Algorithmen II

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Übungen:

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Web:

`http://algo2.iti.kit.edu/AlgorithmenII_WS19.php`

4 Anwendungen von DFS

Tiefensuchschema für $G = (V, E)$

```
unmark all nodes;   init
foreach s ∈ V do
    if s is not marked then
        mark s                              // make s a root and grow
        root(s)                             // a new DFS-tree rooted at it.
        DFS(s, s)
Procedure DFS(u, v : NodeId)                // Explore v coming from u.
    foreach (v, w) ∈ E do
        if w is marked then traverseNonTreeEdge(v, w)
        else    traverseTreeEdge(v, w)
                mark w
                DFS(v, w)
    backtrack(u, v)       // return from v along the incoming edge
```

DFS Nummerierung

init: dfsPos$=1 \quad : 1..n$

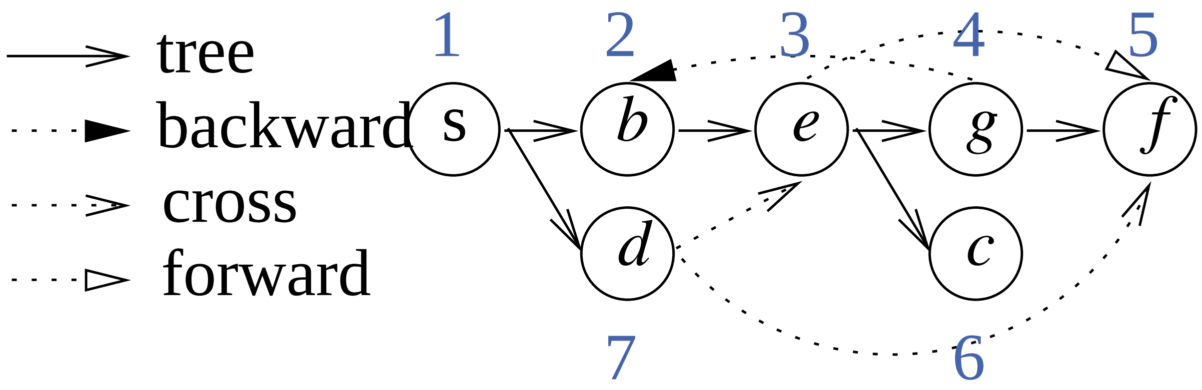
root(s): dfsNum$[s]$:= dfsPos$++$

traverseTreeEdge(v,w): dfsNum$[w]$:= dfsPos$++$

$$u \prec v \Leftrightarrow \text{dfsNum}[u] < \text{dfsNum}[v] \quad .$$

Beobachtung:

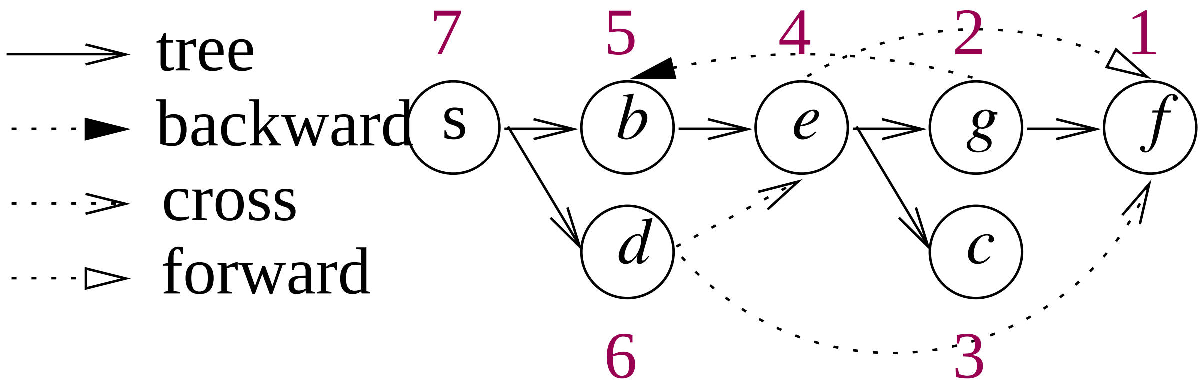
Knoten auf dem Rekursionsstapel sind bzgl., $\prec$ sortiert



Fertigstellungszeit

init: finishingTime$=1$: $1..n$

backtrack(u,v): finishTime$[v]$:= finishingTime$++$



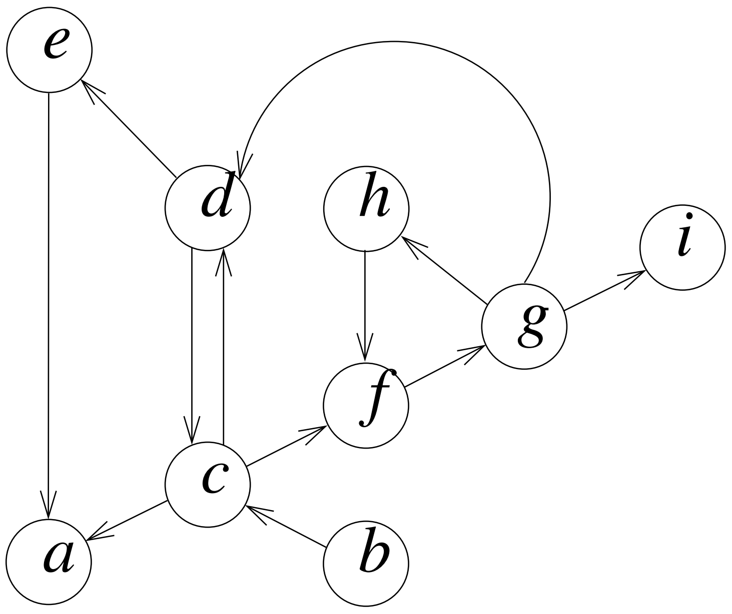
Starke Zusammenhangskomponenten

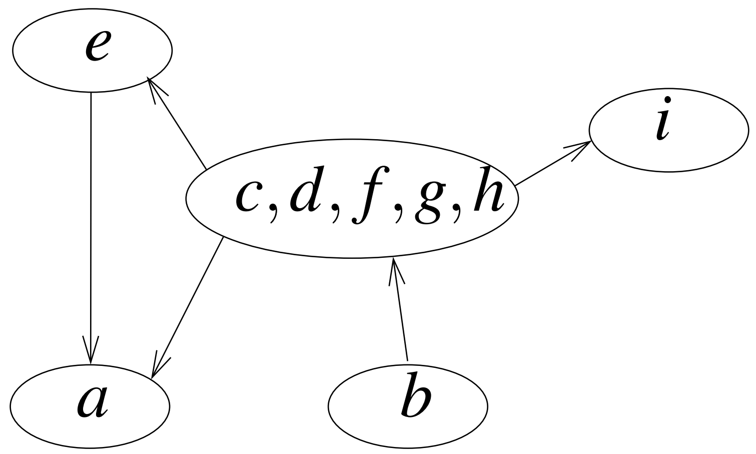
Betrachte die Relation $\overset{*}{\leftrightarrow}$ mit

$u \overset{*}{\leftrightarrow} v$ falls $\exists$ Pfad $\langle u, \ldots, v \rangle$ und $\exists$ Pfad $\langle v, \ldots, u \rangle$.

Beobachtung: $\overset{*}{\leftrightarrow}$ ist Äquivalenzrelation Übung

Die Äquivalenzklassen von $\overset{*}{\leftrightarrow}$ bezeichnet man als starke Zusammenhangskomponenten.





