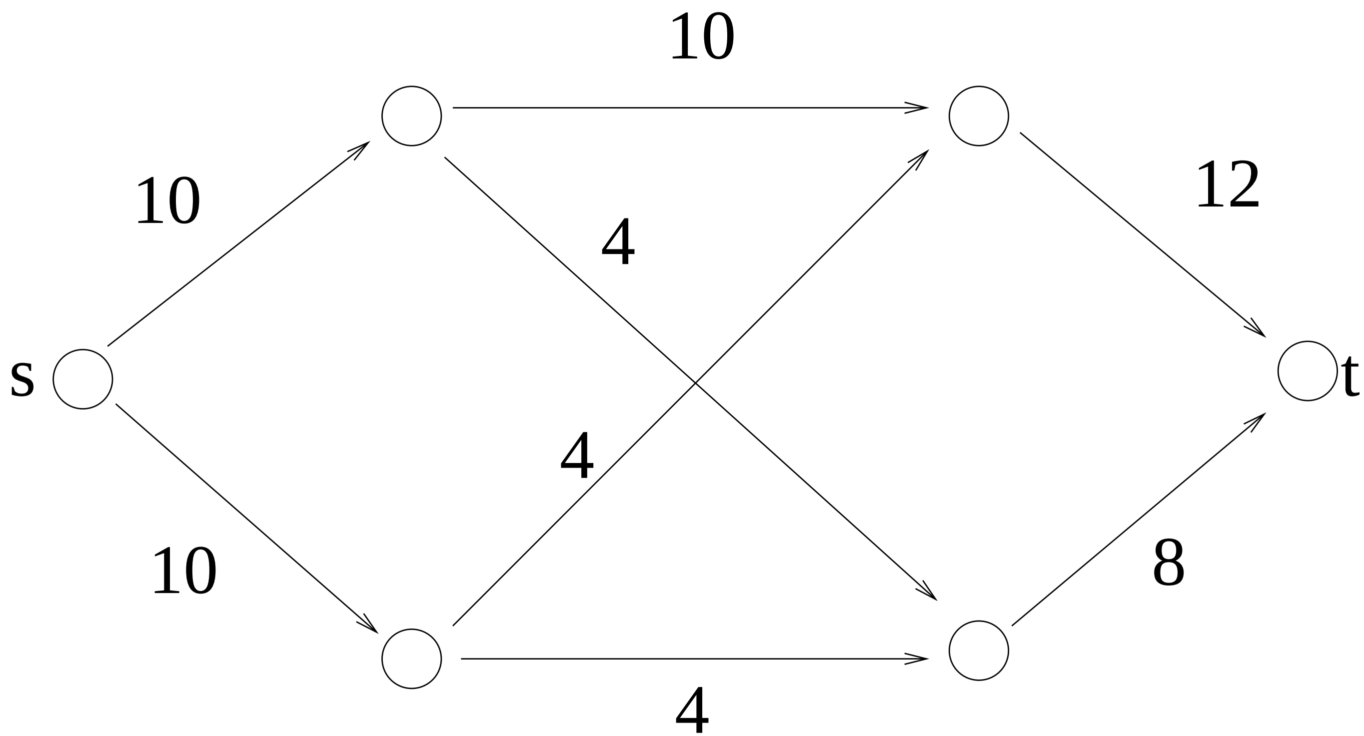
Augmenting Paths (Rough Idea)

Find a path from s to t such that each edge has some spare capacity

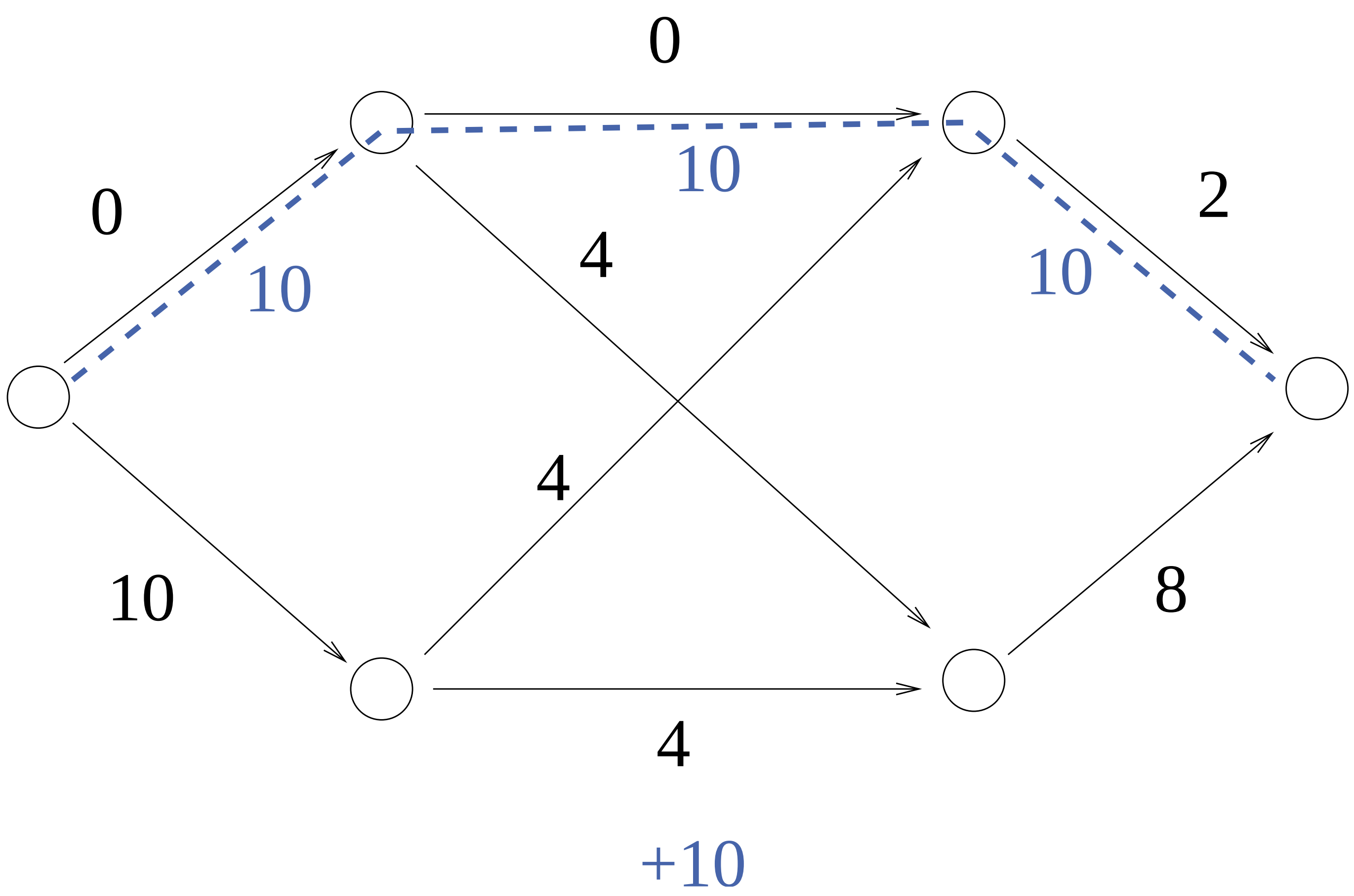
On this path, saturate the edge with the smallest spare capacity

Adjust capacities for all edges (create residual graph) and repeat

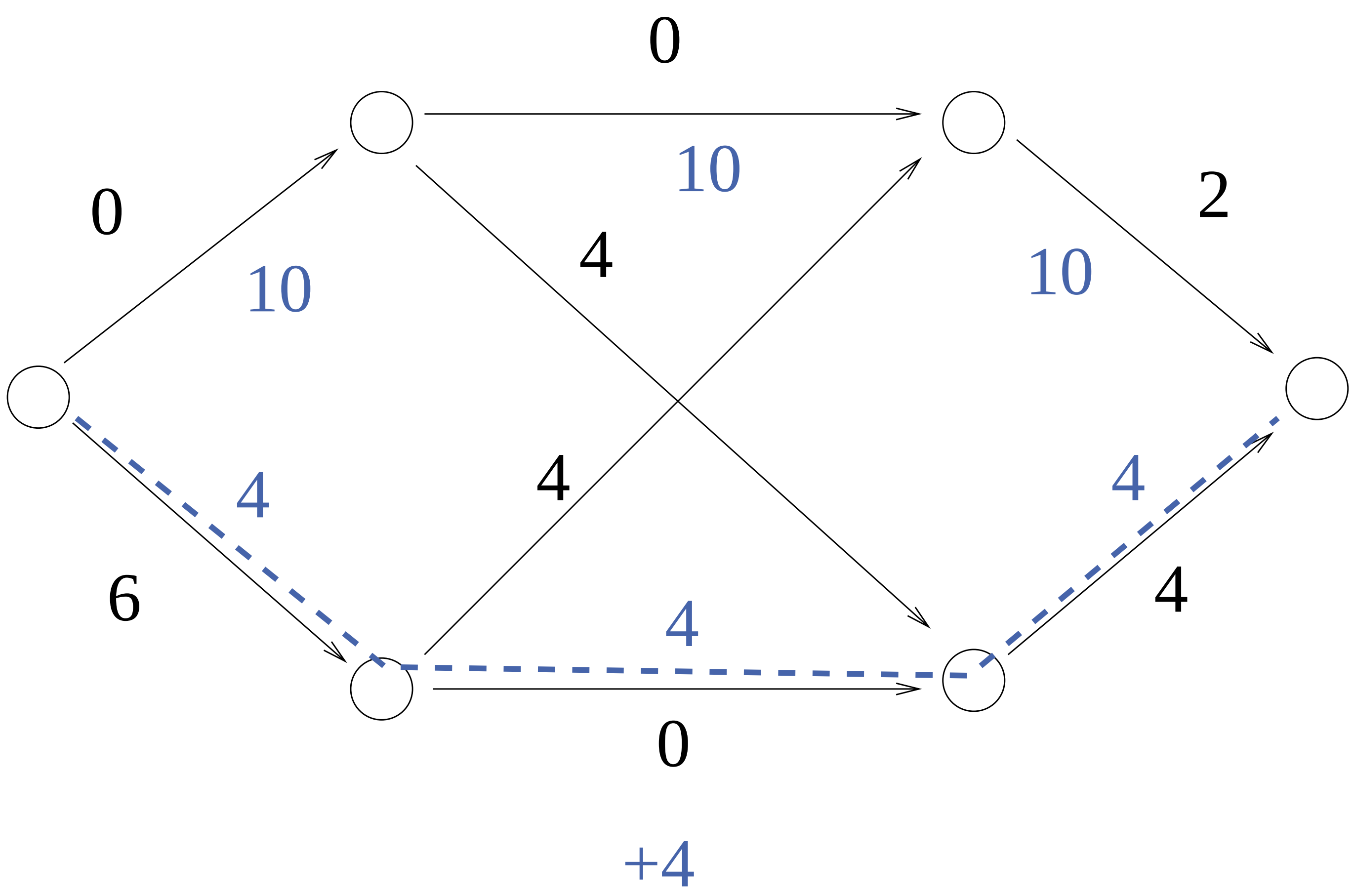
Example



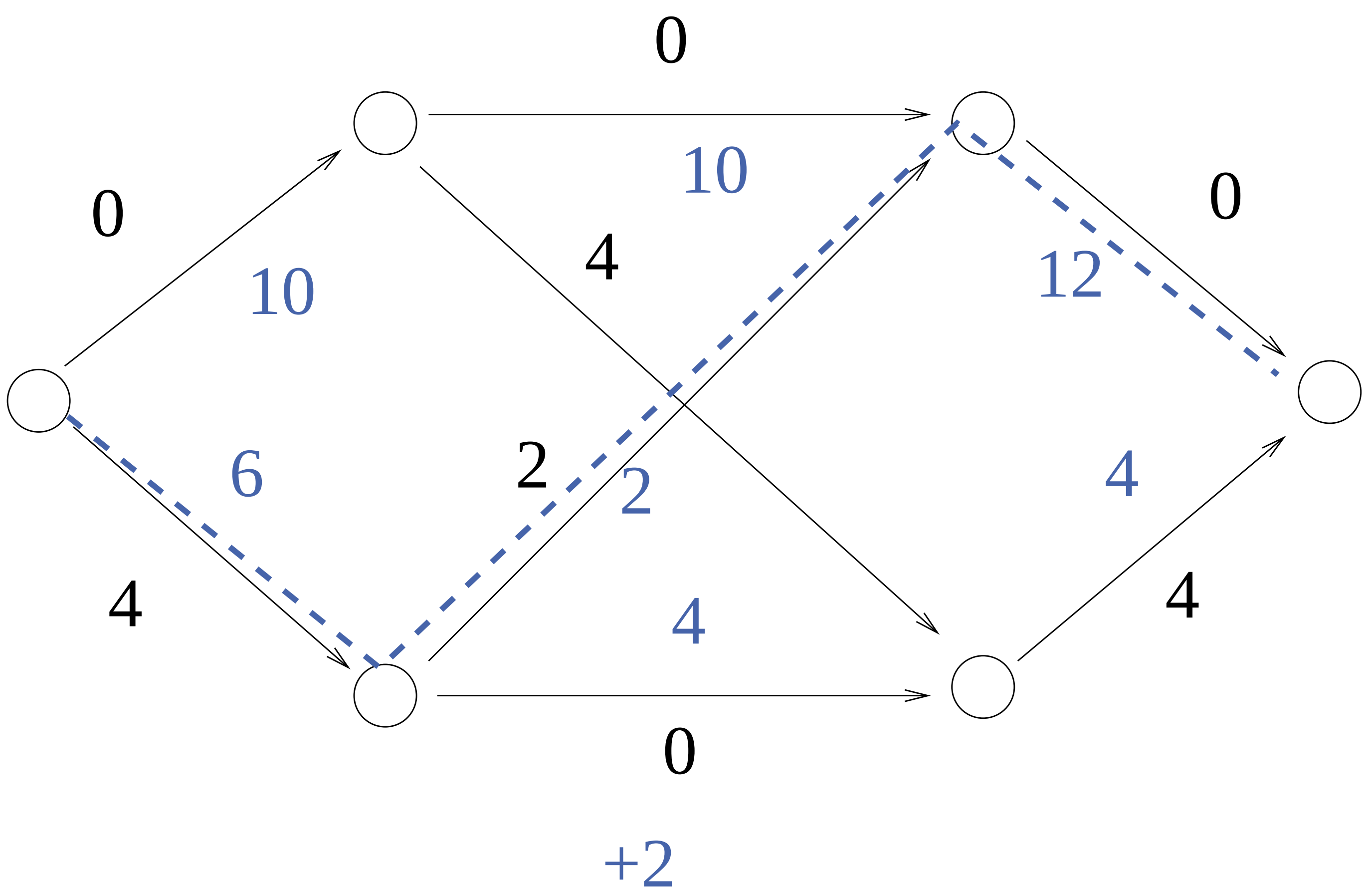
Example



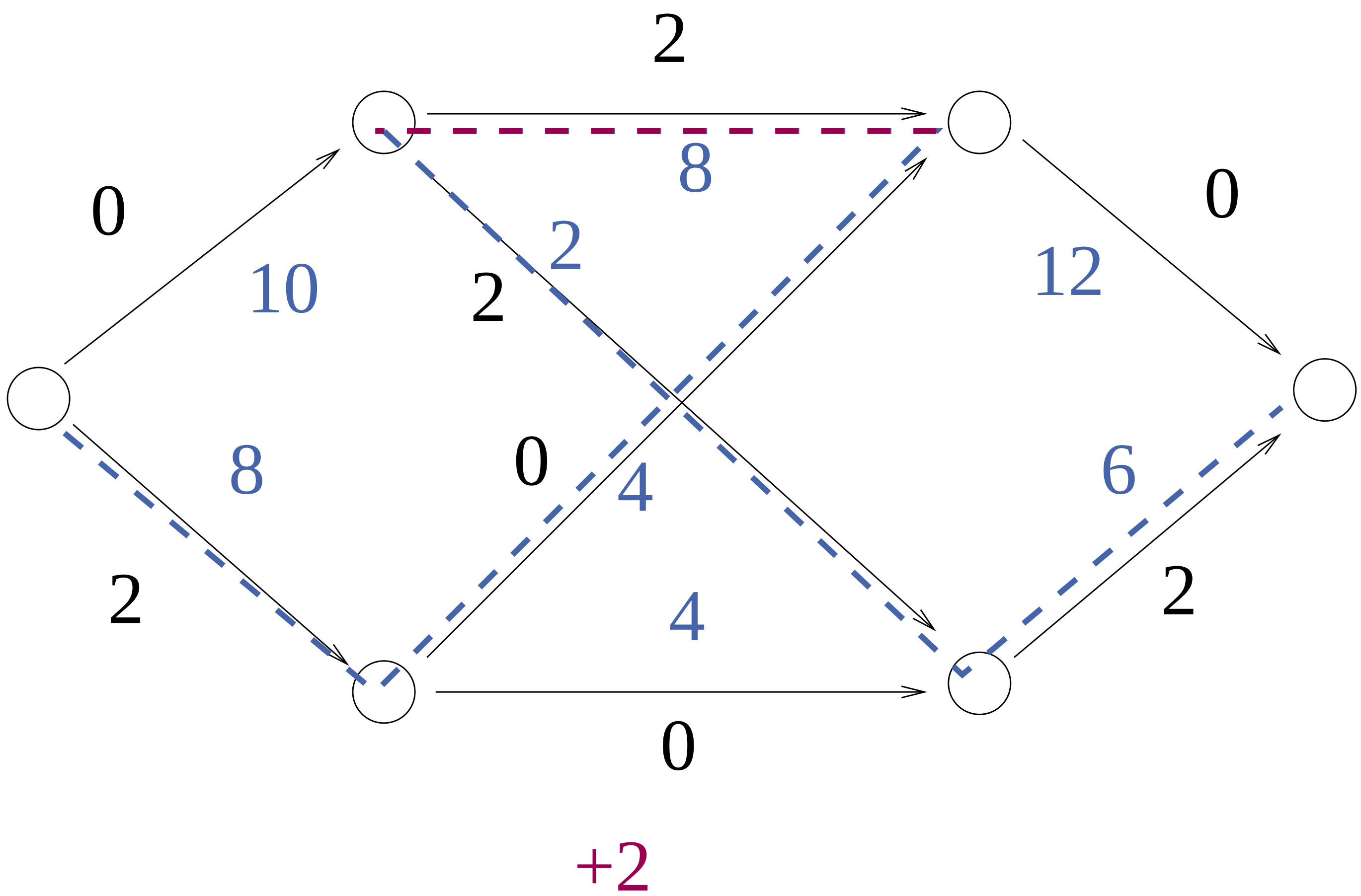
Example



Example



Example

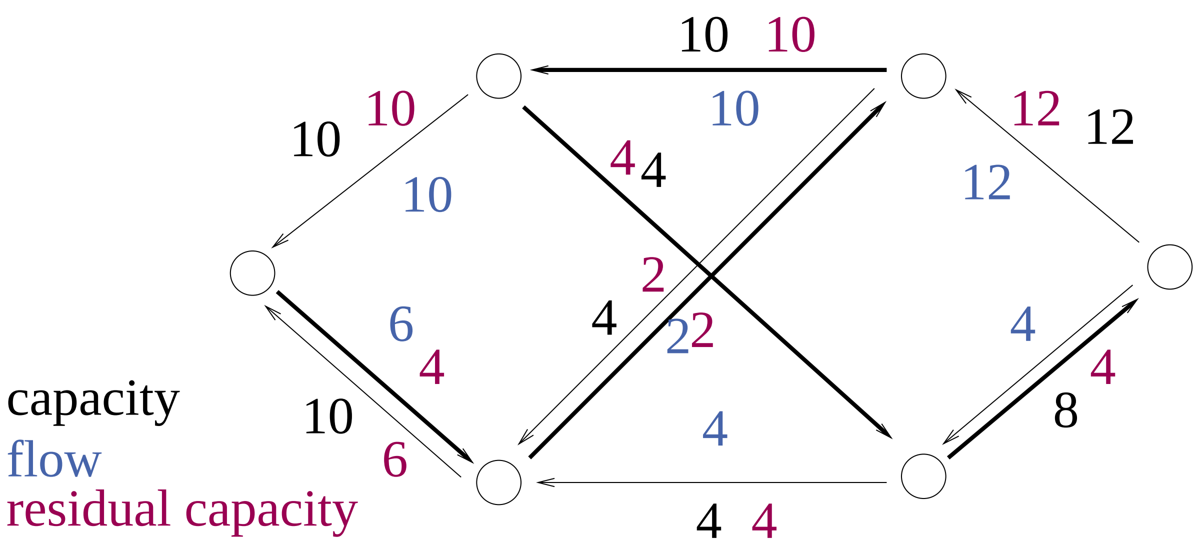


Residual Graph

Given, network $G = (V, E, c)$, flow f

Residual graph $G_f = (V, E_f, c^f)$. For each $e \in E$ we have

$$\begin{cases} e \in E_f \text{ with } c^f_e = c_e - f(e) & \text{if } f(e) < c(e) \\ e^{\text{rev}} \in E_f \text{ with } c^f_{e^{\text{rev}}} = f(e) & \text{if } f(e) > 0 \end{cases}$$