Starke Zusammenhangskomponenten – Abstrakter Algorithmus

$G_c := (V, \emptyset = E_c)$

foreach edge $e \in E$ **do**

  invariant SCCs of G_c are known

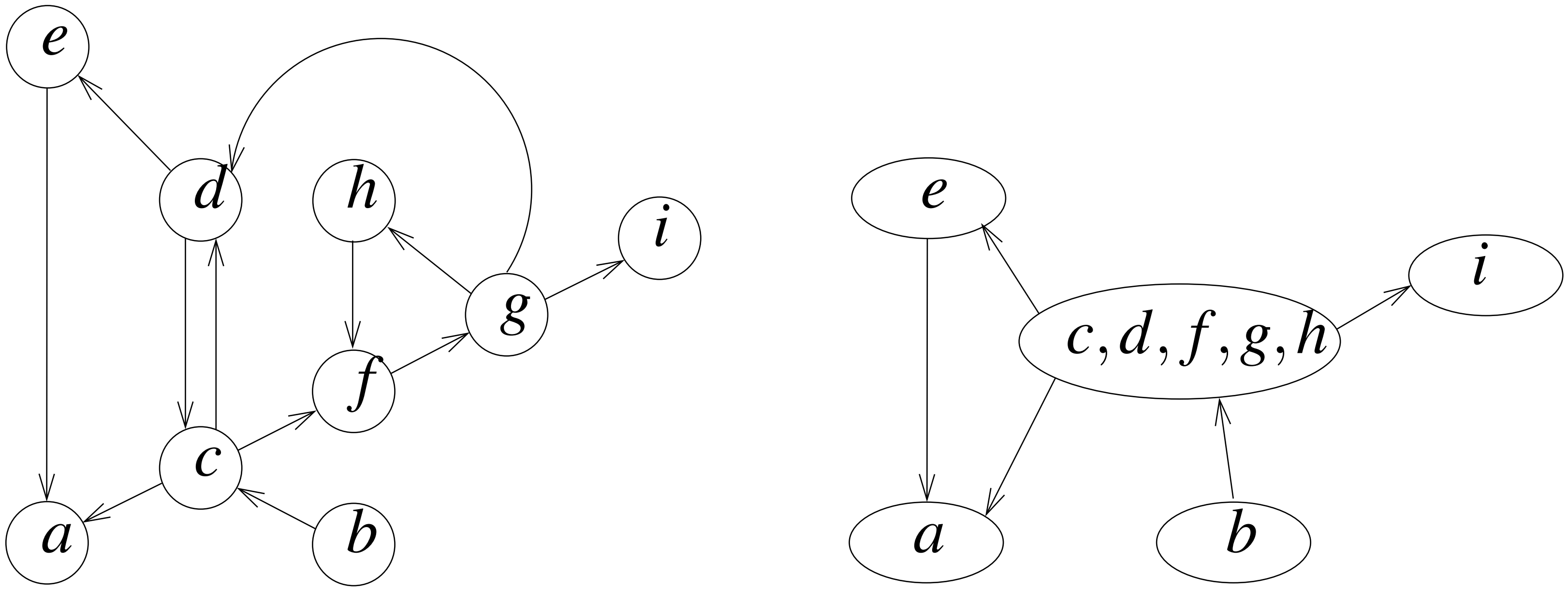
  $E_c := E_c \cup \{e\}$

Schrumpfgraph

$G_c^s = (V^s, E_c^s)$

Knoten: SCCs von G_c.

Kanten: $(C, D) \in E_c^s \Leftrightarrow \exists (c, d) \in E_c : c \in C \wedge d \in D$



Beobachtung: Der Schrumpfgraph ist azyklisch

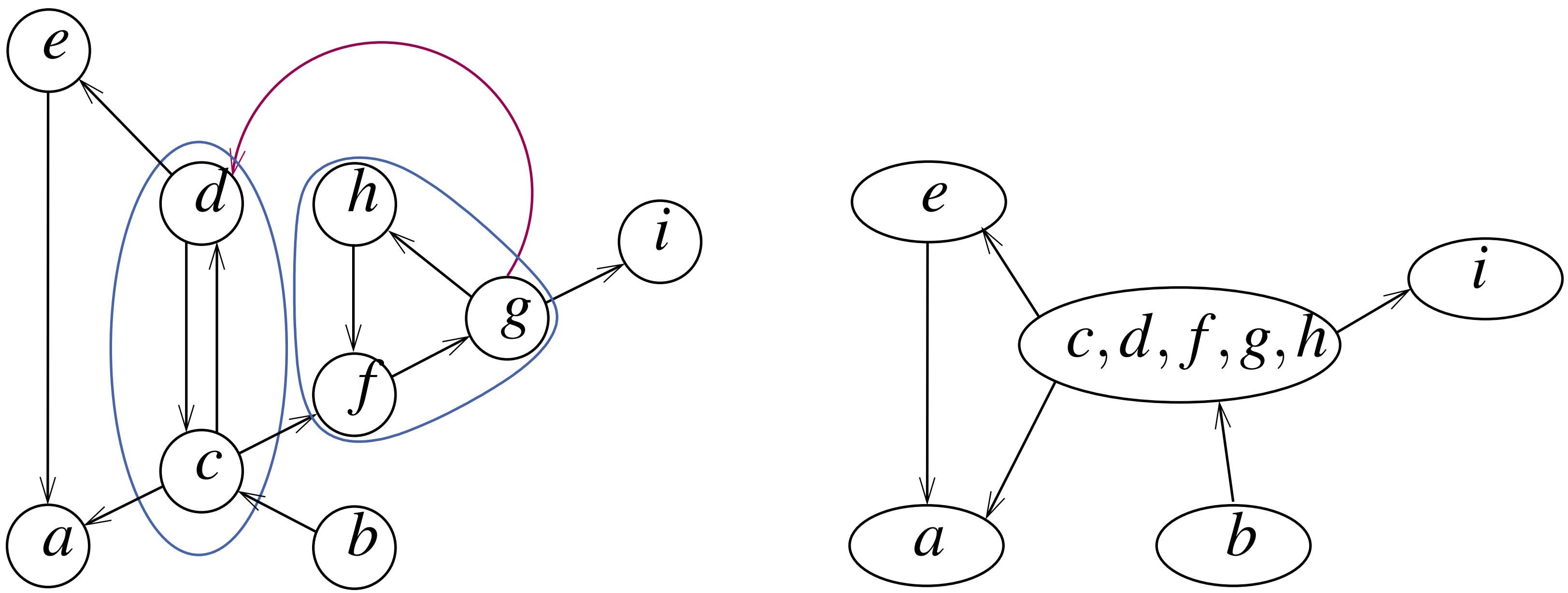
Auswirkungen einer neuen Kante e auf G_c, G_c^s

SCC-intern: Nichts ändert sich

zwischen zwei SCCs:

Kein Kreis: Neue Kante in G_c^s

Kreisschluss: SCCs auf Kreis kollabieren.



Konkreter: SCCs mittels DFS

[Cheriyan/Mehlhorn 96, Gabow 2000]

$V_c =$ markierte Knoten

$E_c =$ bisher explorierte Kanten

Aktive Knoten: markiert aber nicht finished.

SCCs von G_c:

nicht erreicht: Unmarkierte Knoten

offen: enthält aktive Knoten

abgeschlossen: alle Knoten finished

component[w] gibt Repräsentanten einer SCC an.

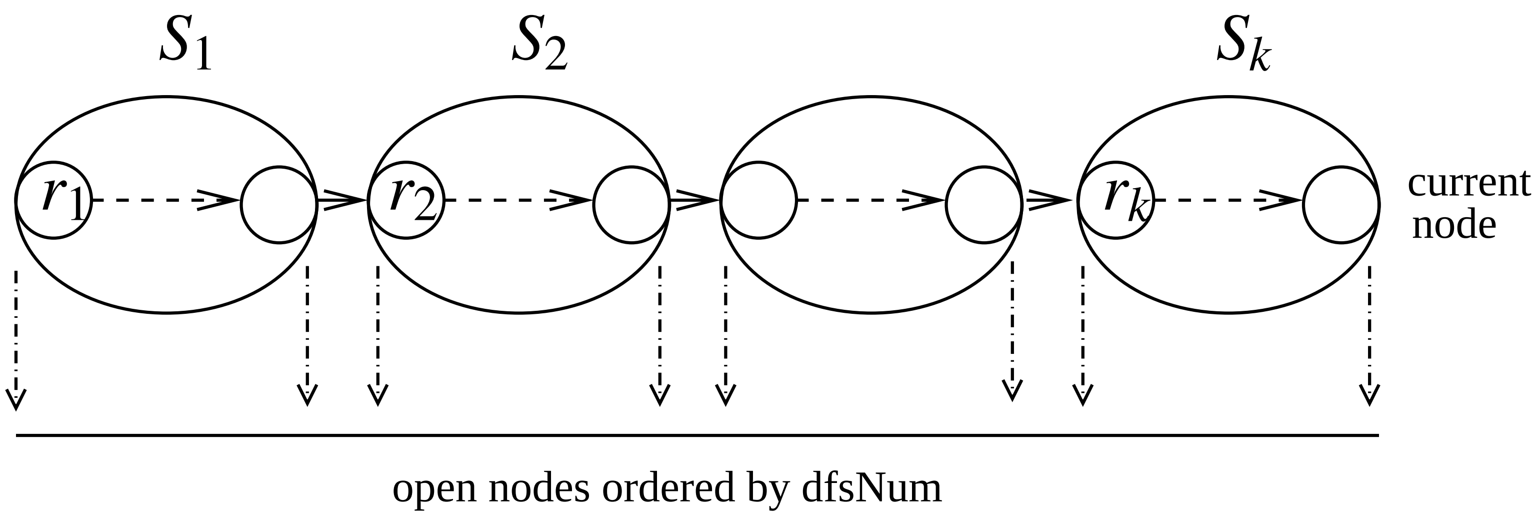
Knoten von offenen (abgeschl.) Komponenten heißen offen (abgeschl.)