Augmenting Paths

Find a path p from s to t such that each edge e has nonzero residual capacity c_e^f

$$\Delta f := \min_{e \in p} c_e^f$$

foreach $(u,v) \in p$ **do**

  if $(u,v) \in E$ **then** $f_{(u,v)} += \Delta f$

  else $f_{(v,u)} -= \Delta f$

Ford Fulkerson Algorithm

Function FFMaxFlow$(G=(V,E),s,t,\mathsf{c}:E\to\mathbb{N}):E\to\mathbb{N}$

$f:=0$

while $\exists$path $p=(s,\ldots,t)$ in G_f **do**

augment f along p

return f

time $\mathrm{O}(m\mathsf{val}(f))$

Ford Fulkerson – Correctness

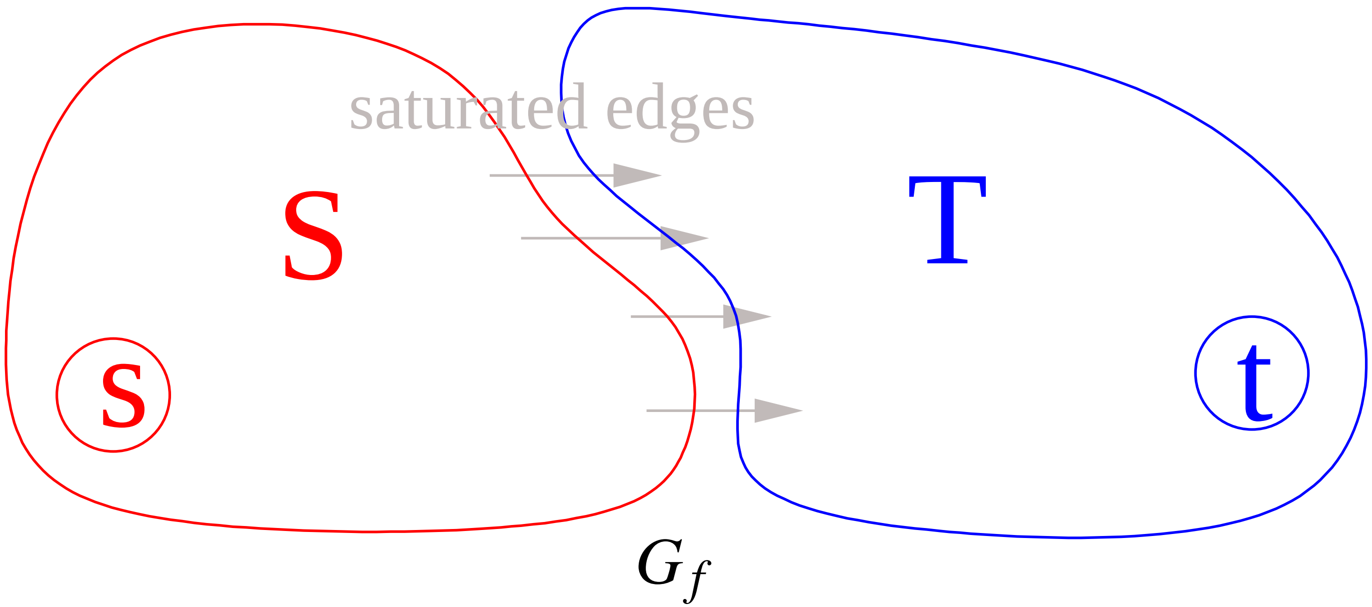
"Clearly" FF computes a feasible flow f. (Invariant)

Todo: flow value is maximal

At termination: no augmenting paths in G_f left.

Consider cut $(S, T := V \setminus S)$ with

$S := \{v \in V : v$ reachable from s in $G_f\}$



A Basic Observations

Lemma 1: For any cut (S,T):

$$\mathbf{val}(f) = \overbrace{\sum_{e \in E \cap S \times T} f_e}^{S \to T \text{ edges}} - \overbrace{\sum_{e \in E \cap T \times S} f_e}^{T \to S \text{ edges}} .$$

Ford Fulkerson – Correctness

Todo: $\mathbf{val}(f)$ is maximal when no augmenting paths in G_f left.

Consider cut $(S, T := V \setminus S)$ with
$S := \{v \in V : v$ reachable from s in $G_f\}$.
Observation: $\forall (u,v) \in E \cap T \times S : f(u,v) = 0$
otherwise $c^f(v,u) > 0$ contradicting the definition of S.

$$\mathbf{val}(f) = \sum_{e \in E \cap S \times T} f_e - \sum_{e \in E \cap T \times S} f_e \qquad \text{Lemma 1}$$

$$= \sum_{e \in E \cap S \times T} f_e \qquad \text{Observation above}$$

$$= \sum_{e \in E \cap S \times T} c_{(u,v)} = (S,T) \text{ cut capacity}$$

see next slide

Max-Flow-Min-Cut theorem

Theorem: Max-flow = min-cut

Proof:

obvious: any-flow $\leq$ max-flow $\leq$ min-cut $\leq$ any-cut

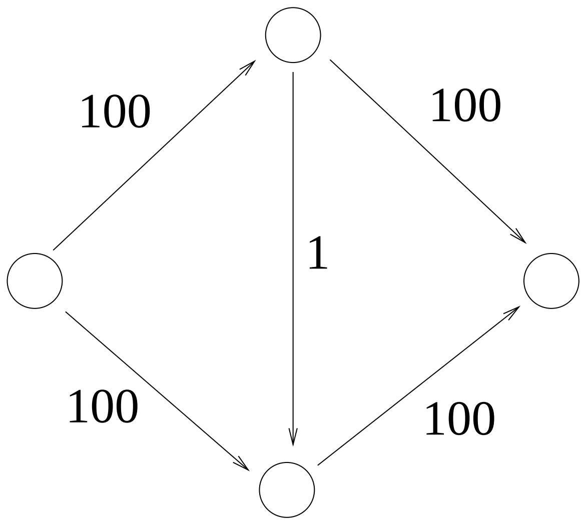
previous slide:

(S,T) flow $=$ (S,T) cut capacity

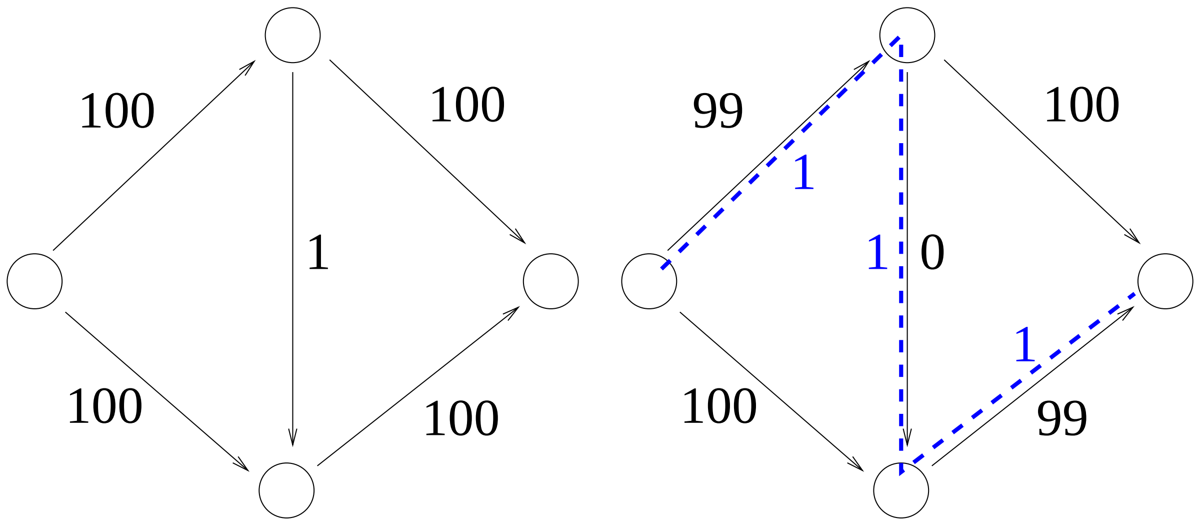
$\Rightarrow$

(S,T) flow $=$ max-flow $=$ min-cut

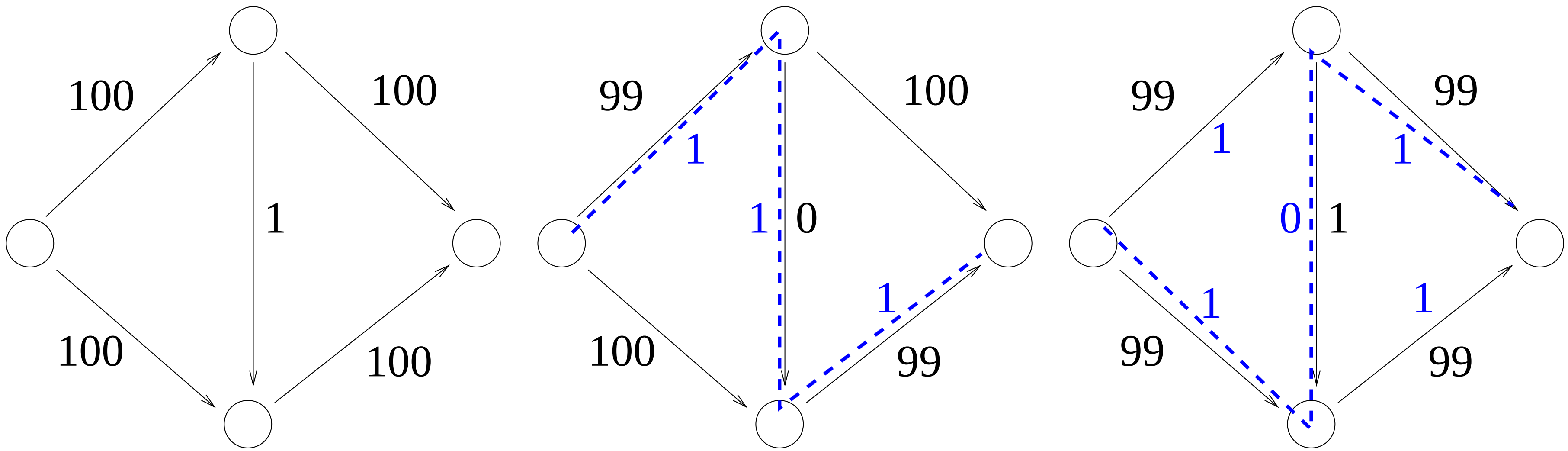
A Bad Example for Ford Fulkerson



A Bad Example for Ford Fulkerson



A Bad Example for Ford Fulkerson

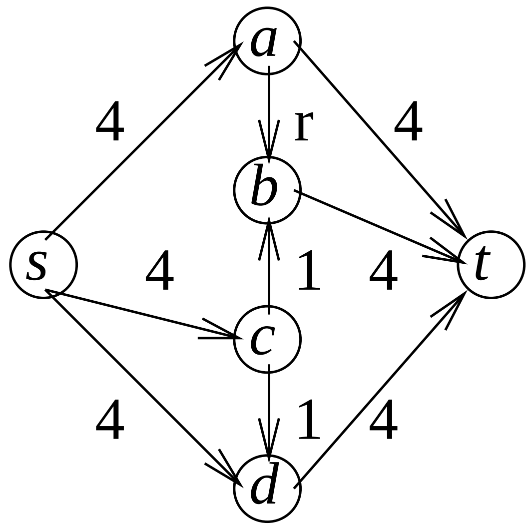


An Even Worse Example for Ford Fulkerson

[U. Zwick, TCS 148, p. 165–170, 1995]

Let $r = \dfrac{\sqrt{5}-1}{2}$.

Consider the graph



And the augmenting paths

$$p_0 = \langle s, c, b, t \rangle$$

$$p_1 = \langle s, a, b, c, d, t \rangle$$

$$p_2 = \langle s, c, b, a, t \rangle$$

$$p_3 = \langle s, d, c, b, t \rangle$$

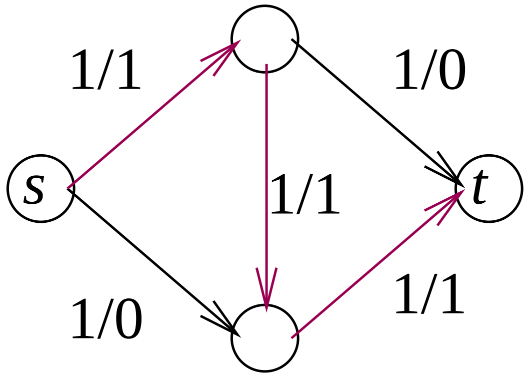
The sequence of augmenting paths $p_0(p_1, p_2, p_1, p_3)^*$ is an infinite sequence of positive flow augmentations.

The flow value does not converge to the maximum value 9.

Blocking Flows

f_b is a blocking flow in H if

$$\forall \text{paths } p = \langle s, \ldots, t \rangle : \exists e \in p : f_b(e) = c(e)$$



Dinitz Algorithm

Function DinitzMaxFlow($G = (V, E), s, t, c : E \to \mathbb{N}$) : $E \to \mathbb{N}$

  f:= 0

  while $\exists$path $p = (s, \ldots, t)$ in G_f **do**

    d=G_f.reverseBFS(t) $: V \to \mathbb{N}$

    $L_f = (V, \{(u, v) \in E_f : d(v) = d(u) - 1\})$ // layer graph

    find a blocking flow f_b in L_f

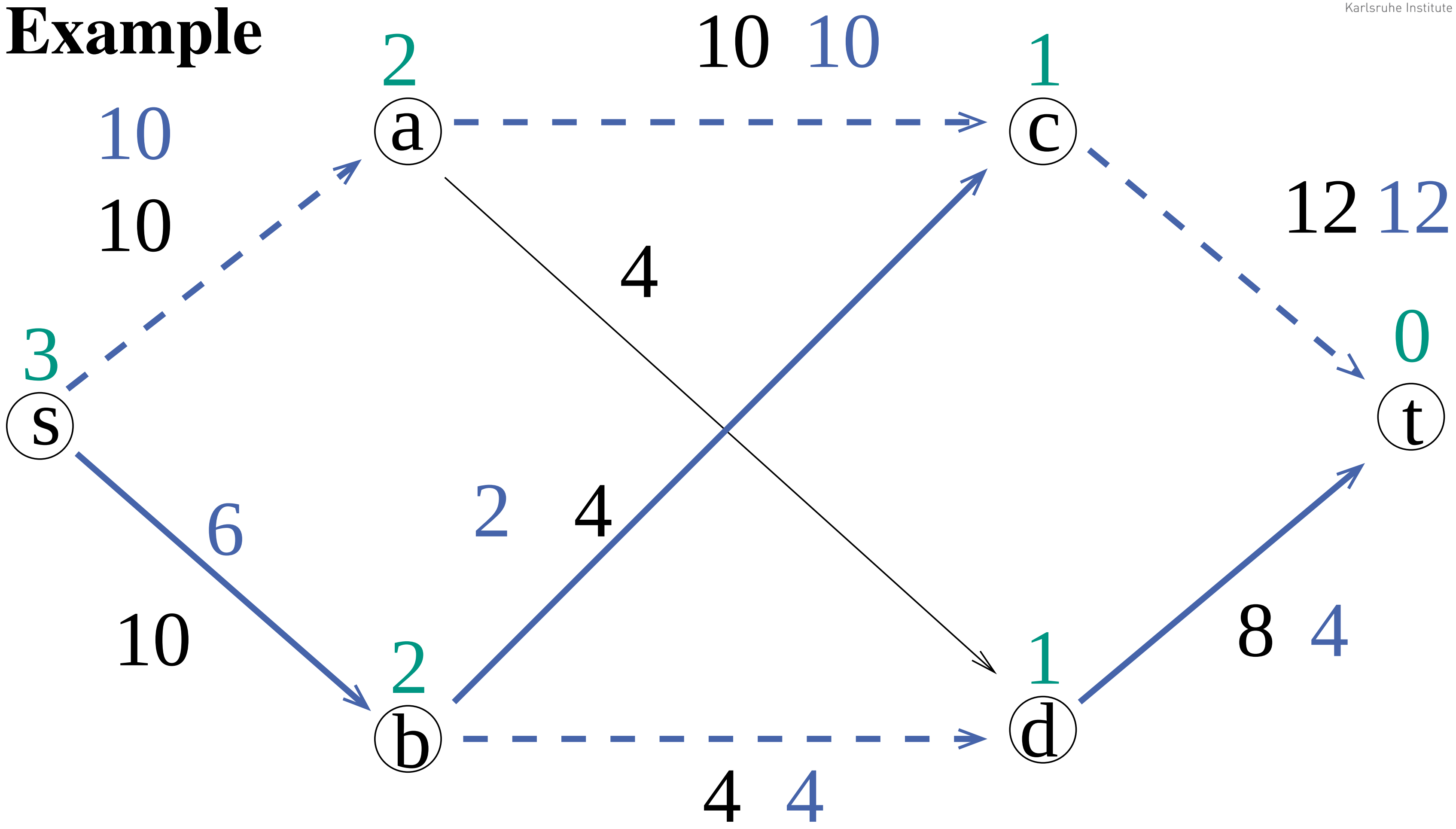
    augment f+=f_b

  return f

Dinitz – Correctness

analogous to Ford-Fulkerson

Example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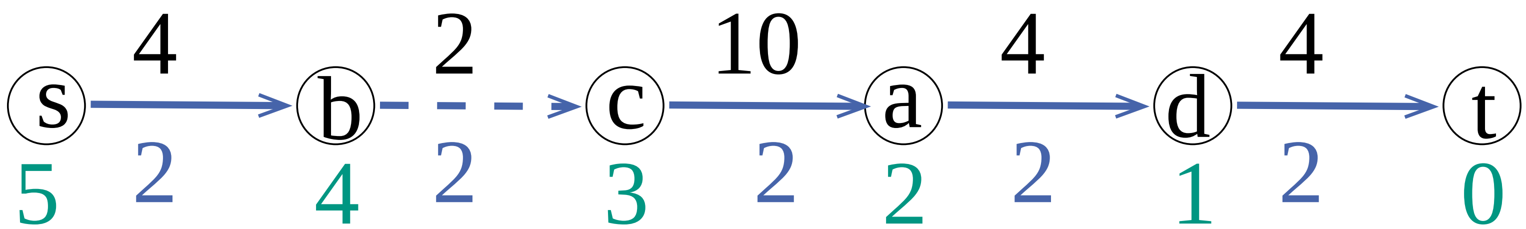
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Computing Blocking Flows