Invarianten von G_c

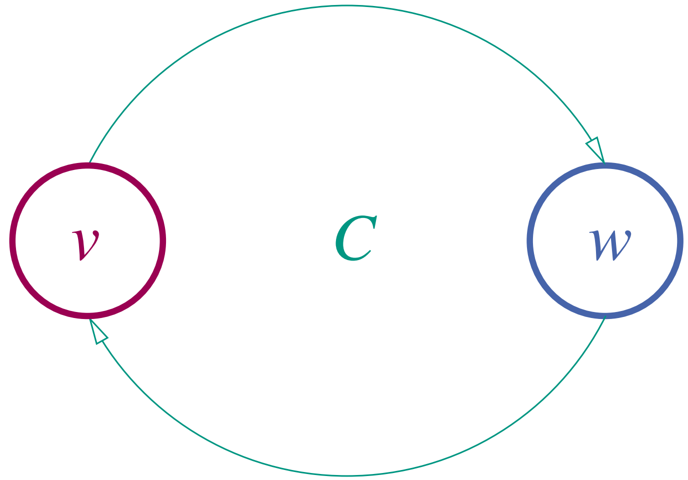
1. Kanten von abgeschlossenen Knoten gehen zu abgeschlossenen Knoten

2. Offene Komponenten $S_1, \ldots, S_k$ bilden Pfad in G_c^s.

3. Repräsentanten partitionieren die offenen Komponenten bzgl. ihrer dfsNum.



Lemma: Abgeschlossene SCCs von G_c sind SCCs von G



Betrachte abgeschlossenen Knoten v

und beliebigen Knoten w

in der SCC von v bzgl. G.

z.Z.: w ist abgeschlossen und

in der gleichen SCC von G_c wie v.

Betrachte Kreis C durch v, w.

Inv. 1: Knoten von C sind abgeschlossen.

Abgeschl. Knoten sind finished.

Kanten aus finished Knoten wurden exploriert.

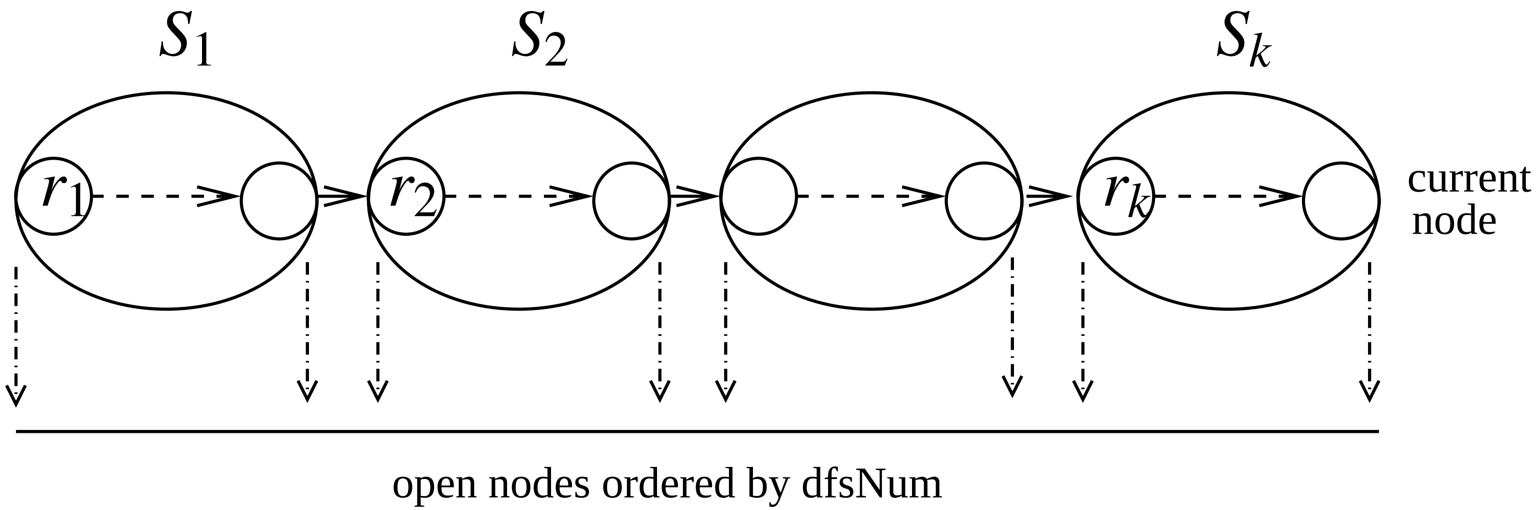
Also sind alle Kanten von C in G_c. □

Repräsentation offener Komponenten

Zwei Stapel aufsteigend sortiert nach dfsNum

oReps: Repräsentanten offener Komponenten

oNodes: Alle offenen Knoten



init

component : NodeArray **of** NodeId // SCC representatives

oReps=$\langle\rangle$: Stack **of** NodeId // representatives of open SCCs

oNodes=$\langle\rangle$: Stack **of** NodeId // all nodes in open SCCs

Alle Invarianten erfüllt.

(Weder offene noch geschlossene Knoten)

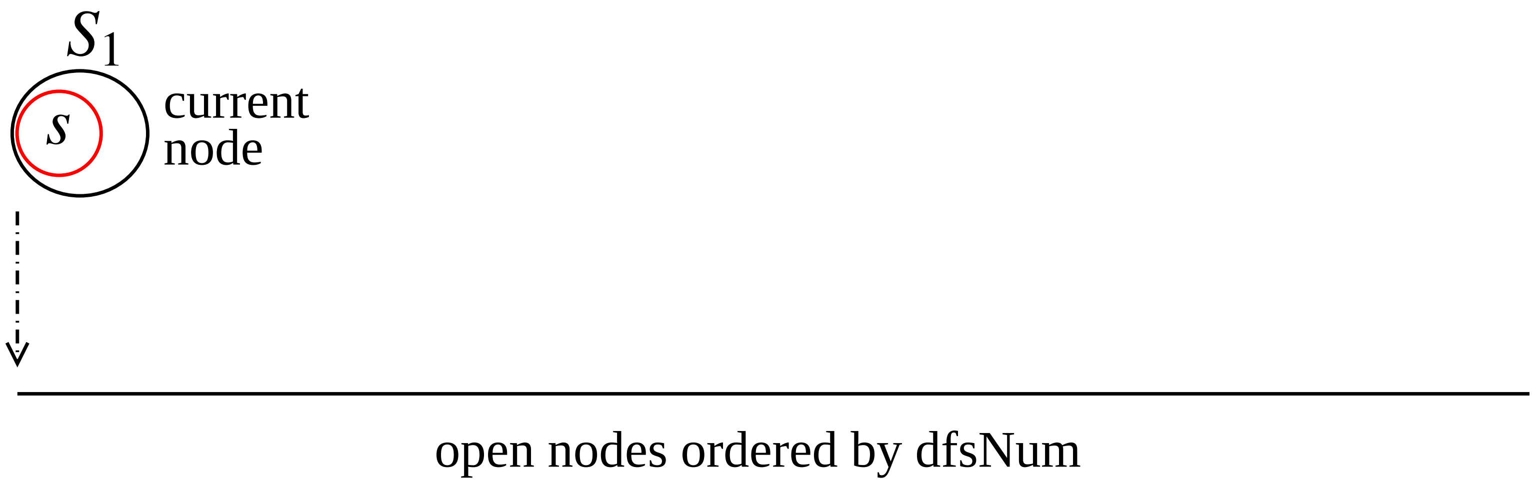
root(s)

oReps.push(s) // new open

oNodes.push(s) // component

$\{s\}$ ist die einzige offene Komponente.

Alle Invarianten bleiben gültig



traverseTreeEdge(v, w)

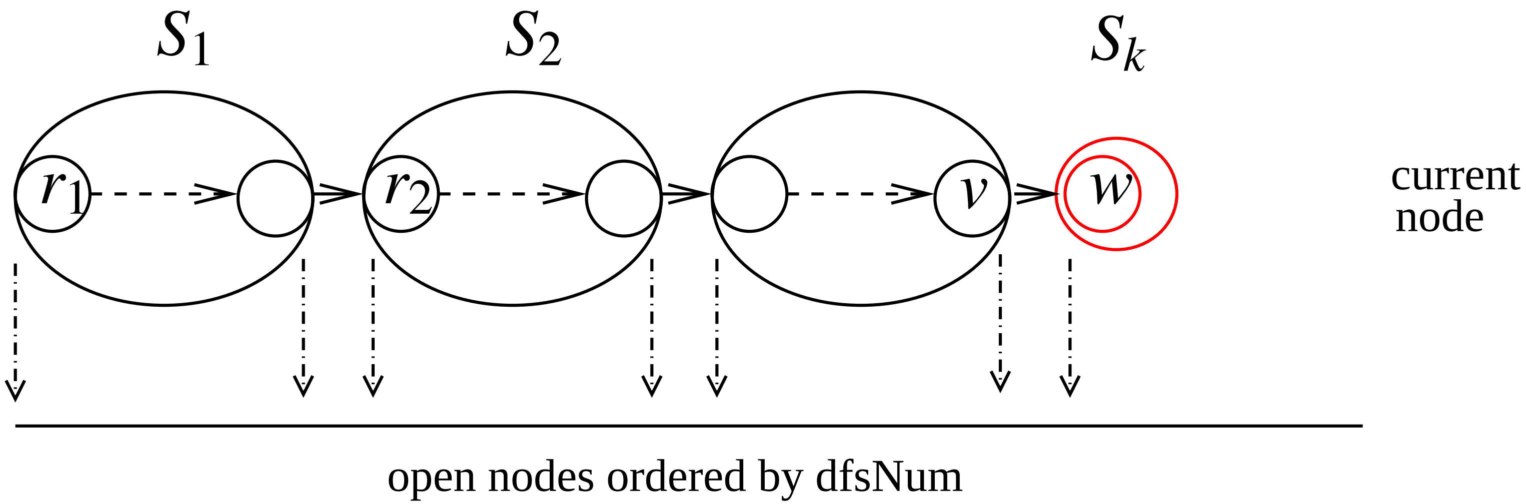
oReps.push(w) // new open

oNodes.push(w) // component

$\{w\}$ ist neue offene Komponente.

dfsNum$(w) >$ alle anderen.

$\leadsto$ Alle Invarianten bleiben gültig



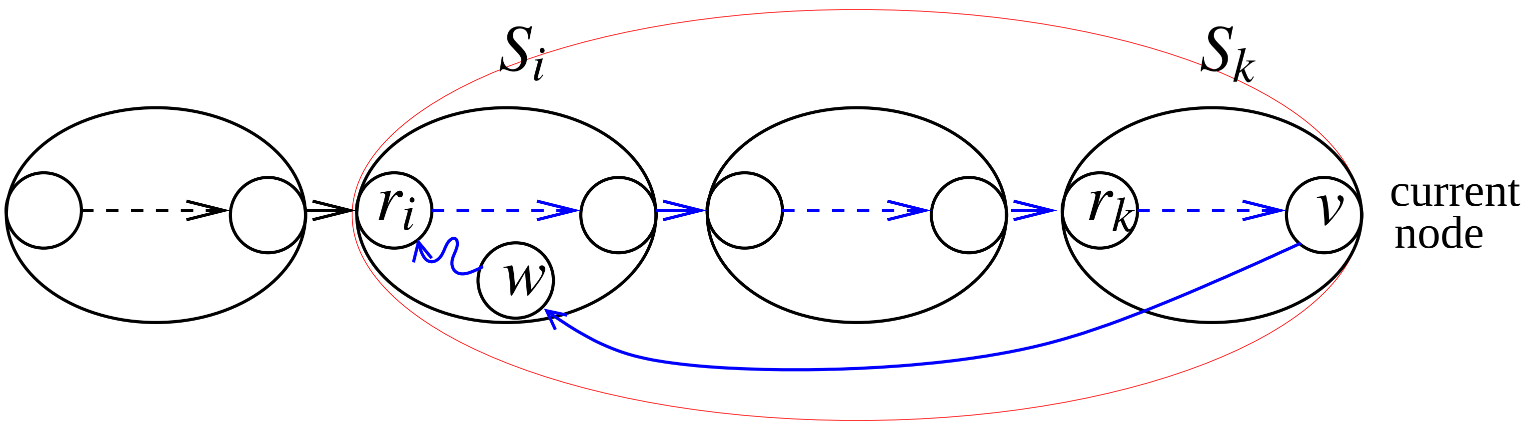
traverseNonTreeEdge(v, w)

if $w \in$ oNodes **then**

while $w \prec$ oReps.top **do** oReps.pop

$w \notin$ oNodes $\rightsquigarrow$ w is abgeschlossen $\overset{Lemma(*)}{\rightsquigarrow}$ Kante uninteressant

$w \in$ oNodes: kollabiere offene SCCs auf Kreis



backtrack(u,v)

if $v =$ oReps.top **then**

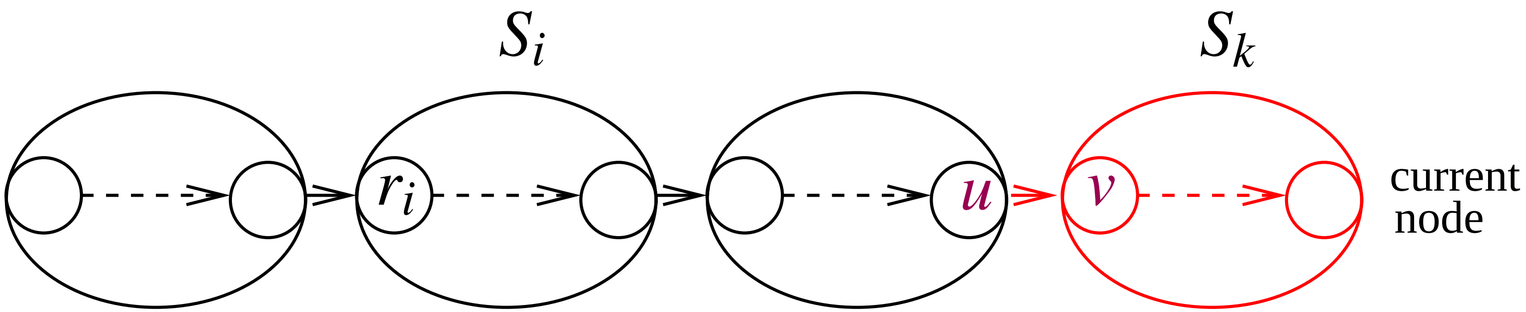
    oReps.pop // close

    repeat // component

        $w :=$ oNodes.pop

        component$[w] := v$

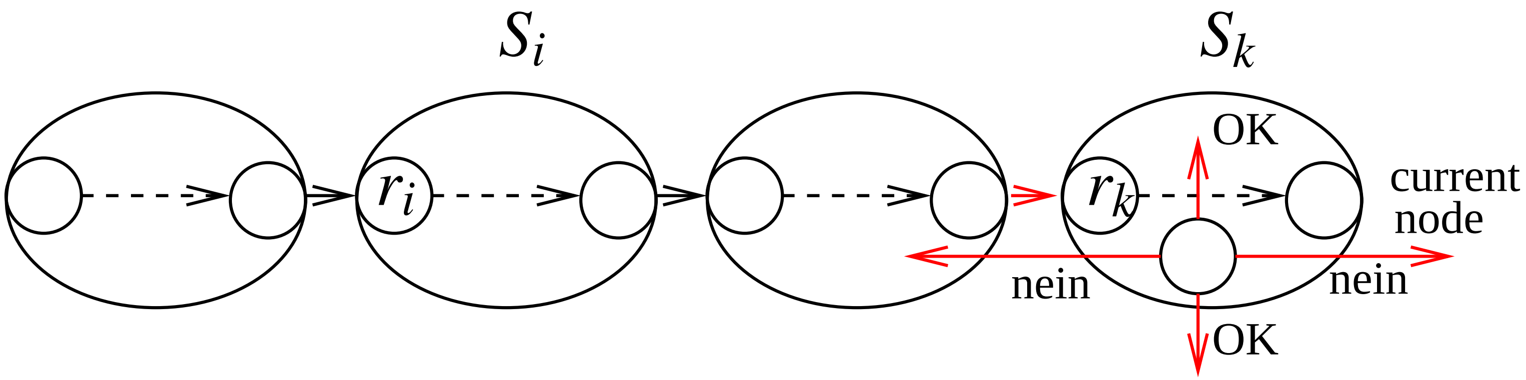
    until $w = v$



z.Z. Invarianten bleiben erhalten. . .

```
backtrack(u, v)
    if v = oReps.top then
        oReps.pop                                   // close
        repeat                                      // component
            w:= oNodes.pop
            component[w]:= v
        until w = v
```