Idee: wiederholte DFS nach augmentierenden Pfaden

Function blockingFlow($L_f = (V,E)$) : $E \to \mathbb{N}$

$p=\langle s\rangle$: Path; f_b=0 : Flow

loop // Round

v:= p.last()

if $v = t$ **then** // breakthrough

δ:= $\min\{c(e) - f_b(e) : e \in p\}$

foreach $e \in p$ **do**

$f_b(e)$+=δ

if $f_b(e) = c(e)$ **then** remove e from E

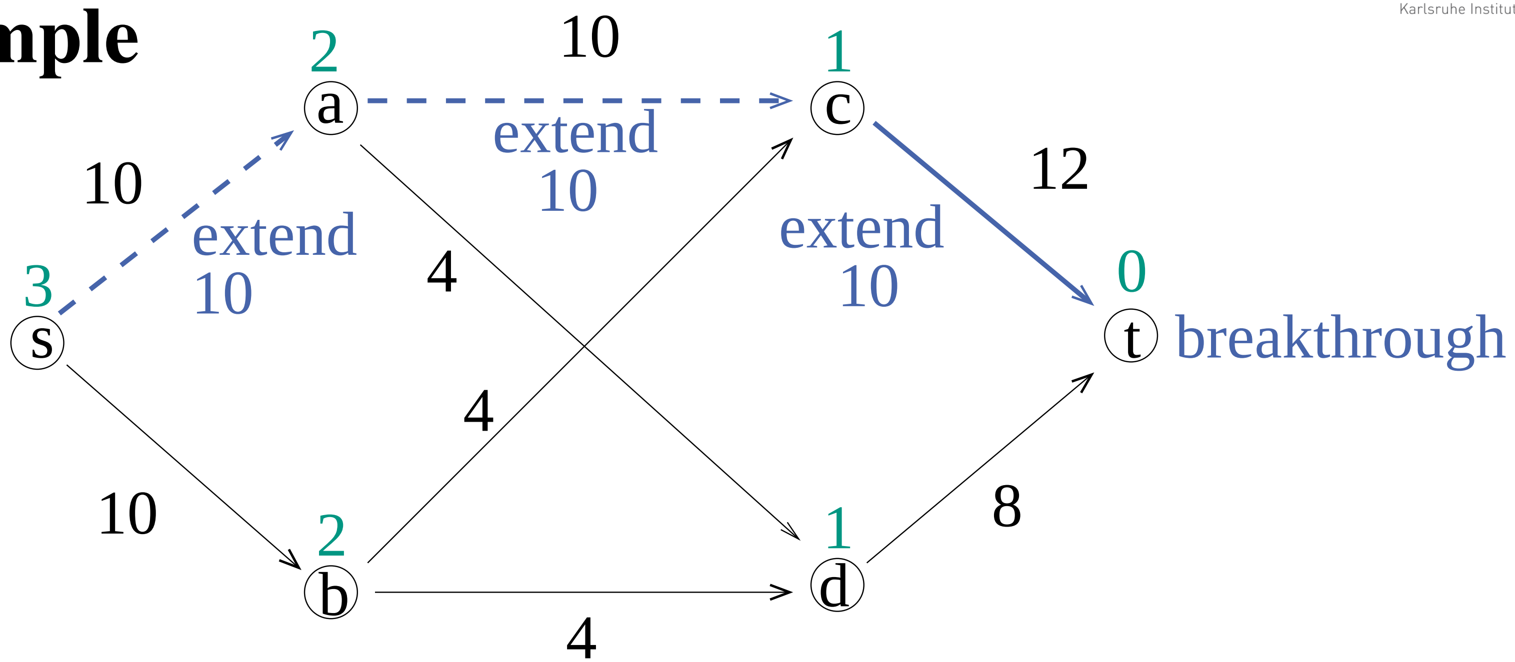
p:= $\langle s\rangle$

else if $\exists e = (v,w) \in E$ **then** p.pushBack(w) // extend

else if $v = s$ **then return** f_b // done

else delete the last edge from p in p and E // retreat

Example



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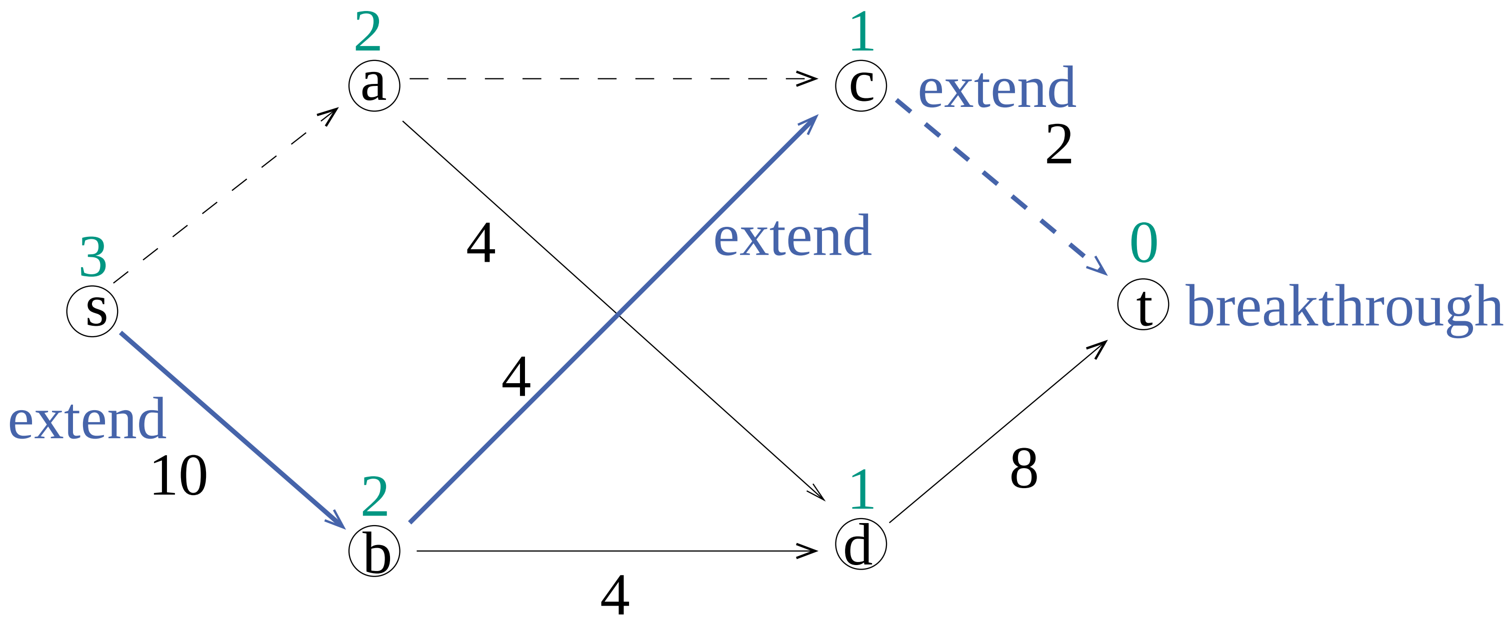

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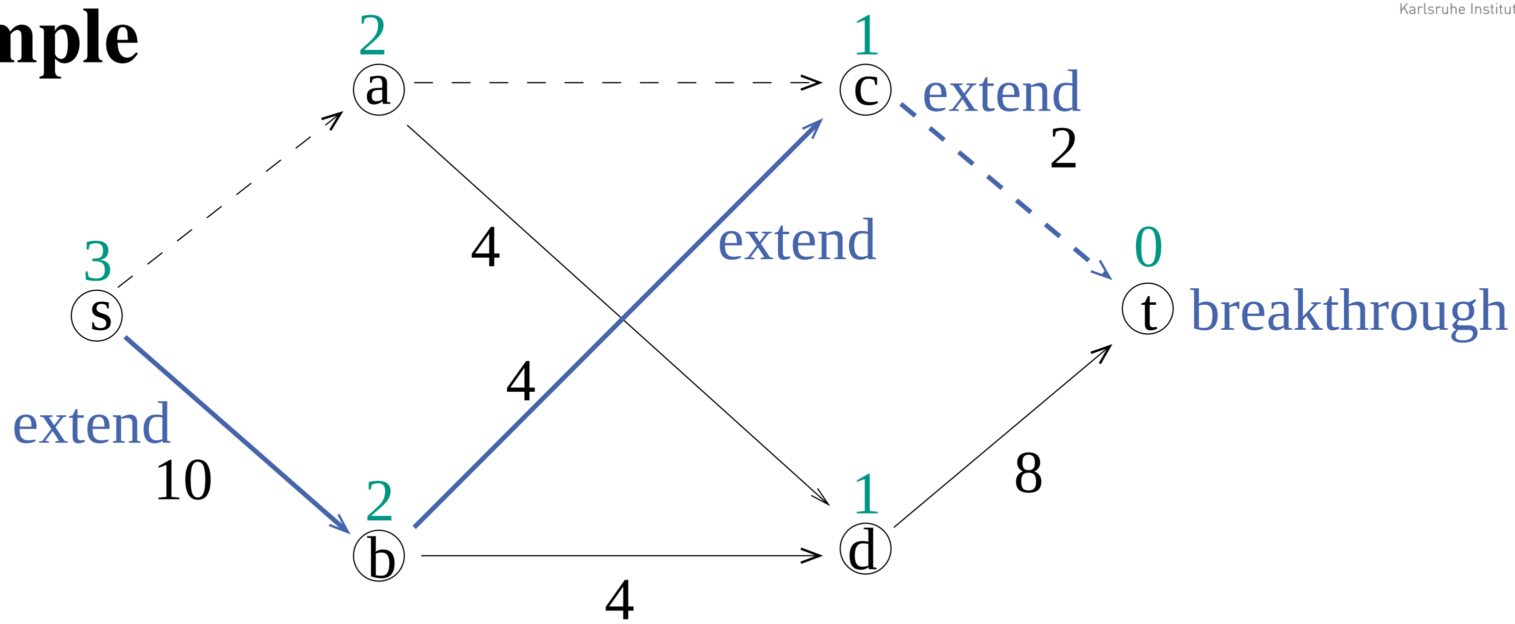
Example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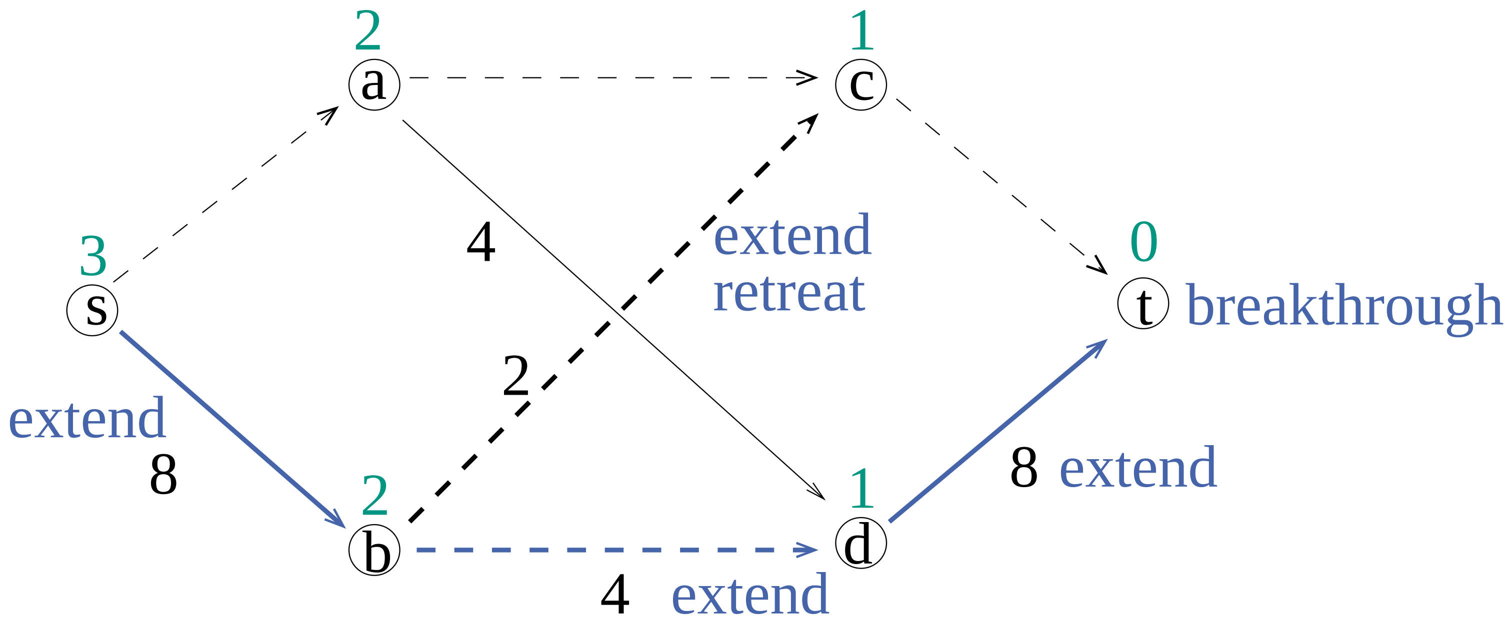
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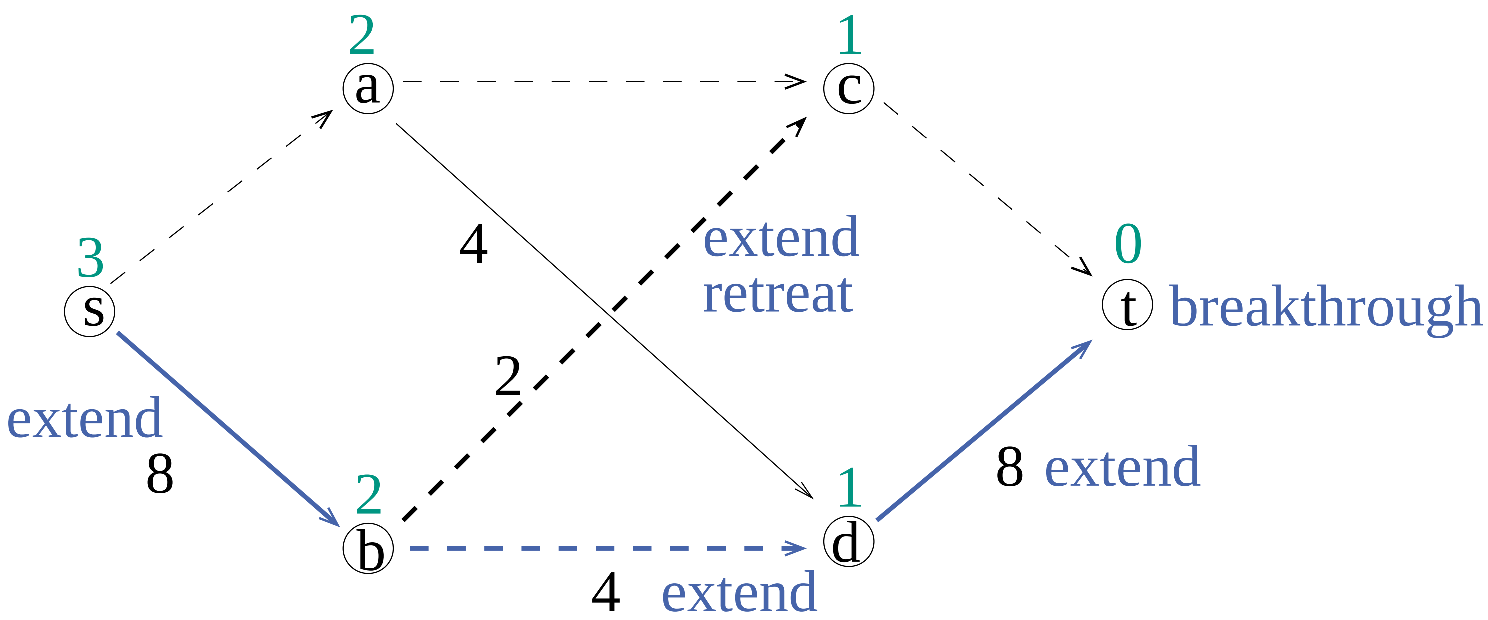
Example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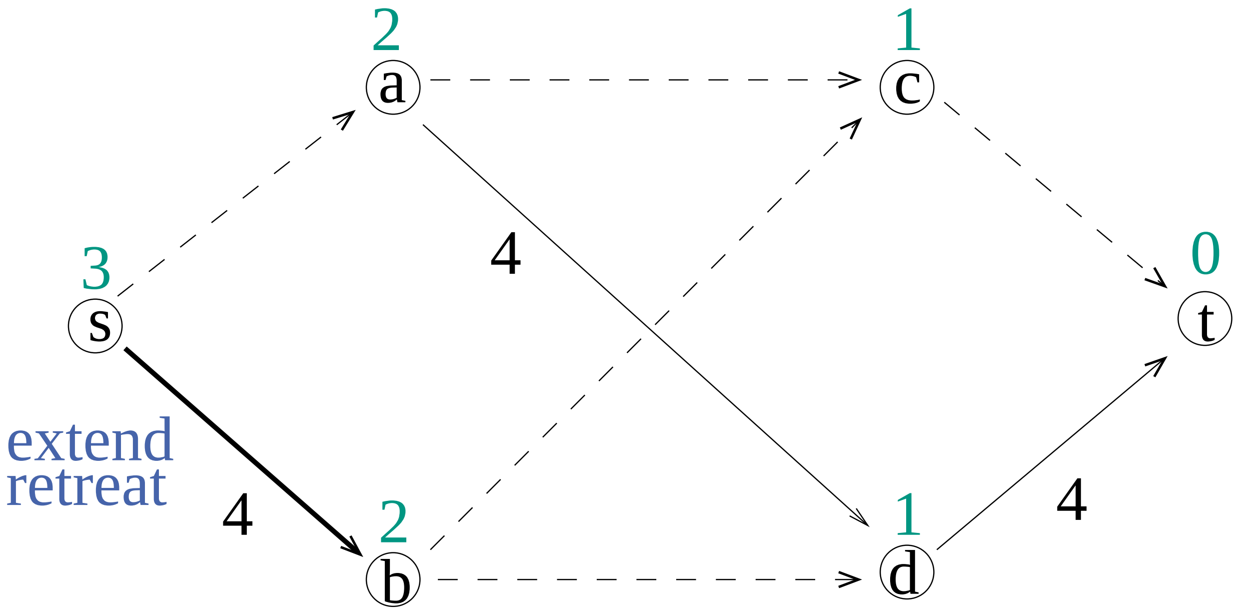
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Blocking Flows Analysis 1

- running time $\#_{extends} + \#_{retreats} + n \cdot \#_{breakthroughs}$
- $\#_{breakthroughs} \leq m$ – ≥ 1 edge is saturated
- $\#_{retreats} \leq m$ – one edge is removed
- $\#_{extends} \leq \#_{retreats} + n \cdot \#_{breakthroughs}$
  – a retreat cancels 1 extend, a breakthrough cancels $\leq n$ extends

time is $O(m + nm) = O(nm)$

Blocking Flows Analysis 2

Unit capacities:

breakthroughs saturates all edges on p, i.e., amortized constant cost per edge.

time $O(m+n)$

Blocking Flows Analysis 3

Dynamic trees: breakthrough (!), retreat, extend in time $\mathrm{O}(\log n)$

time $O((m+n)\log n)$

"Theory alert": In practice, this seems to be slower

(few breakthroughs, many retreat, extend ops.)