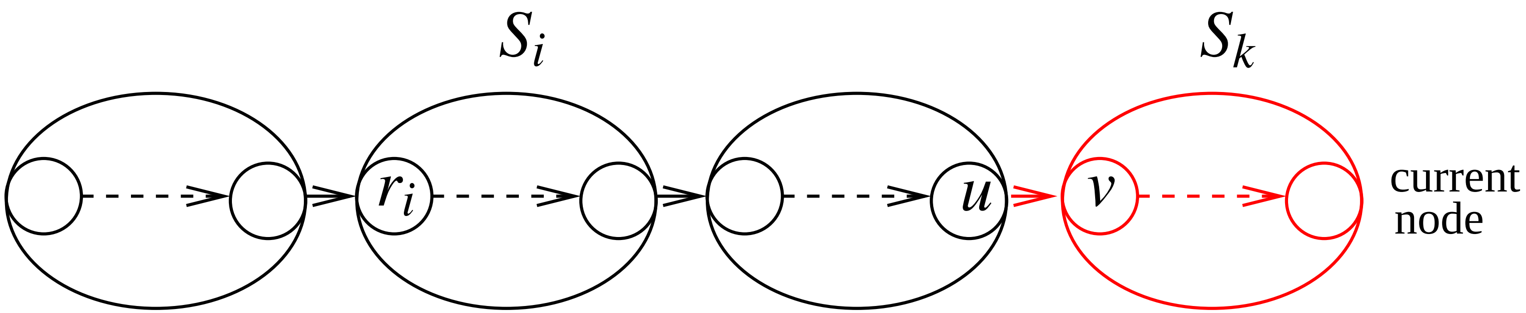
Inv. 1: Kanten von abgeschlossenen Knoten gehen zu abgeschlossenen Knoten.



```
backtrack(u, v)
    if v = oReps.top then
        oReps.pop                                   // close
        repeat                                      // component
            w:= oNodes.pop
            component[w]:= v
        until w = v
```

Inv. 2: Offene Komponenten $S_1, \ldots, S_k$ bilden Pfad in G_c^s

OK. (S_k wird ggf. entfernt)



backtrack(u, v)

    if $v =$ oReps.top **then**

        oReps.pop // close

        repeat // component

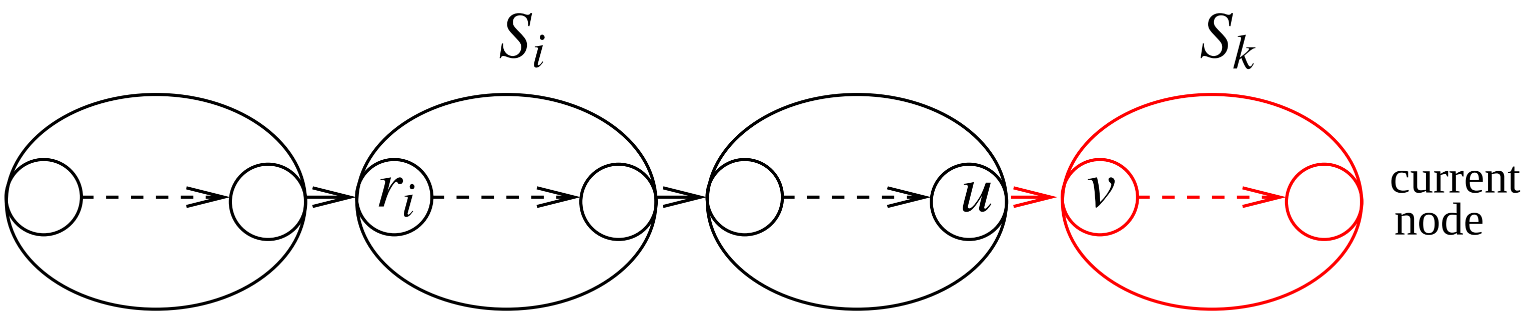
            w:= oNodes.pop

            component$[w]$:= v

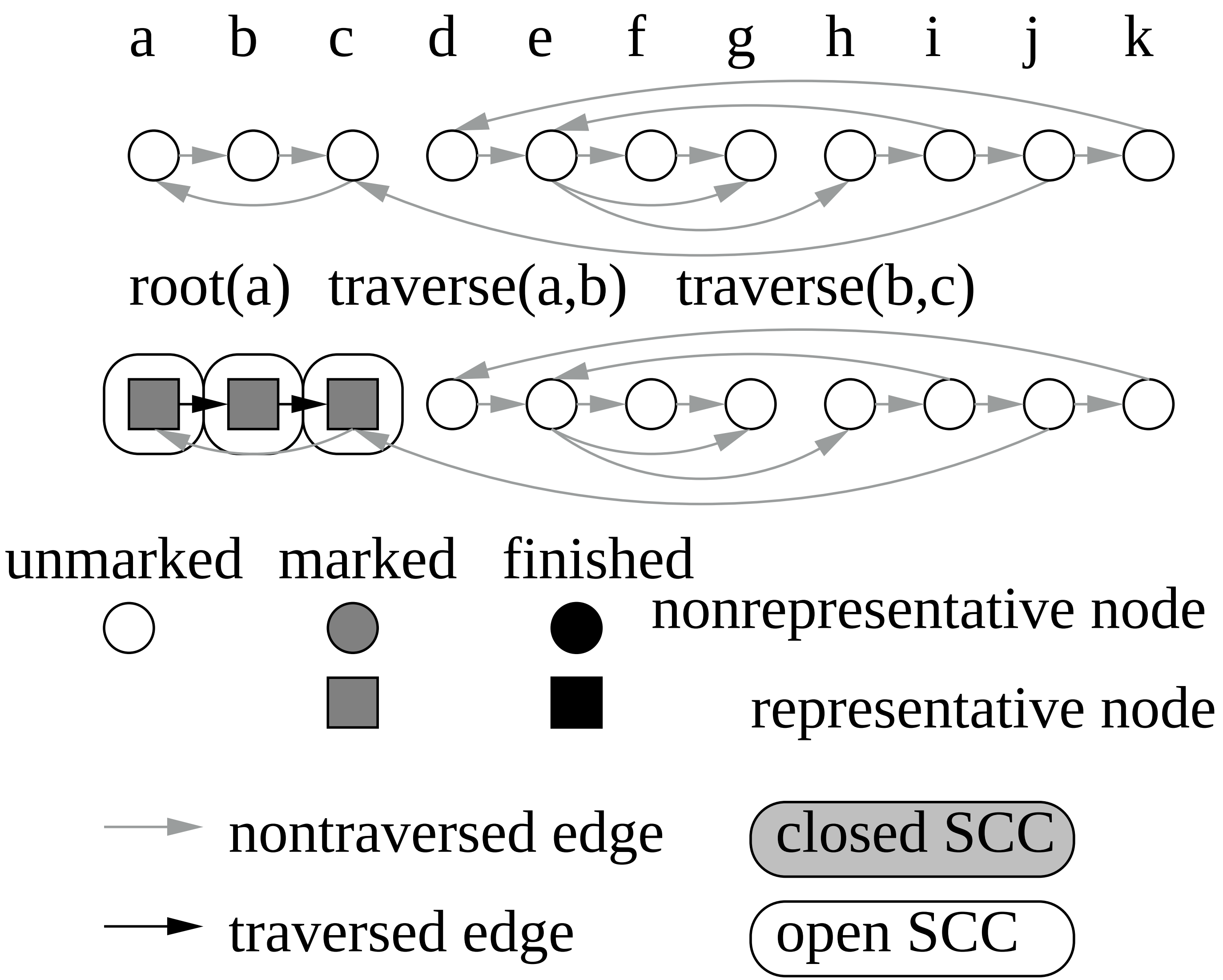
        until $w = v$

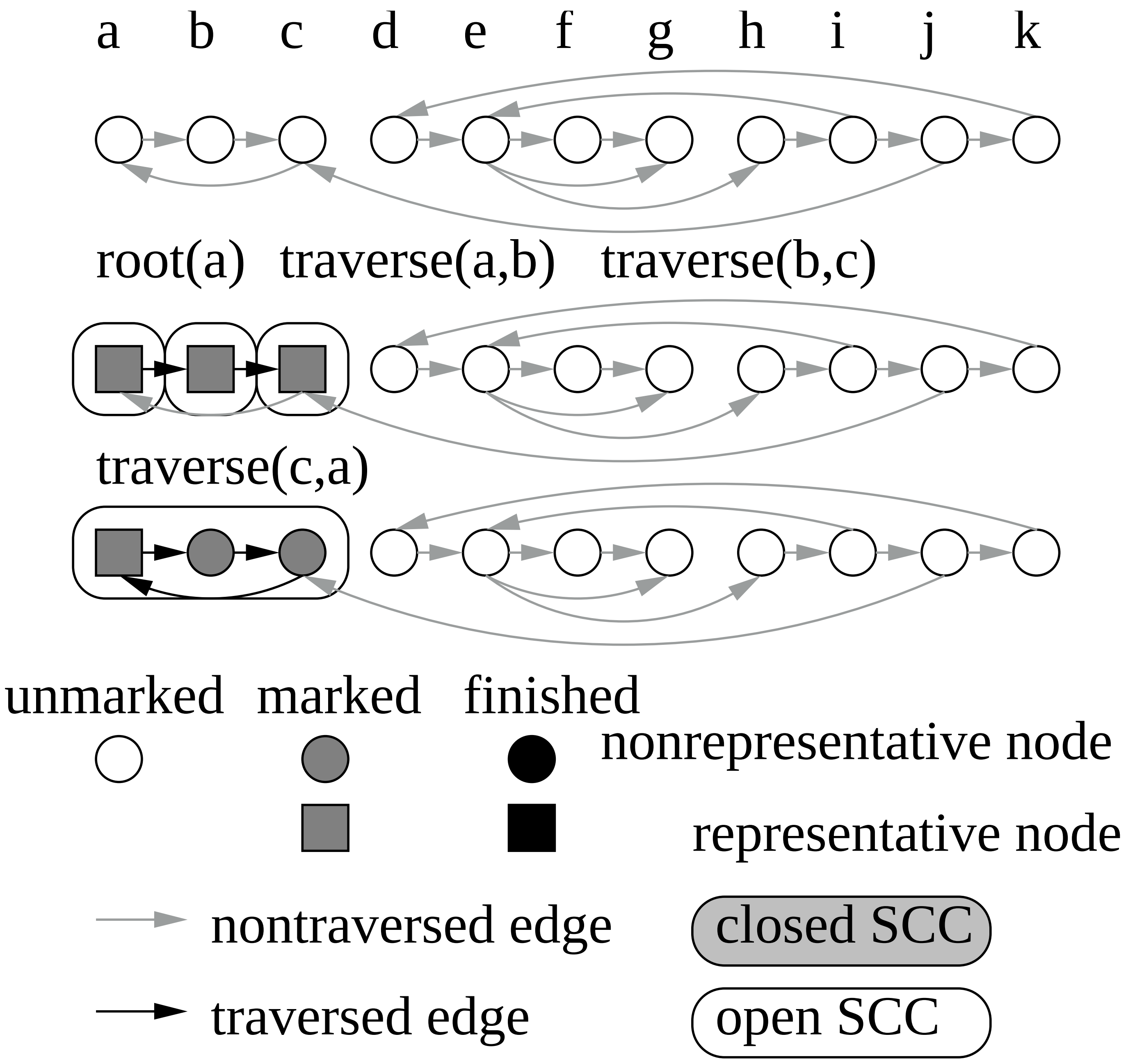
Inv. 3: Repräsentanten partitionieren die offenen Komponenten bzgl. ihrer dfsNum.

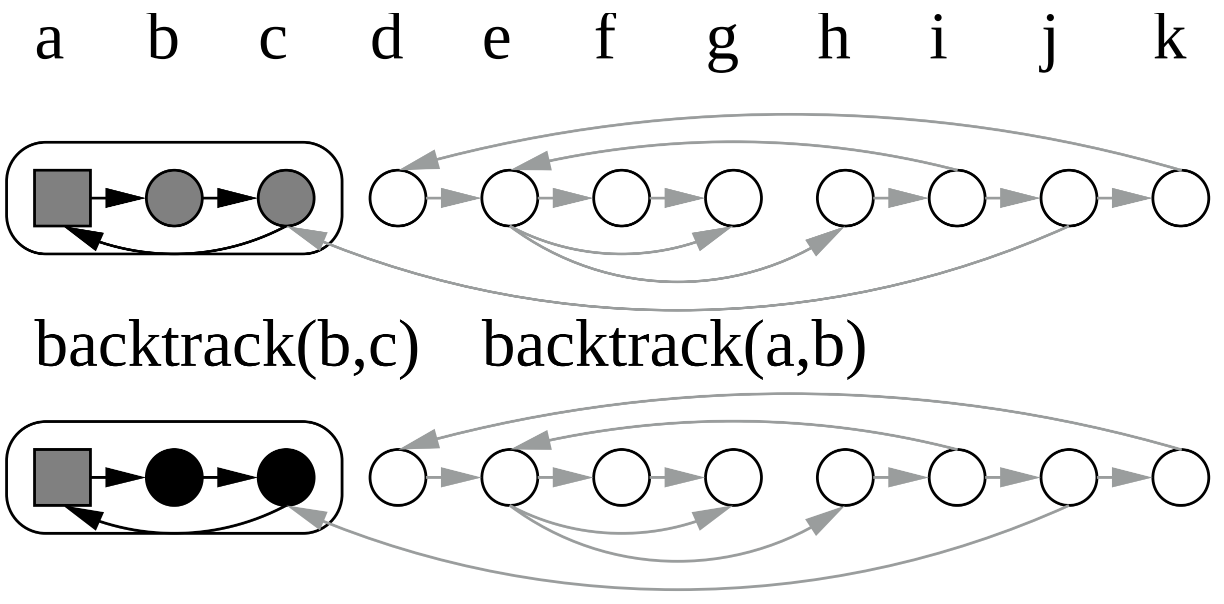
OK. (S_k wird ggf. entfernt)