Dinitz Analysis 1

Lemma 1. *$d(s)$ increases by at least one in each round.*

Beweis. not here $\square$

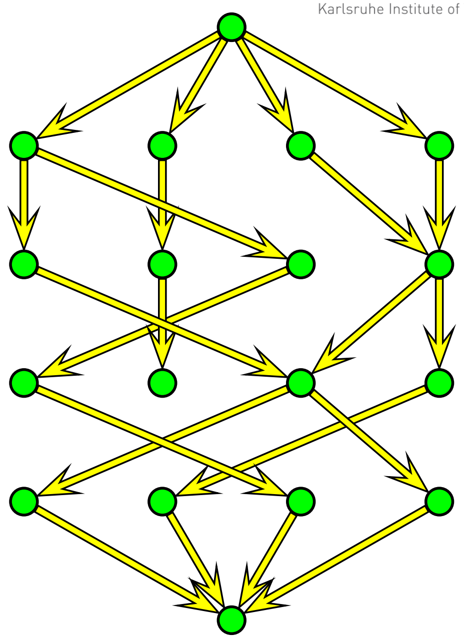
Dinitz Analysis 2

- $\leq n$ rounds
- time $\mathrm{O}(mn)$ each

time $\mathrm{O}\left(mn^2\right)$ (strongly polynomial)

time $\mathrm{O}(mn\log n)$ with dynamic trees

Dinitz Analysis 3 – Unit Capacities



Lemma 2. *At most $2\sqrt{m}$ BF computations:*

Beweis. Consider iteration $k = \sqrt{m}$.
Cut in layergraph induces cut in residual graph of capacity at most $\sqrt{m}$.
At most $\sqrt{m}$ additional phases.

$\square$

Total time: $\mathrm{O}((m+n)\sqrt{m})$

more detailed analysis: $\mathrm{O}\left(m \min\left\{m^{1/2}, n^{2/3}\right\}\right)$

Dinitz Analysis 4 – Unit Networks

Unit capacity $+ \forall v \in V$: $\min\{\text{indegree}(v), \text{outdegree}(v)\} = 1$:

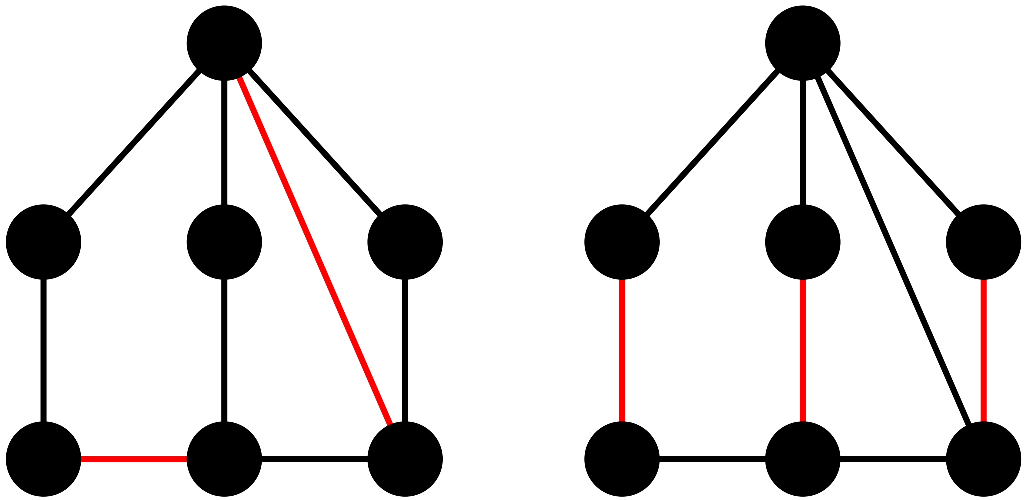
time: $\mathrm{O}((m+n)\sqrt{n})$

Matching

$M \subseteq E$ is a matching in the undirected graph $G = (V, E)$ iff (V, M) has maximum degree ≤ 1.

M is maximal if $\nexists e \in E \setminus M : M \cup \{e\}$ is a matching.

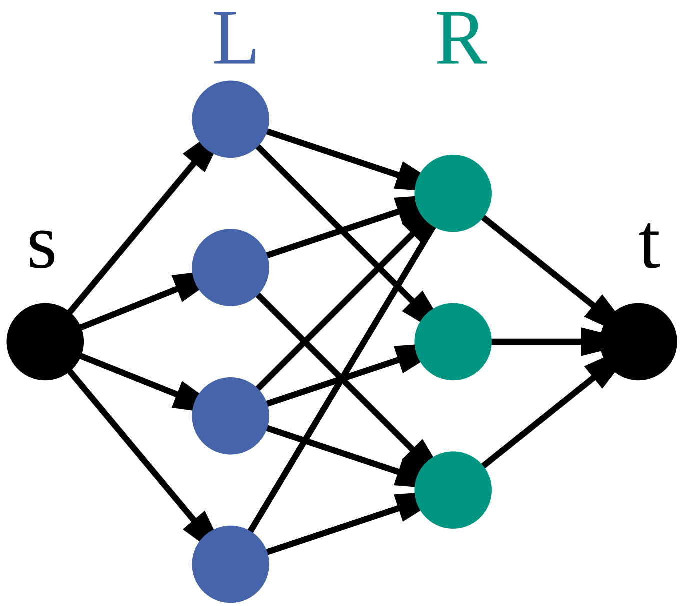
M has maximum cardinality if $\nexists$matching $M' : |M'| > |M|$



Maximum Cardinality Bipartite Matching

in $(L \cup R, E)$. Model as a unit network maximum flow problem

$$(\{s\} \cup L \cup R \cup \{t\}, \{(s,u) : u \in L\} \cup E \cup \{(v,t) : v \in R\})$$

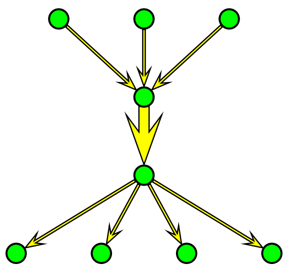


Dinitz algorithm yields $\mathrm{O}((n+m)\sqrt{n})$ algorithm

Similar Performance for Weighted Graphs?

time: $\mathrm{O}\left(m \min\left\{m^{1/2}, n^{2/3}\right\}\log C\right)$ [Goldberg Rao 97]

Problem: Fat edges between layers ruin the argument



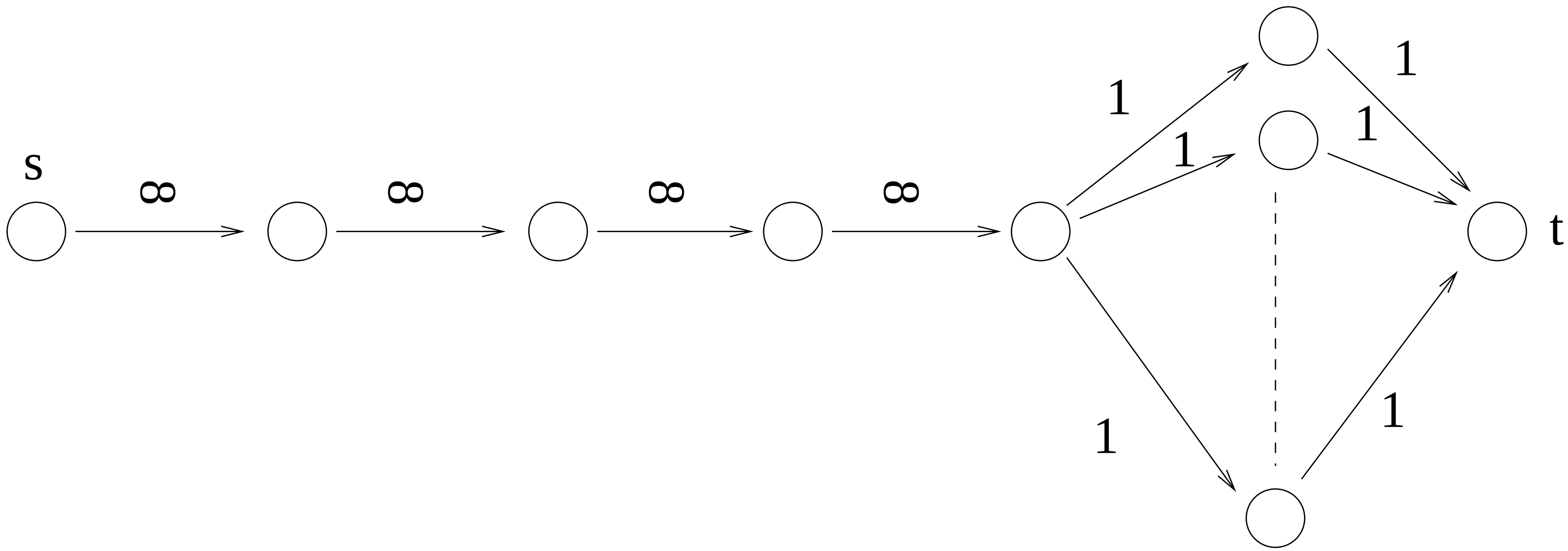
Idea: scale a parameter Δ from small to large

contract SCCs of fat edges (capacity $> \Delta$)

Experiments [Hagerup, Sanders Träff 98]:

Sometimes best algorithm usually slower than preflow push

Disadvantage of augmenting paths algorithms



Preflow-Push Algorithms

Preflow f: a flow where the flow conservation constraint is relaxed to

$$\text{excess}(v) := \overbrace{\sum_{(u,v)\in E} f_{(u,v)}}^{\text{inflow}} - \overbrace{\sum_{(v,w)\in E} f_{(v,w)}}^{\text{outflow}} \geq 0 \ .$$

$v \in V \setminus \{s,t\}$ is active iff $\text{excess}(v) > 0$

Procedure push$(e = (v,w), \delta)$

assert $\delta > 0 \quad \wedge \quad \text{excess}(v) \geq \delta$

assert residual capacity of $e \geq \delta$

$\text{excess}(v) -= \delta$

$\text{excess}(w) += \delta$

if e is reverse edge **then** $f(\text{reverse}(e)) -= \delta$

else $f(e) += \delta$

Level Function

Idea: make progress by pushing towards t

Maintain

an approximation $d(v)$ of the BFS distance from v to t in G_f.

invariant $d(t) = 0$

invariant $d(s) = n$

invariant $\forall (v,w) \in E_f : d(v) \leq d(w) + 1$ // no steep edges

Edge directions of $e = (v,w)$

steep: $d(w) < d(v) - 1$

downward: $d(w) < d(v)$

horizontal: $d(w) = d(v)$

upward: $d(w) > d(v)$

Procedure genericPreflowPush(G=(V,E), f)

  forall $e = (s, v) \in E$ **do** push$(e, c(e))$ // saturate

  $d(s) := n$

  $d(v) := 0$ for all other nodes

  while $\exists v \in V \setminus \{s, t\}$: excess$(v) > 0$ **do** // active node

    if $\exists e = (v, w) \in E_f : d(w) < d(v)$ **then** // eligible edge

      choose some $\delta \leq \min\left\{\text{excess}(v), c_e^f\right\}$

      push(e, δ) // no new steep edges

    else $d(v){+}{+}$ // relabel. No new steep edges

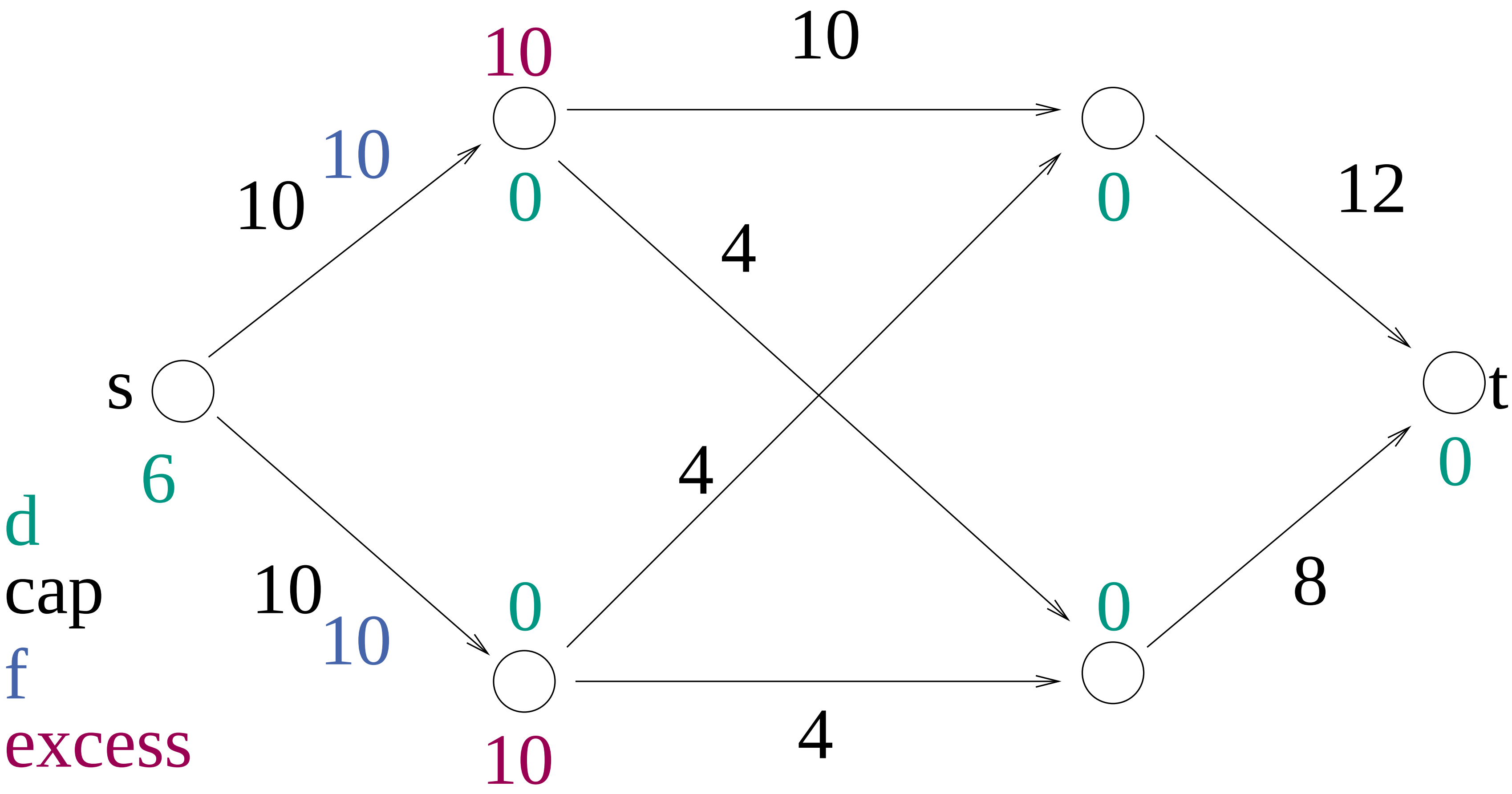
Obvious choice for $\delta : \delta = \min\left\{\text{excess}(v), c_e^f\right\}$

Saturating push: $\delta = c_e^f$

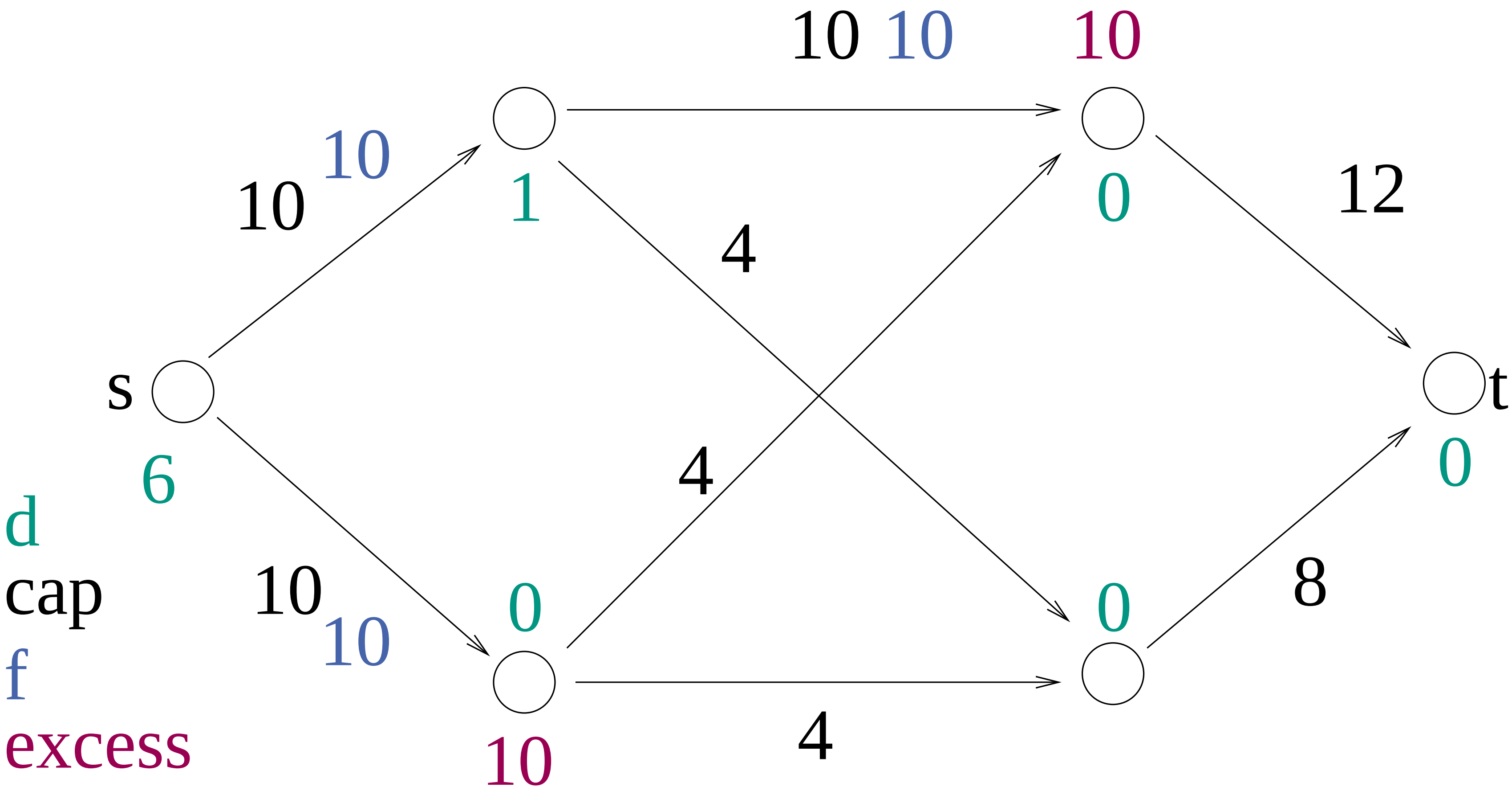
nonsaturating push: $\delta < c_e^f$

To be filled in: How to select active nodes and eligible edges?