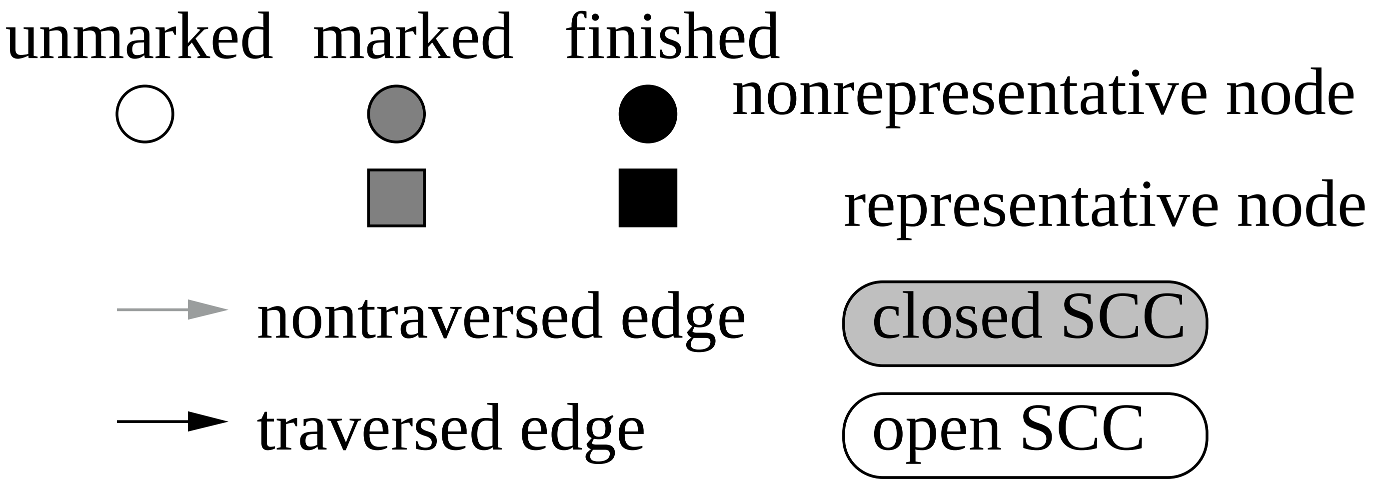


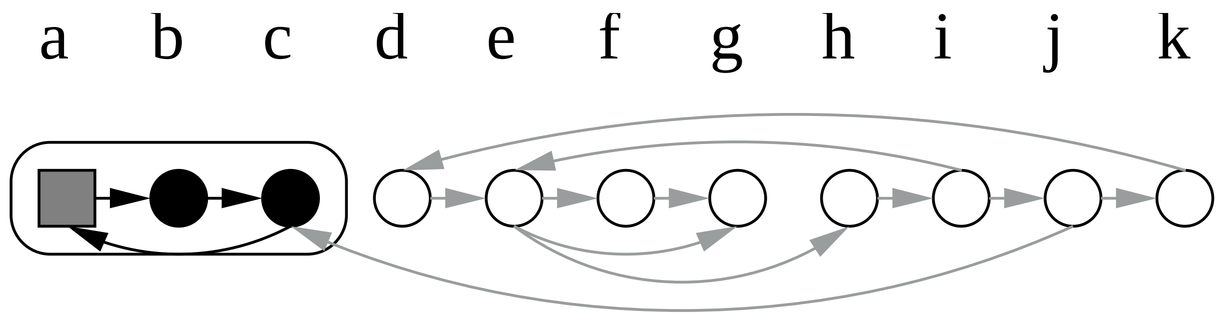
Beispiel


a b c d e f g h i j k
root(a) traverse(a,b) traverse(b,c)
unmarked
marked
finished
nonrepresentative node
representative node
nontraversed edge
traversed edge
closed SCC
open SCC

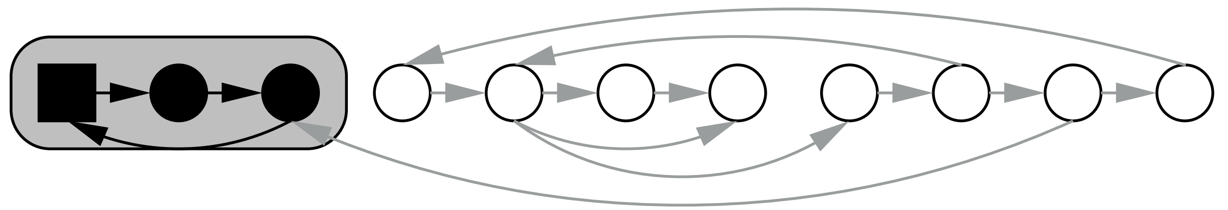

a b c d e f g h i j k
root(a) traverse(a,b) traverse(b,c)
traverse(c,a)
unmarked
marked
finished
nonrepresentative node
representative node
nontraversed edge
traversed edge
closed SCC
open SCC

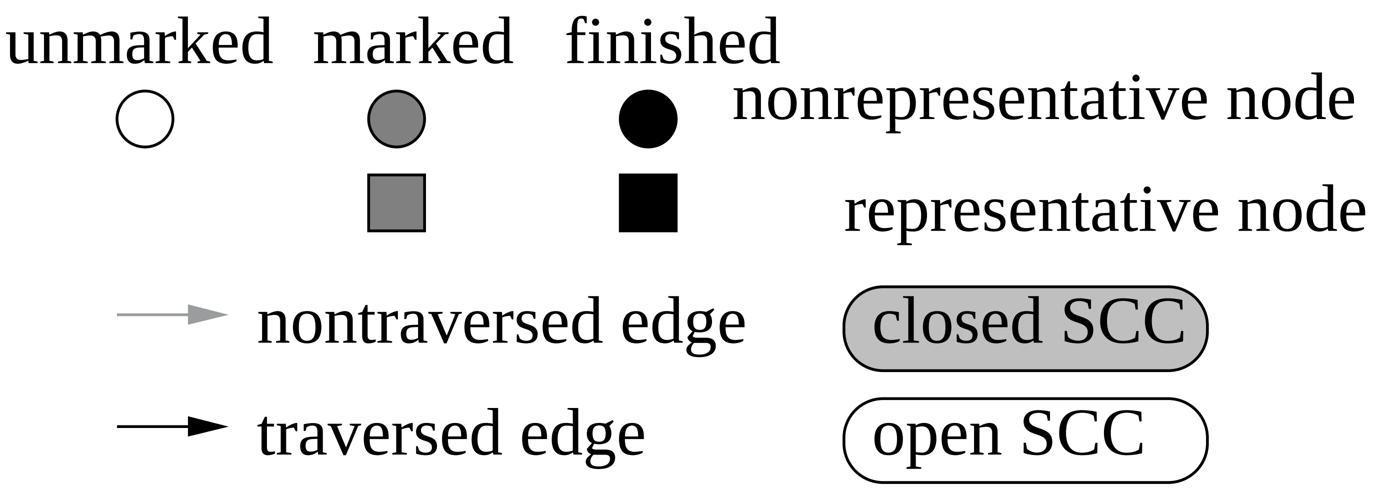

a b c d e f g h i j k
backtrack(b,c) backtrack(a,b)

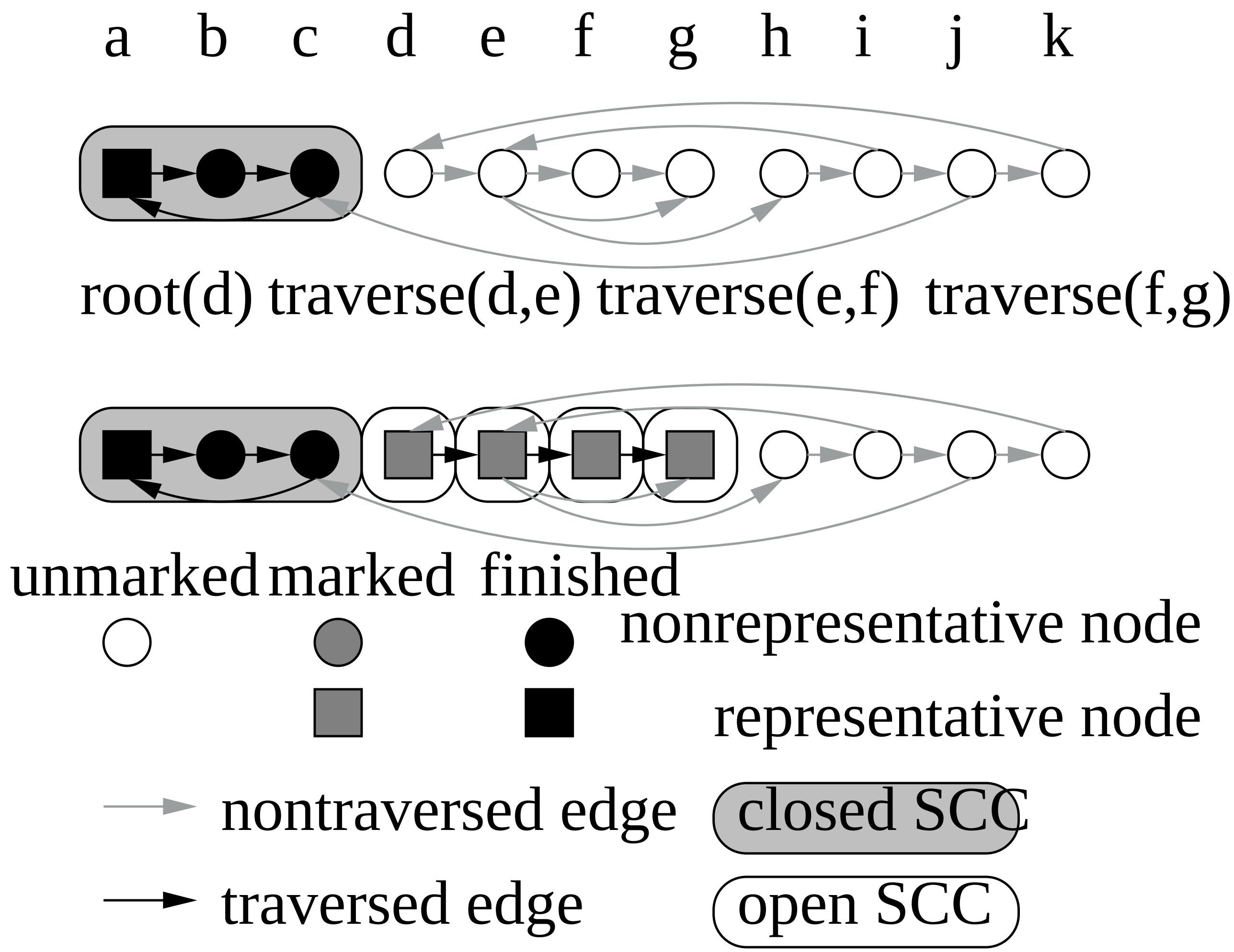

unmarked
marked
finished
nonrepresentative node
representative node
nontraversed edge
closed SCC
traversed edge
open SCC