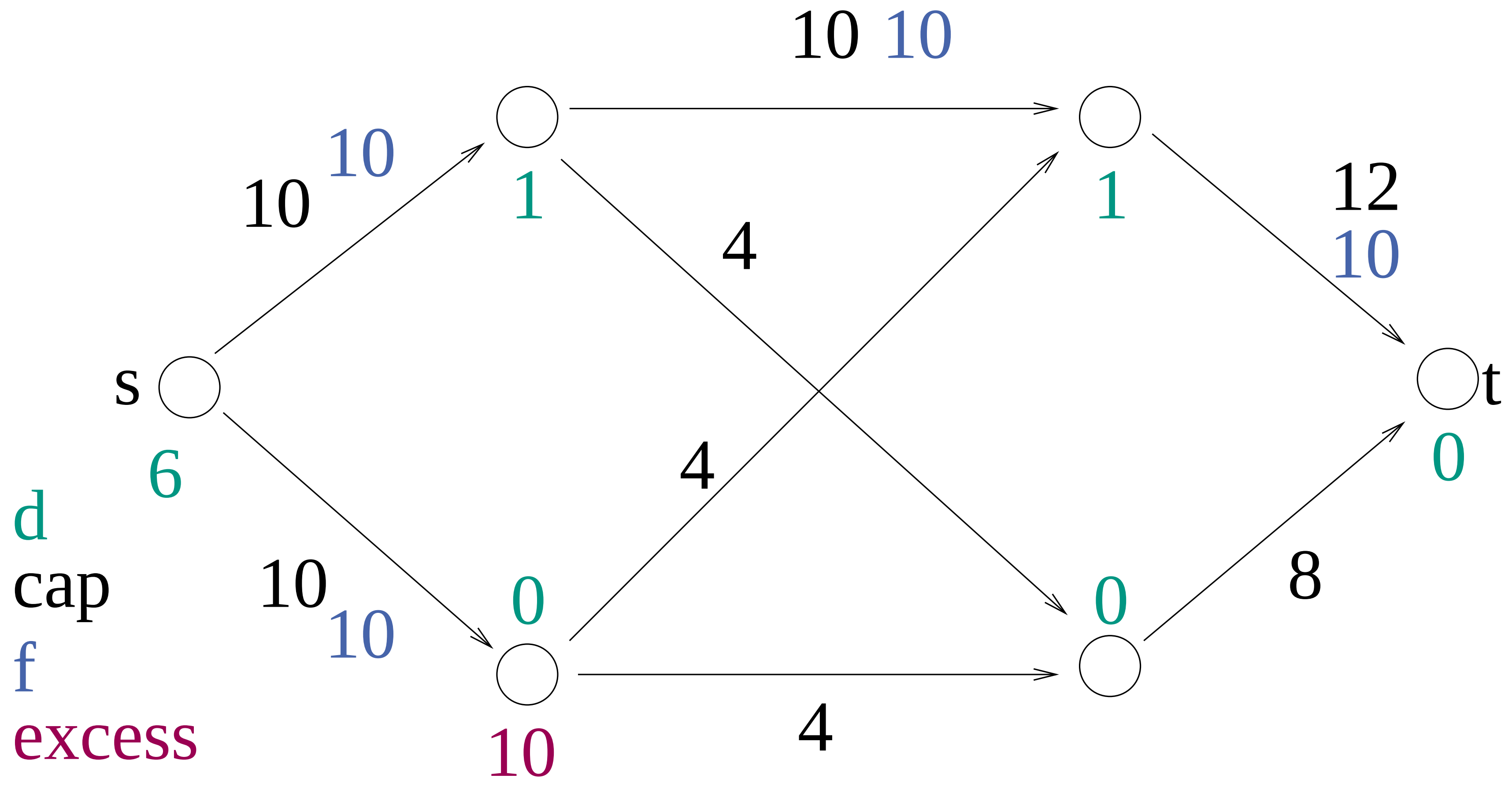
Example



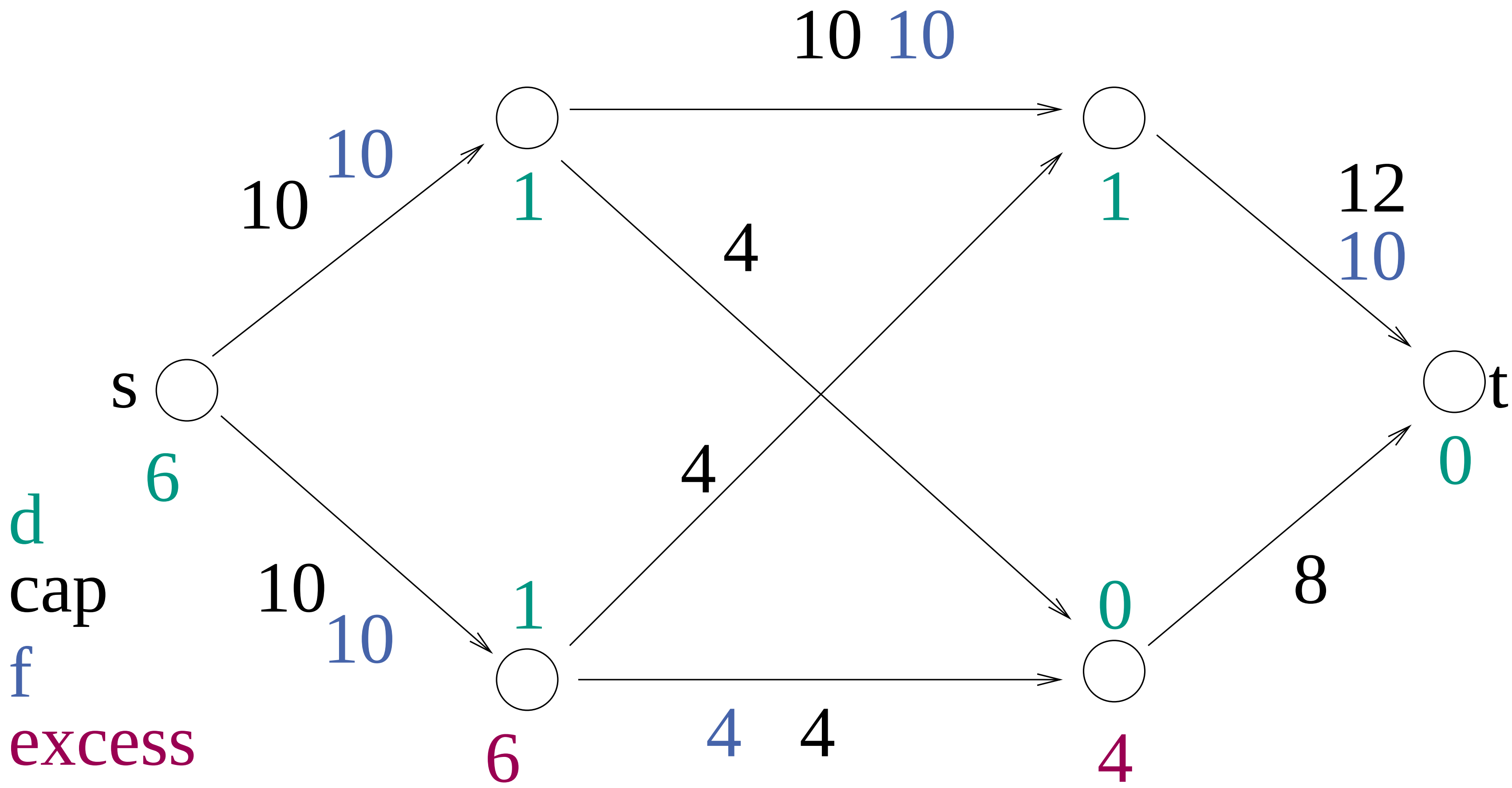
Example



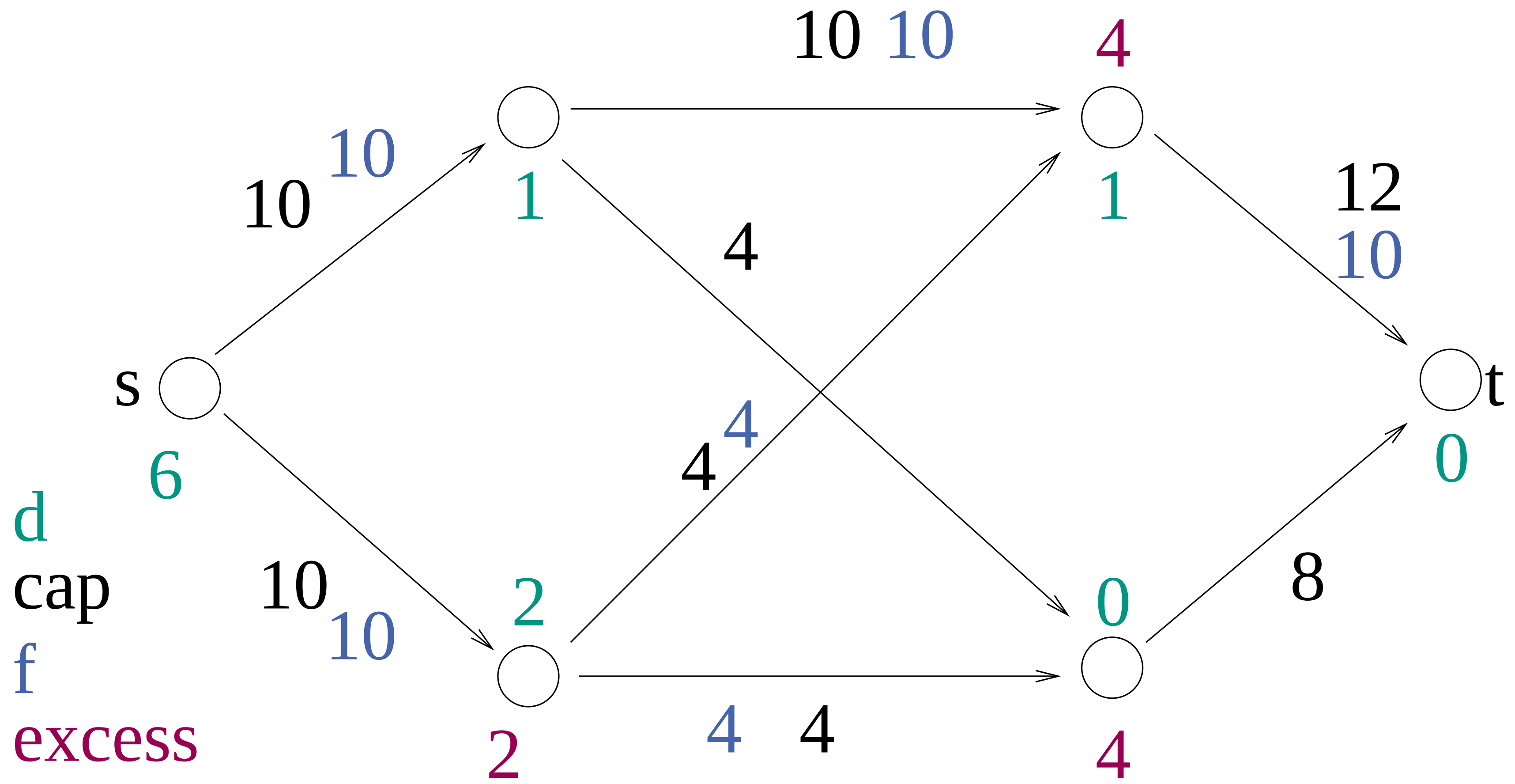
Example



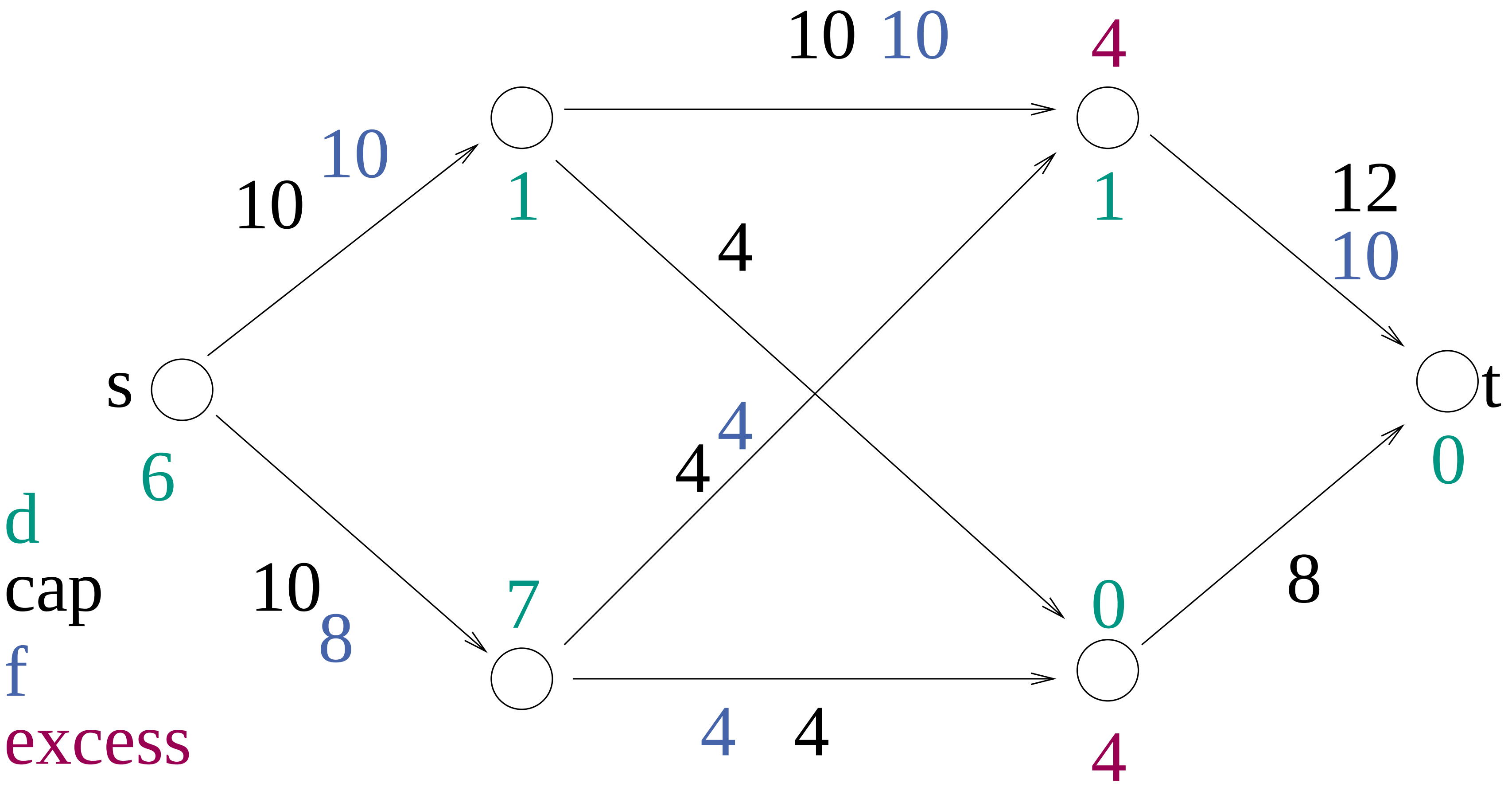
Example



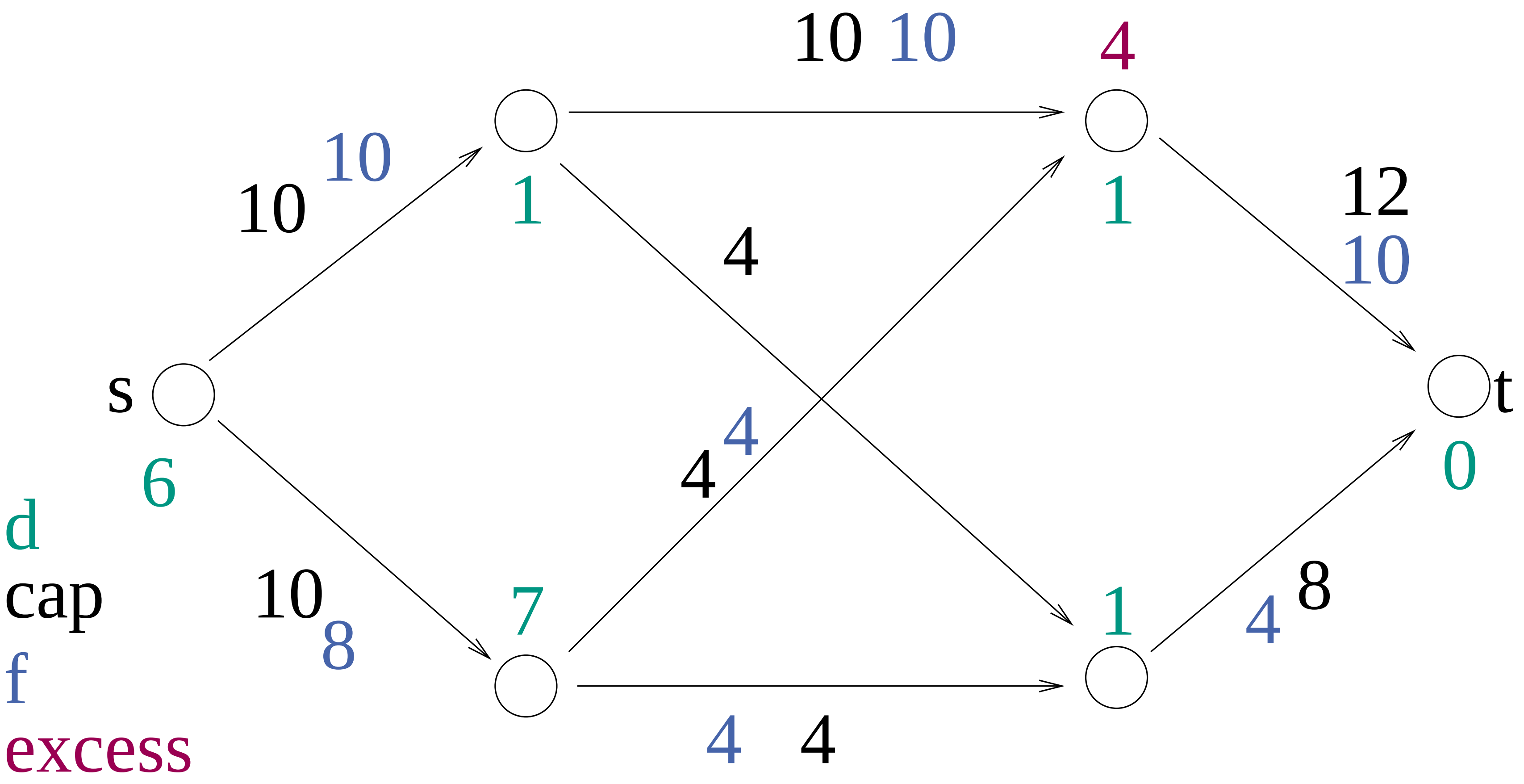
Example



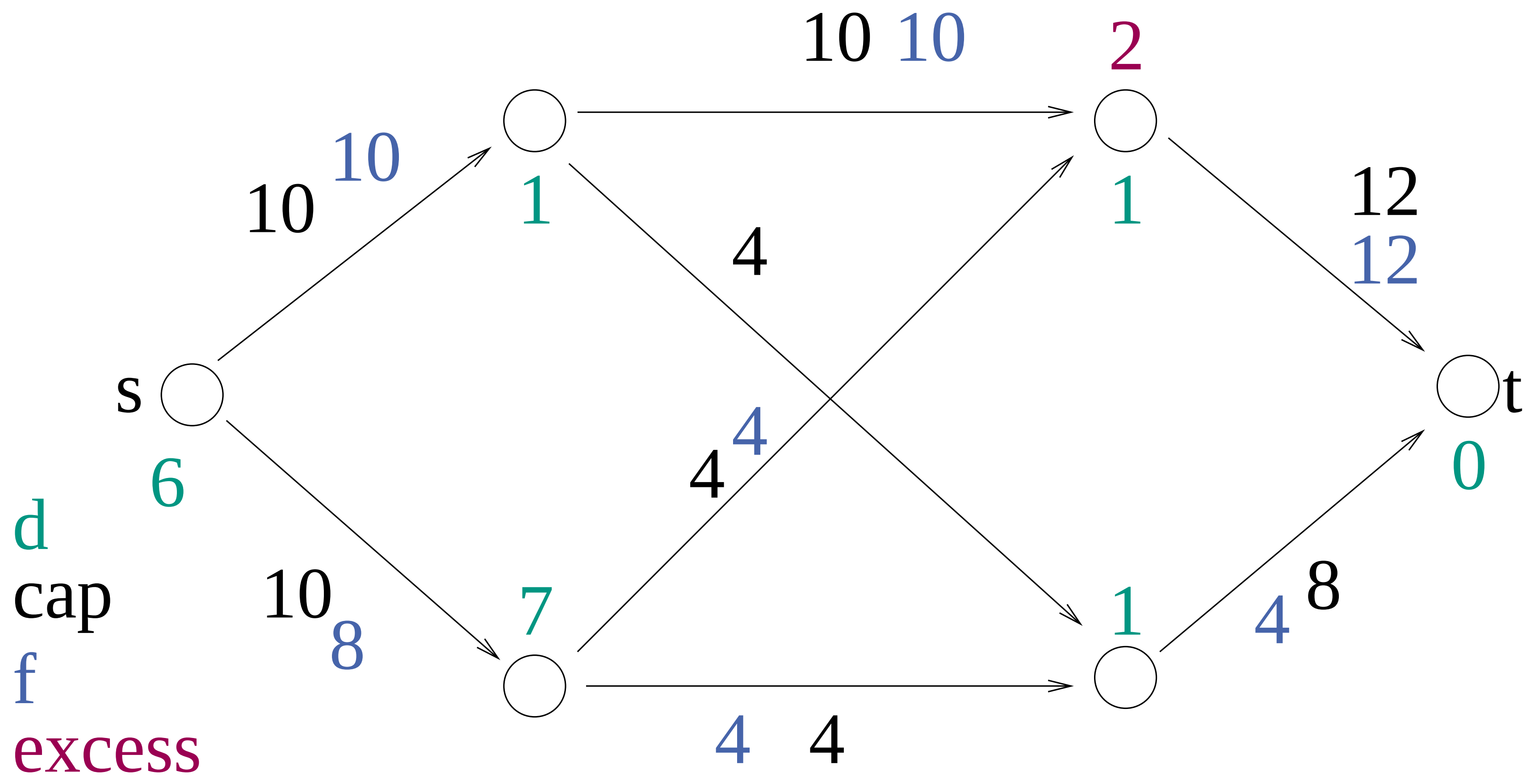
Example



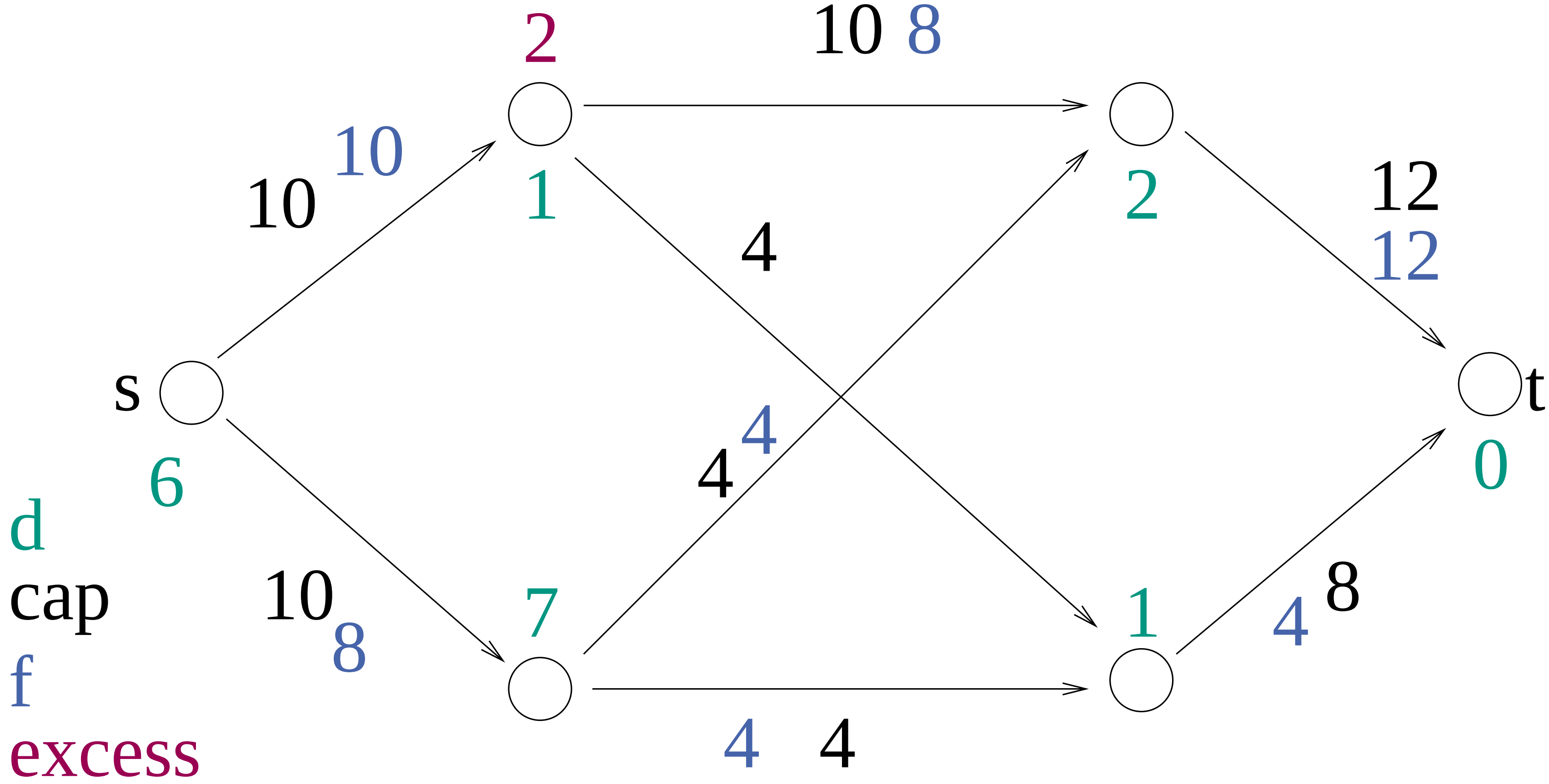
Example



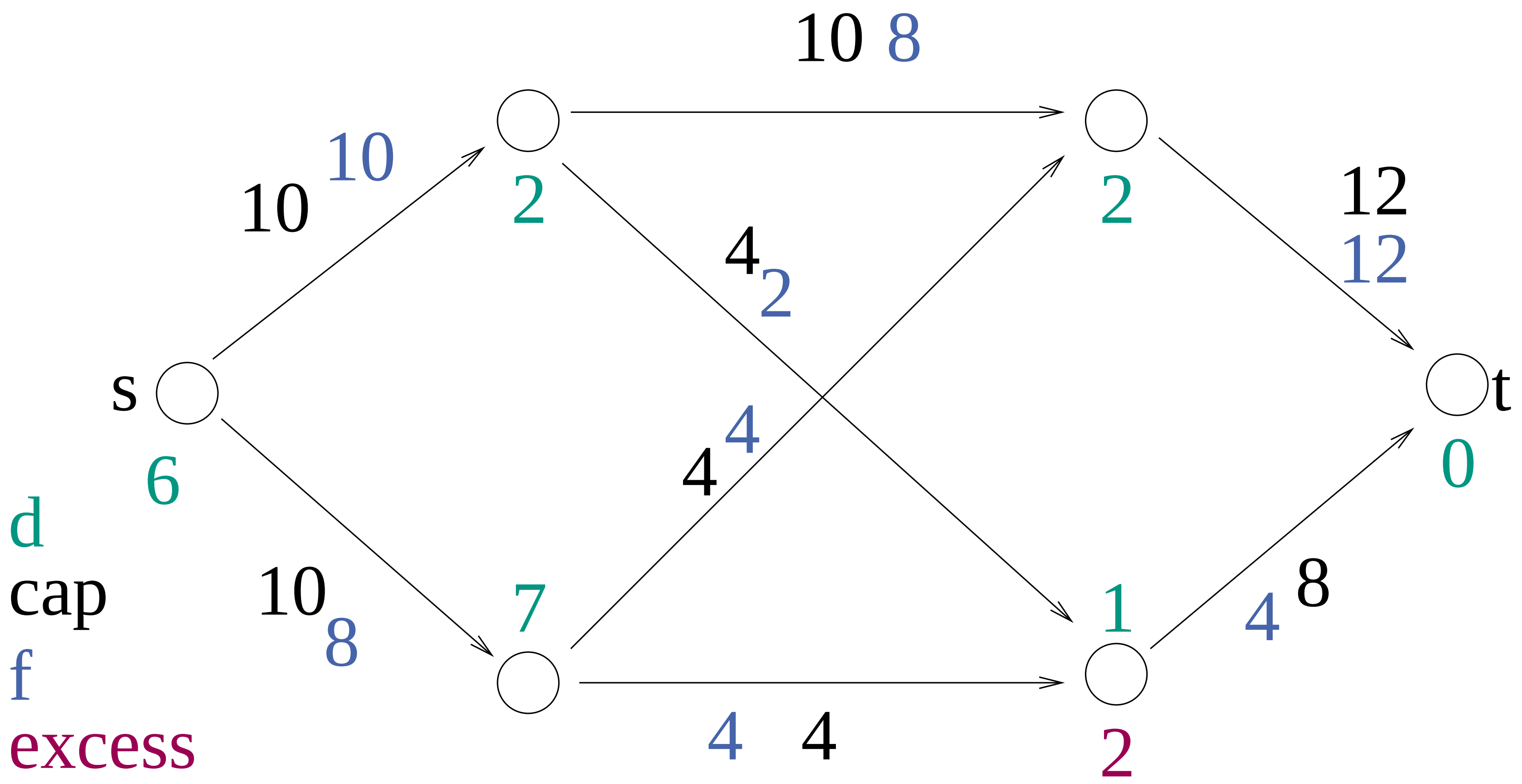
Example



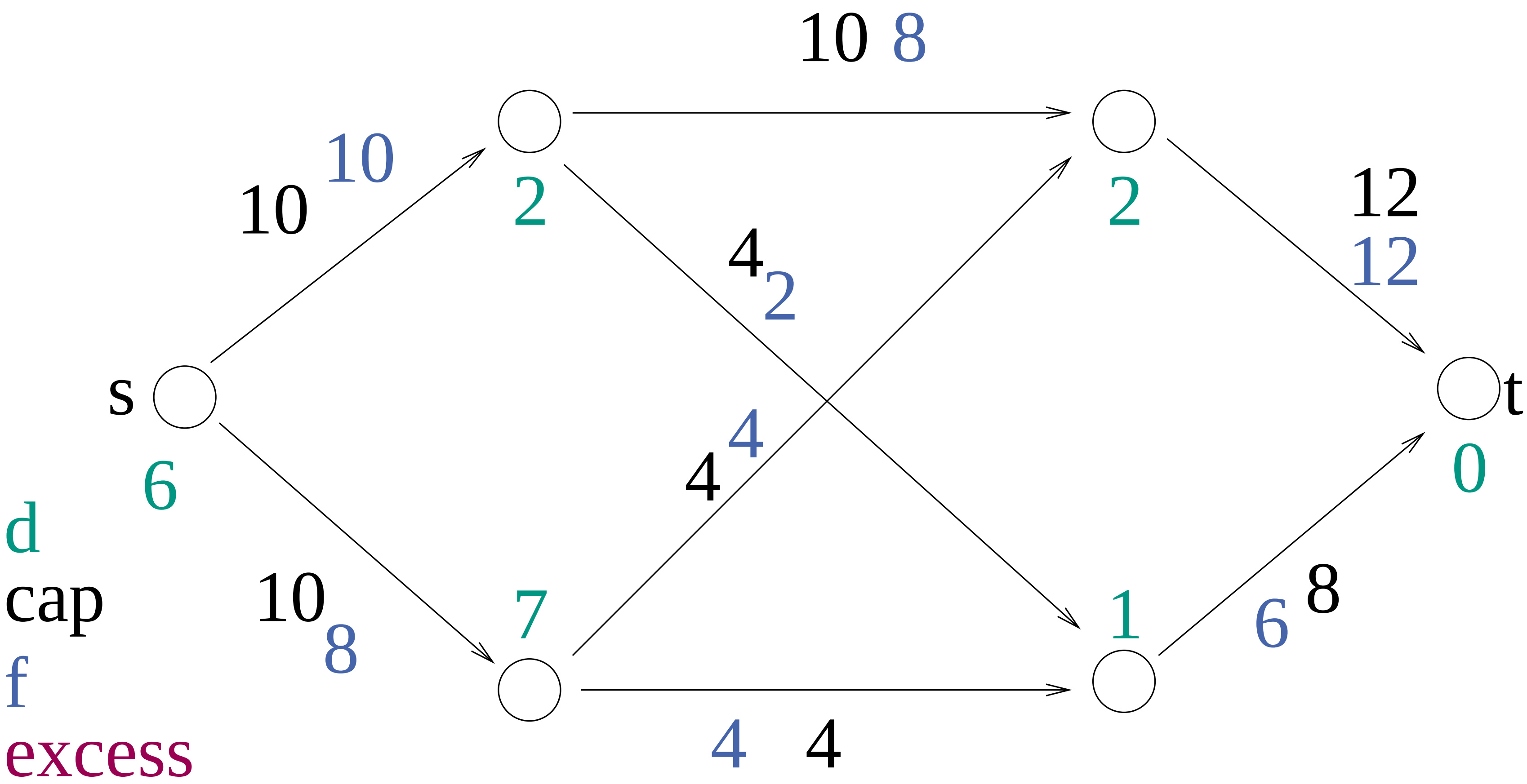
Example



Example



Example



12 pushes in total

Partial Correctness

Lemma 3. *When* genericPreflowPush *terminates* f *is a maximal flow.*

Beweis.

f is a flow since $\forall v \in V \setminus \{s,t\}$: excess$(v) = 0$.

To show that f is maximal, it suffices to show that
$\nexists$ path $p = \langle s, \ldots, t \rangle \in G_f$ (Max-Flow Min-Cut Theorem):
Since $d(s) = n$, $d(t) = 0$, p would have to contain steep edges.
That would be a contradiction. $\square$

Lemma 4. *For any cut* (S,T),

$$\sum_{u\in S} excess(u) = \sum_{e\in E\cap(T\times S)} f(e) - \sum_{e\in E\cap(S\times T)} f(e),$$

Proof:

$$\sum_{u\in S} excess(u) = \sum_{u\in S}\left(\sum_{(v,u)\in E} f((v,u)) - \sum_{(u,v)\in E} f((u,v))\right)$$

Contributions of edge e to sum:

S to T: $-f(e)$

T to S: $f(e)$

within S: $f(e)-f(e)=0$

within T: 0 ■

Lemma 5.

$\forall$ *active nodes v* : excess$(v) > 0 \Rightarrow \exists$ *path* $\langle v, \ldots, s\rangle \in G_f$

Intuition: what got there can always go back.

Beweis. $S := \{u \in V : \exists \text{ path } \langle v, \ldots u\rangle \in G_f\}$, $T := V \setminus S$. Then

$$\sum_{u \in S} excess(u) = \sum_{e \in E \cap (T \times S)} f(e) - \sum_{e \in E \cap (S \times T)} f(e),$$

$\forall (u,w) \in E_f : u \in S \Rightarrow w \in S$ by Def. of G_f, S

$\Rightarrow \forall e = (u,w) \in E \cap (T \times S) : f(e) = 0$ Otherwise $(w,u) \in E_f$

Hence, $\sum_{u \in S} excess(u) \leq 0$

Only the negative excess of s can outweigh excess$(v) > 0$.

Hence $s \in S$. $\square$

Lemma 6.

$\forall v \in V : d(v) < 2n$

Beweis.

Suppose v is lifted to $d(v) = 2n$.

By the Lemma 2, there is a (simple) path p to s in G_f.

p has at most $n-1$ nodes

$d(s) = n$.

Hence $d(v) < 2n$. Contradiction (no steep edges). $\square$

Lemma 7. *# Relabel operations* $\leq 2n^2$

Beweis. $d(v) \leq 2n$, i.e., v is relabeled at most $2n$ times.
Hence, at most $|V| \cdot 2n = 2n^2$ relabel operations. $\square$

Lemma 8. *# saturating pushes $\leq nm$*

Beweis.

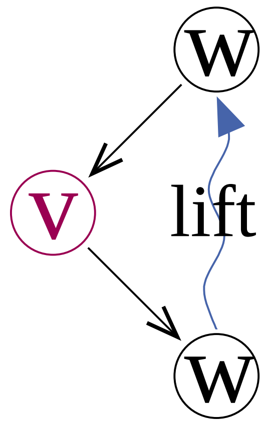
We show that there are at most n sat. pushes over any edge $e = (v, w)$.

A saturating push(e, δ) removes e from E_f.

Only a push on (w, v) can reinsert e into E_f.

For this to happen, w must be lifted at least two levels.

Hence, at most $2n/2 = n$ saturating pushes over (v, w)



$\square$

Lemma 9. *# nonsaturating pushes* $= \mathrm{O}\left(n^2 m\right)$
if $\delta = \min\left\{\mathsf{excess}(v), c_e^f\right\}$
for arbitrary node and edge selection rules.
(arbitrary-preflow-push)

Beweis. $\Phi := \sum_{\{v : v \text{ is active}\}} d(v)$. (Potential)

$\Phi = 0$ initially and at the end (no active nodes left!)

Operation	$\Delta(\Phi)$	How many times?	Total effect
relabel	1	$\leq 2n^2$	$\leq 2n^2$
saturating push	$\leq 2n$	$\leq nm$	$\leq 2n^2 m$
nonsaturating push	≤ -1		

$\Phi \geq 0$ always. □

Searching for Eligible Edges

Every node v maintains a currentEdge pointer to its sequence of outgoing edges in G_f.

invariant no edge $e = (v, w)$ to the left of currentEdge is eligible

Reset currentEdge at a relabel $(\leq 2n\times)$

Invariant cannot be violated by a push over a reverse edge $e' = (w, v)$ since this only happens when e' is downward,

i.e., e is upward and hence not eligible.

Lemma 10.

Total cost for searching $\leq \sum_{v \in V} 2n \cdot \mathrm{degree}(v) = 4nm = \mathrm{O}(nm)$

Satz 11. *Arbitrary Preflow Push finds a maximum flow in time* $O(n^2m)$.

Beweis.

Lemma 3: partial correctness

Initialization in time $O(n+m)$.

Maintain set (e.g., stack, FIFO) of active nodes.

Use reverse edge pointers to implement push.

Lemma 7: $2n^2$ relabel operations

Lemma 8: nm saturating pushes

Lemma 9: $O(n^2m)$ nonsaturating pushes

Lemma 10: $O(nm)$ search time for eligible edges

Total time $O(n^2m)$ □

FIFO Preflow push

Examine a node: Saturating pushes until nonsaturating push or relabel.

Examine all nodes in phases (or use FIFO queue).

Theorem: time $\mathrm{O}\left(n^3\right)$

Proof: not here

Highest Level Preflow Push

Always select active nodes that maximize $d(v)$

Use bucket priority queue (insert, increaseKey, deleteMax)

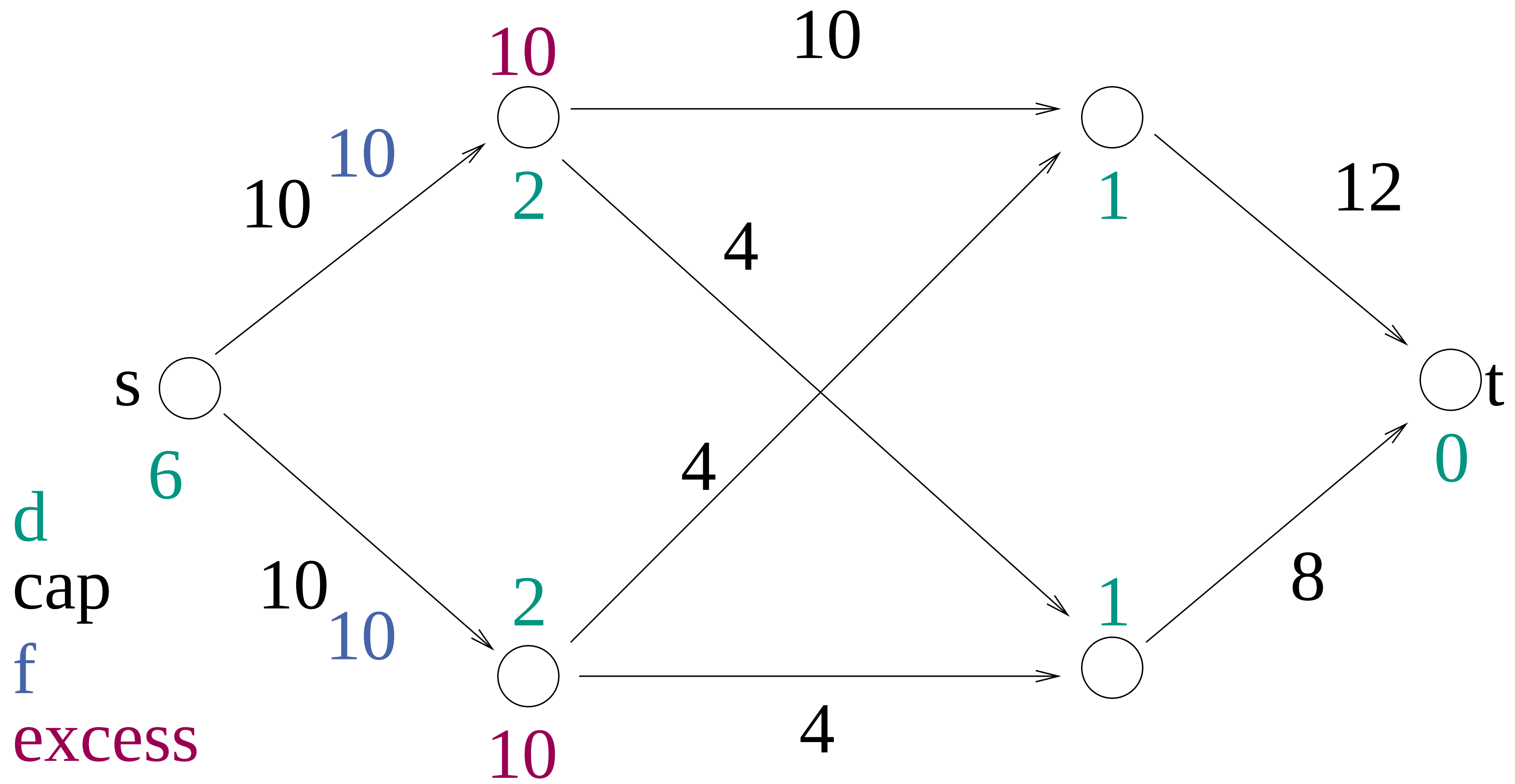
not monotone (!) but relabels "pay" for scan operations

Lemma 12. *At most $n^2\sqrt{m}$ nonsaturating pushes.*

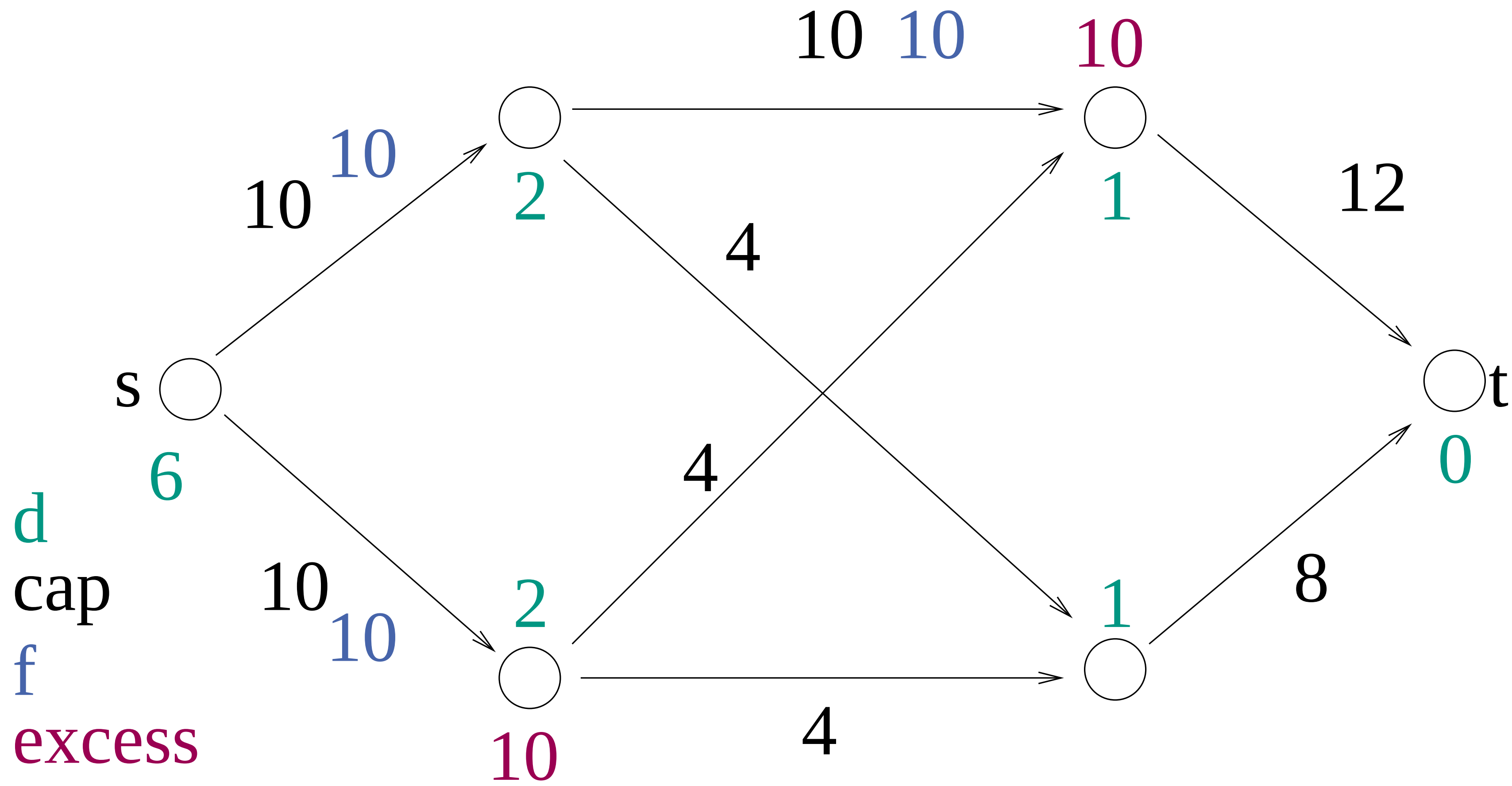
Beweis. later $\square$

Satz 13. *Highest Level Preflow Push finds a maximum flow in time* $\mathrm{O}\left(n^2\sqrt{m}\right)$.

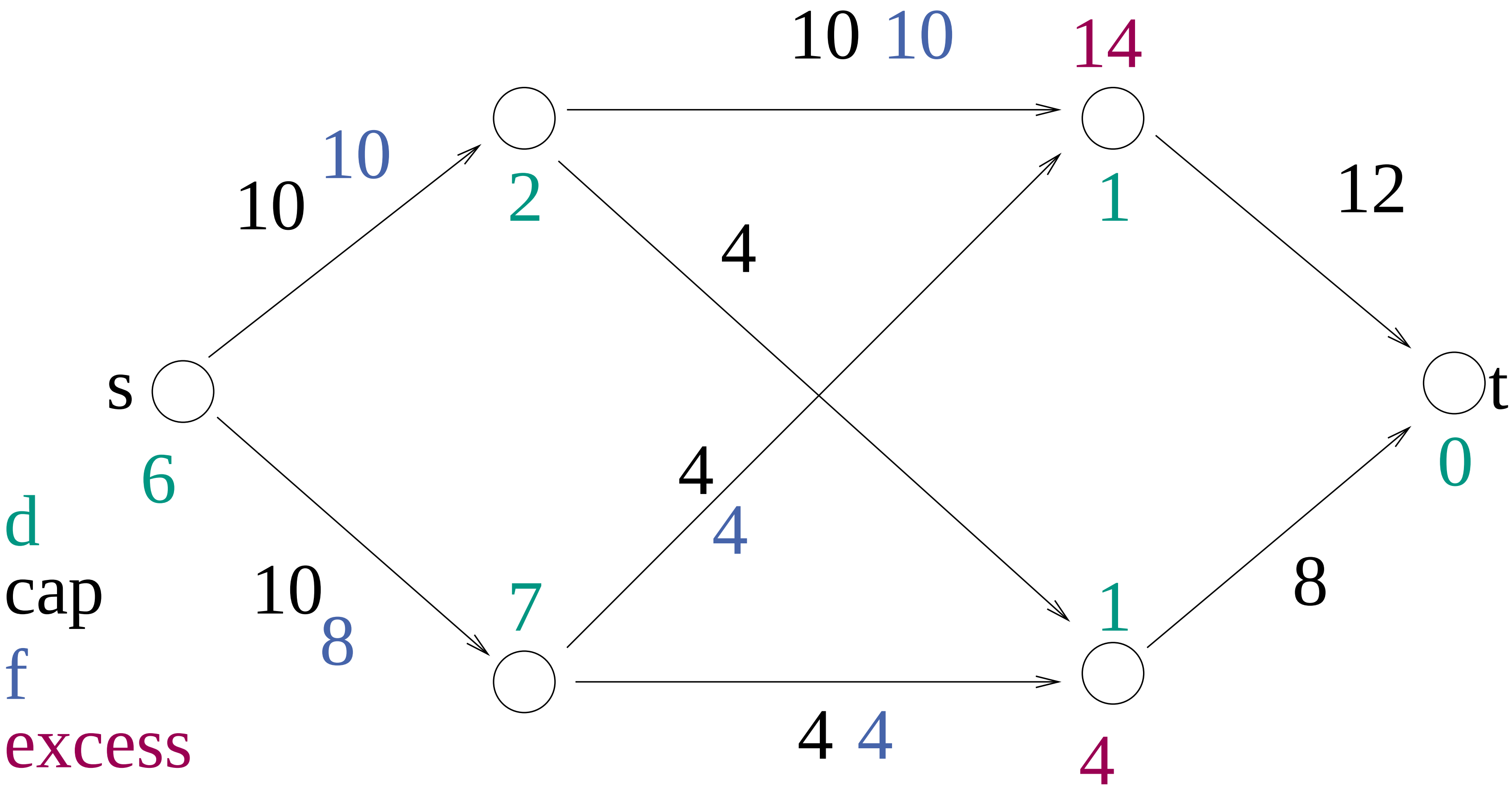
Example



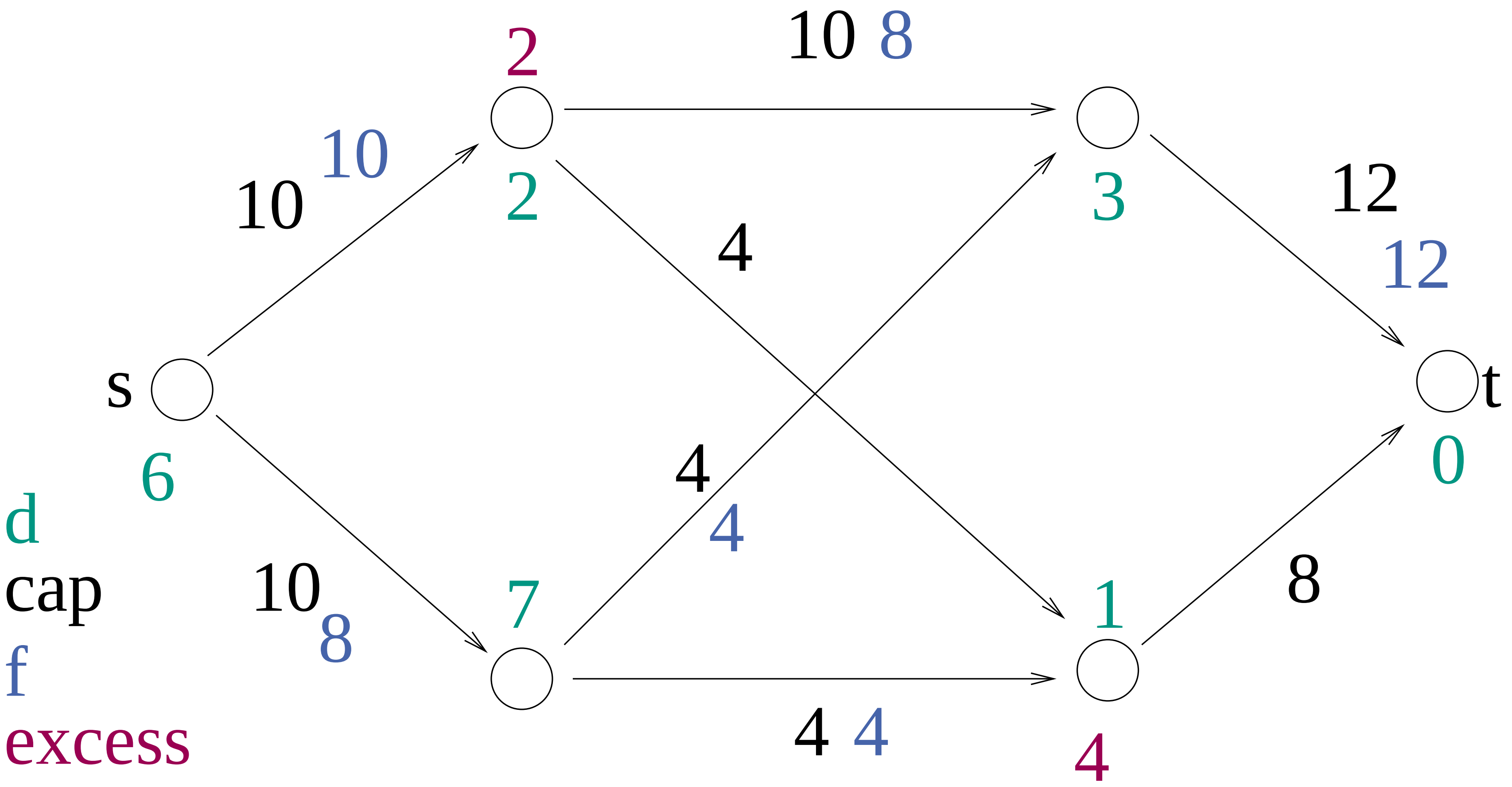
Example



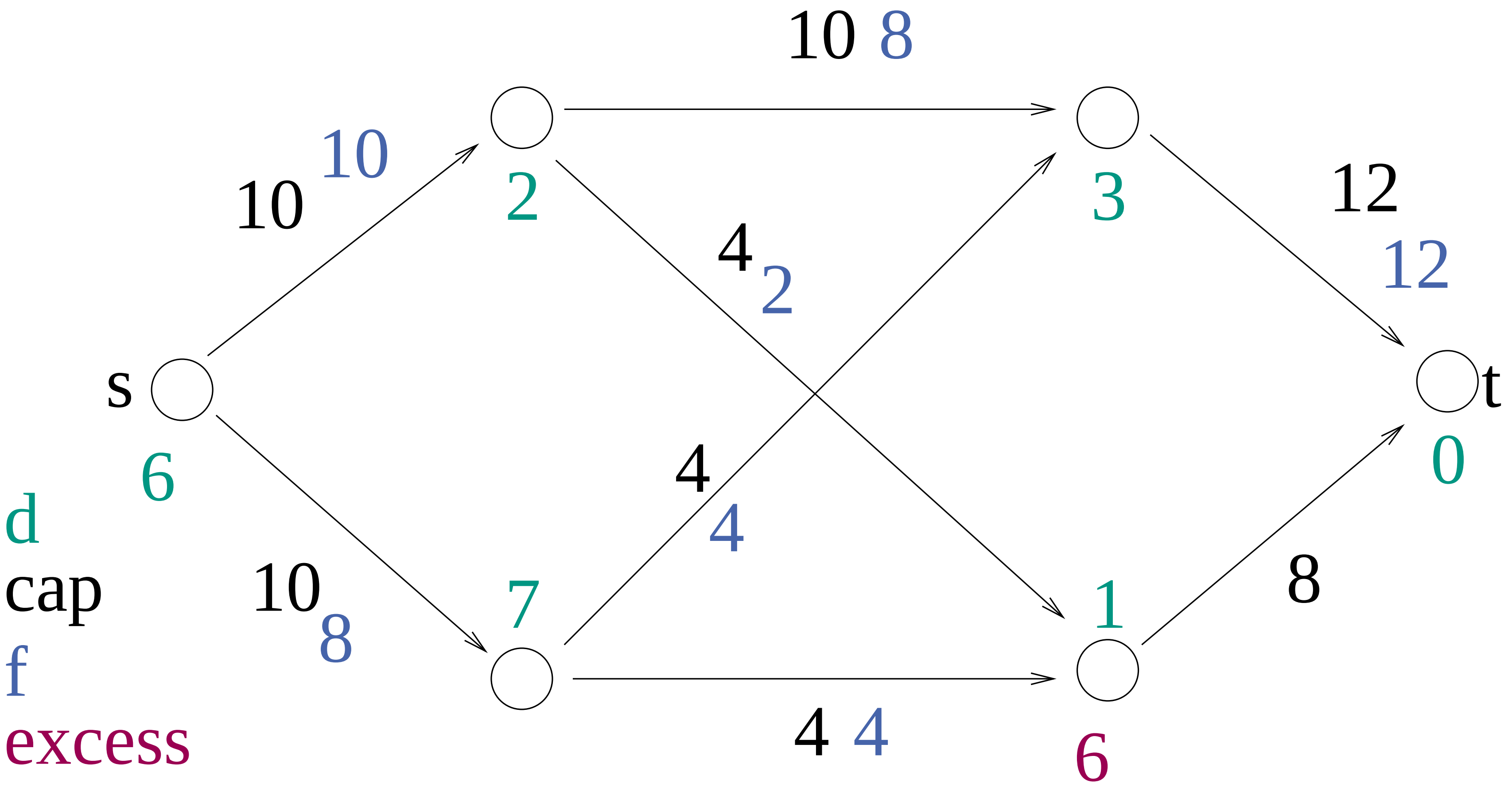
Example



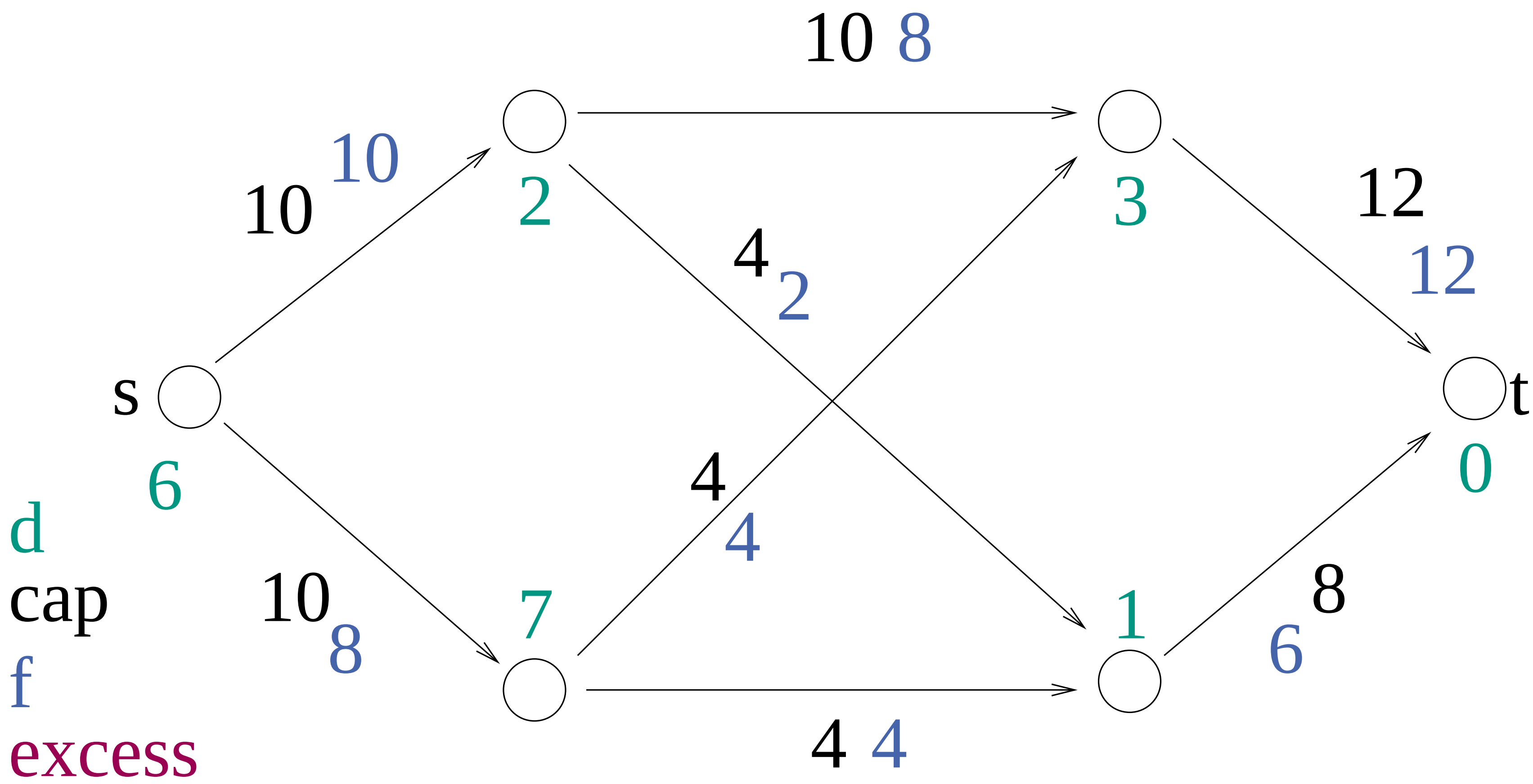
Example



Example



Example



9 pushes in total, 3 less than before

Proof of Lemma 12

$K := \sqrt{m}$ tuning parameter

$d'(v) := \frac{|\{w : d(w) \leq d(v)\}|}{K}$ scaled number of dominated nodes

$\Phi := \sum_{\{v:v \text{ is active}\}} d'(v)$. (Potential)

$d^* := \max\{d(v) : v \text{ is active}\}$ (highest level)

phase:= all pushes between two consecutive changes of d^*

expensive phase: more than K pushes

cheap phase: otherwise

Claims:

1. $\leq 4n^2K$ nonsaturating pushes in all cheap phases together
2. $\Phi \geq 0$ always, $\Phi \leq n^2/K$ initially (obvious)
3. a relabel or saturating push increases Φ by at most n/K.
4. a nonsaturating push does not increase Φ.
5. an expensive phase with $Q \geq K$ nonsaturating pushes decreases Φ by at least Q.

Operation	Amount
Relabel	$2n^2$
Sat.push	nm

Lemma 7+Lemma 8+2.+3.+4.:$\Rightarrow$

total possible decrease $\leq (2n^2+nm)\frac{n}{K}+\frac{n^2}{K}$

This $+5.:\leq \frac{2n^3+n^2+mn^2}{K}$ nonsaturating pushes in expensive phases

This $+1.:\leq \frac{2n^3+n^2+mn^2}{K}+4n^2K = \mathrm{O}\left(n^2\sqrt{m}\right)$ nonsaturating pushes overall for $K=\sqrt{m}$ $\square$

Claims:

1. $\leq 4n^2K$ nonsaturating pushes in all cheap phases together

We first show that there are at most $4n^2$ phases

(changes of $d^* = \max\{d(v) : v \text{ is active}\}$).

$d^* = 0$ initially, $d^* \geq 0$ always.

Only relabel operations increase d^*, i.e.,

$\leq 2n^2$ increases by Lemma 7 and hence

$\leq 2n^2$ decreases

$\leq 4n^2$ changes overall

By definition of a cheap phase, it has at most K pushes.

Claims:

1. $\leq 4n^2K$ nonsaturating pushes in all cheap phases together
2. $\Phi \geq 0$ always, $\Phi \leq n^2/K$ initially (obvious)
3. a relabel or saturating push increases Φ by at most n/K.

Let v denote the relabeled or activated node.

$$d'(v) := \frac{|\{w : d(w) \leq d(v)\}|}{K} \leq \frac{n}{K}$$

A relabel of v can increase only the d'-value of v.

A saturating push on (u, w) may activate only w.

Claims:

1. $\leq 4n^2K$ nonsaturating pushes in all cheap phases together
2. $\Phi \geq 0$ always, $\Phi \leq n^2/K$ initially (obvious)
3. a relabel or saturating push increases Φ by at most n/K.
4. a nonsaturating push does not increase Φ.

v is deactivated (excess(v) is now 0)

w may be activated

but $d'(w) \leq d'(v)$ (we do not push flow away from the sink)

Claims:

1. $\leq 4n^2K$ nonsaturating pushes in all cheap phases together
2. $\Phi \geq 0$ always, $\Phi \leq n^2/K$ initially (obvious)
3. a relabel or saturating push increases Φ by at most n/K.
4. a nonsaturating push does not increase Φ.
5. an expensive phase with $Q \geq K$ nonsaturating pushes decreases Φ by at least Q.

During a phase d^* remains constant

Each nonsat. push decreases the number of active nodes at level d^*

Hence, $|\{w : d(w) = d^*\}| \geq Q \geq K$ during an expensive phase

Each nonsat. push across (v,w) decreases Φ by

$\geq d'(v) - d'(w) \geq |\{w : d(w) = d^*\}|/K \geq K/K = 1$ ■

Claims:

1. $\leq 4n^2K$ nonsaturating pushes in all cheap phases together
2. $\Phi \geq 0$ always, $\Phi \leq n^2/K$ initially (obvious)
3. a relabel or saturating push increases Φ by at most n/K.
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Operation	Amount
Relabel	$2n^2$
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Lemma 7+Lemma 8+2.+3.+4.:$\Rightarrow$

total possible decrease $\leq (2n^2 + nm)\frac{n}{K} + \frac{n^2}{K}$

This $+5. :\leq \frac{2n^3+n^2+mn^2}{K}$ nonsaturating pushes in expensive phases

This $+1. :\leq \frac{2n^3+n^2+mn^2}{K} + 4n^2K = \mathrm{O}\left(n^2\sqrt{m}\right)$ nonsaturating pushes overall for $K = \sqrt{m}$ $\square$

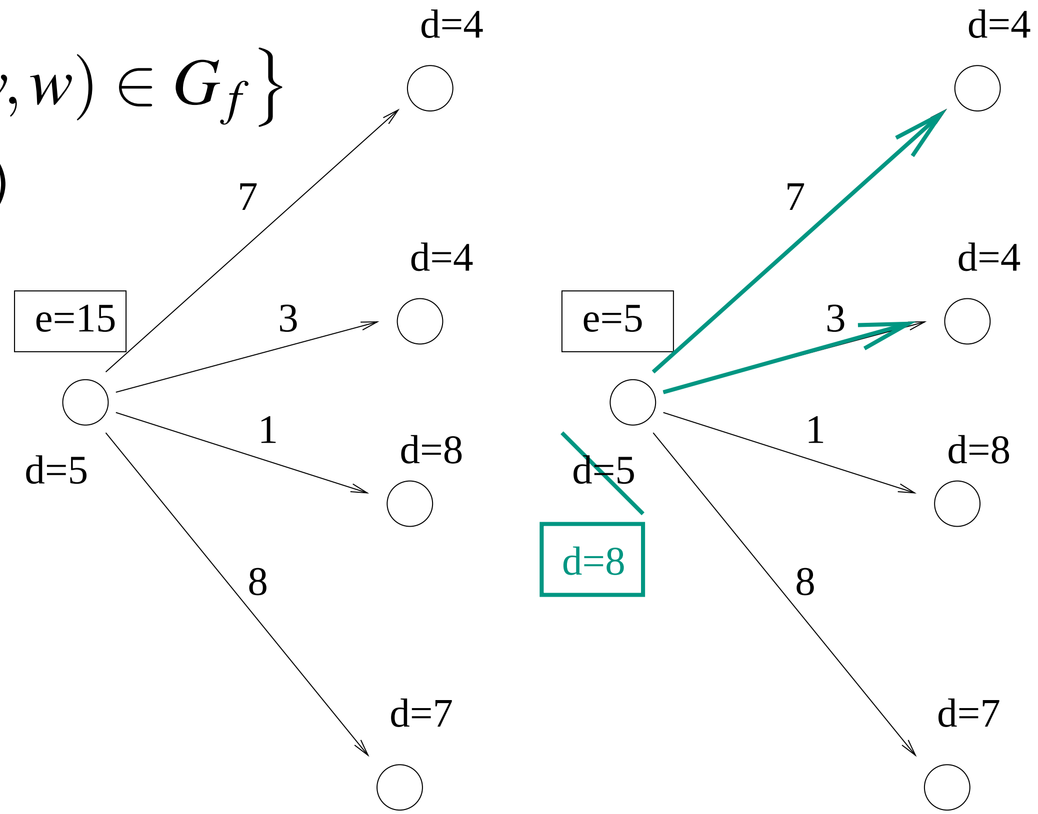
Heuristic Improvements

Naive algorithm has best case $\Omega\left(n^2\right)$. Why? We can do better.

aggressive local relabeling:

$$d(v) := 1 + \min\left\{d(w) : (v,w) \in G_f\right\}$$

(like a sequence of relabels)



Heuristic Improvements

Naive algorithm has best case $\Omega\left(n^2\right)$. Why? We can do better.

aggressive local relabeling: $d(v) := 1 + \min\left\{d(w) : (v,w) \in G_f\right\}$
(like a sequence of relabels)

global relabeling: (initially and every $\mathrm{O}(m)$ edge inspections):
$d(v) := G_f$.reverseBFS(t) for nodes that can reach t in G_f.

Special treatment of nodes with $d(v) \geq n$. (Returning flow is easy)

Gap Heuristics. No node can connect to t across an empty level:

if $\{v : d(v) = i\} = \emptyset$ **then foreach** v with $d(v) > i$ **do** $d(v) := n$

Experimental results

We use four classes of graphs:

- Random: n nodes, $2n+m$ edges; all edges (s,v) and (v,t) exist
- Cherkassky and Goldberg (1997) (two graph classes)
- Ahuja, Magnanti, Orlin (1993)

Timings: Random Graphs

Rule	BASIC	Ln	LRH	GRH	GAP	LEDA
FF	5.84	6.02	4.75	0.07	0.07	—
	33.32	33.88	26.63	0.16	0.17	—
HL	6.12	6.3	4.97	0.41	0.11	0.07
	27.03	27.61	22.22	1.14	0.22	0.16
MF	5.36	5.51	4.57	0.06	0.07	—
	26.35	27.16	23.65	0.19	0.16	—

$n \in \{1000, 2000\}, m = 3n$

FF=FIFO node selection, HL=hightest level, MF=modified FIFO

Ln= $d(v) \geq n$ is special,

LRH=local relabeling heuristic, GRH=global relabeling heuristics

Timings: CG1

Rule	BASIC	Ln	LRH	GRH	GAP	LEDA
FF	3.46	3.62	2.87	0.9	1.01	—
	15.44	16.08	12.63	3.64	4.07	—
HL	20.43	20.61	20.51	1.19	1.33	0.8
	192.8	191.5	193.7	4.87	5.34	3.28
MF	3.01	3.16	2.3	0.89	1.01	—
	12.22	12.91	9.52	3.65	4.12	—

$n \in \{1000, 2000\}, m = 3n$

FF=FIFO node selection, HL=hightest level, MF=modified FIFO

Ln= $d(v) \geq n$ is special,

LRH=local relabeling heuristic, GRH=global relabeling heuristics

Timings: CG2

Rule	BASIC	Ln	LRH	GRH	GAP	LEDA
FF	50.06	47.12	37.58	1.76	1.96	—
	239	222.4	177.1	7.18	8	—
HL	42.95	41.5	30.1	0.17	0.14	0.08
	173.9	167.9	120.5	0.36	0.28	0.18
MF	45.34	42.73	37.6	0.94	1.07	—
	198.2	186.8	165.7	4.11	4.55	—

$n \in \{1000, 2000\}, m = 3n$

FF=FIFO node selection, HL=hightest level, MF=modified FIFO

Ln= $d(v) \geq n$ is special,

LRH=local relabeling heuristic, GRH=global relabeling heuristics

Timings: AMO

Rule	BASIC	Ln	LRH	GRH	GAP	LEDA
FF	12.61	13.25	1.17	0.06	0.06	—
	55.74	58.31	5.01	0.1399	0.1301	—
HL	15.14	15.8	1.49	0.13	0.13	0.07
	62.15	65.3	6.99	0.26	0.26	0.14
MF	10.97	11.65	0.04999	0.06	0.06	—
	46.74	49.48	0.1099	0.1301	0.1399	—

$n \in \{1000, 2000\}, m = 3n$

FF$=$FIFO node selection, HL$=$hightest level, MF$=$modified FIFO

Ln$= d(v) \geq n$ is special,

LRH$=$local relabeling heuristic, GRH$=$global relabeling heuristics

Asymptotics, $n \in \{5000, 10000, 20000\}$

Gen	Rule	GRH			GAP			LEDA		
rand	FF	0.16	0.41	1.16	0.15	0.42	1.05	—	—	—
	HL	1.47	4.67	18.81	0.23	0.57	1.38	0.16	0.45	1.09
	MF	0.17	0.36	1.06	0.14	0.37	0.92	—	—	—
CG1	FF	3.6	16.06	69.3	3.62	16.97	71.29	—	—	—
	HL	4.27	20.4	77.5	4.6	20.54	80.99	2.64	12.13	48.52
	MF	3.55	15.97	68.45	3.66	16.5	70.23	—	—	—
CG2	FF	6.8	29.12	125.3	7.04	29.5	127.6	—	—	—
	HL	0.33	0.65	1.36	0.26	0.52	1.05	0.15	0.3	0.63
	MF	3.86	15.96	68.42	3.9	16.14	70.07	—	—	—
AMO	FF	0.12	0.22	0.48	0.11	0.24	0.49	—	—	—
	HL	0.25	0.48	0.99	0.24	0.48	0.99	0.12	0.24	0.52
	MF	0.11	0.24	0.5	0.11	0.24	0.48	—	—	—

Zusammenfassung Flows und Matchings

- Natürliche Verallgemeinerung von kürzesten Wegen:
  ein Pfad $\rightsquigarrow$ viele Pfade
- viele Anwendungen
- "schwierigste/allgemeinste" Graph-Probleme, die sich mit kombinatorischen Algorithmen in Polynomialzeit lösen lassen
- Beispiel für nichttriviale Algorithmenanalyse
- Potentialmethode ($\neq$ Knotenpotentiale)
- Algorithm Engineering: practical case $\neq$ worst case.
  Heuristiken/Details/Eingabeeigenschaften wichtig
- Datenstrukturen: bucket queues, graph representation, (dynamic trees)