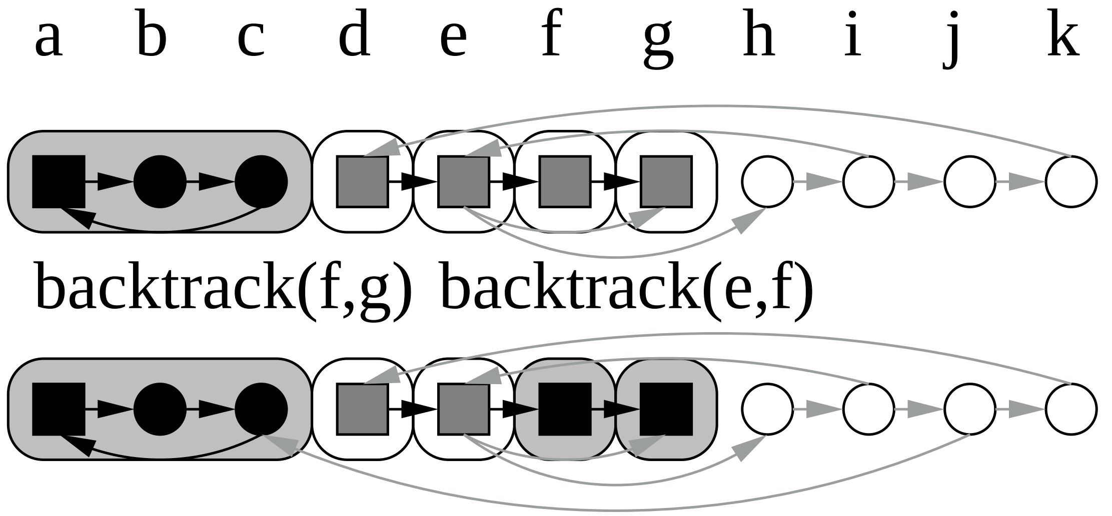


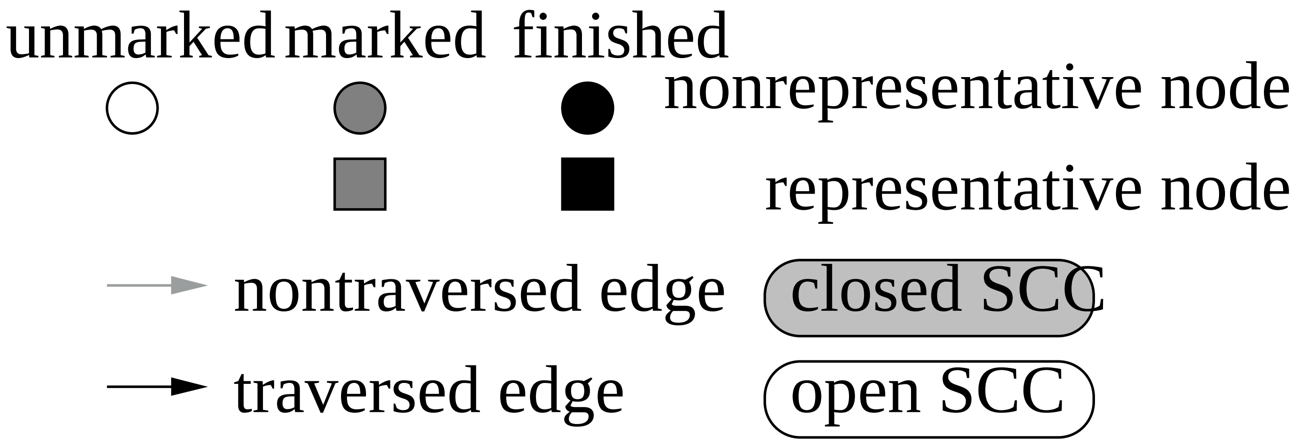
backtrack(a,a)

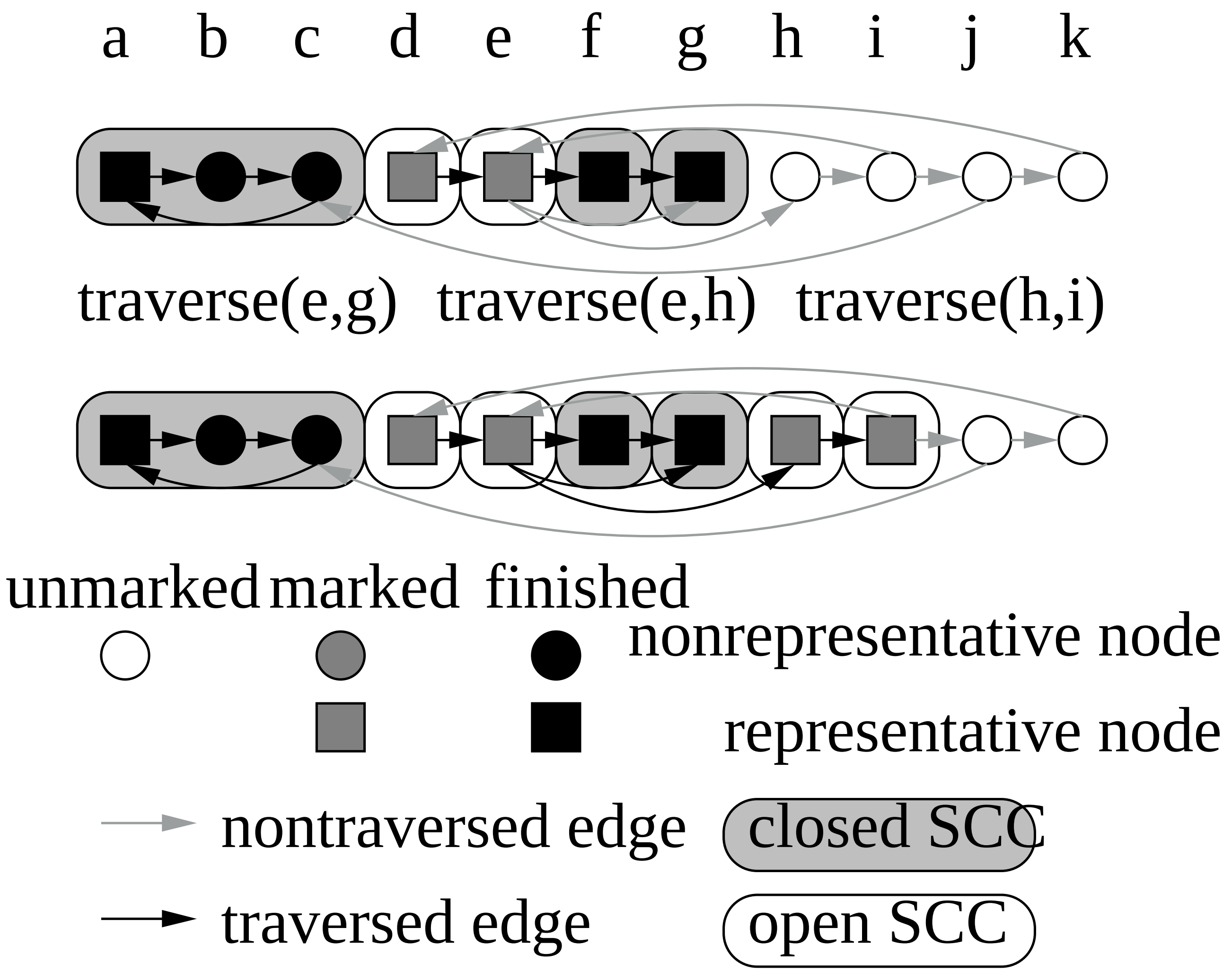


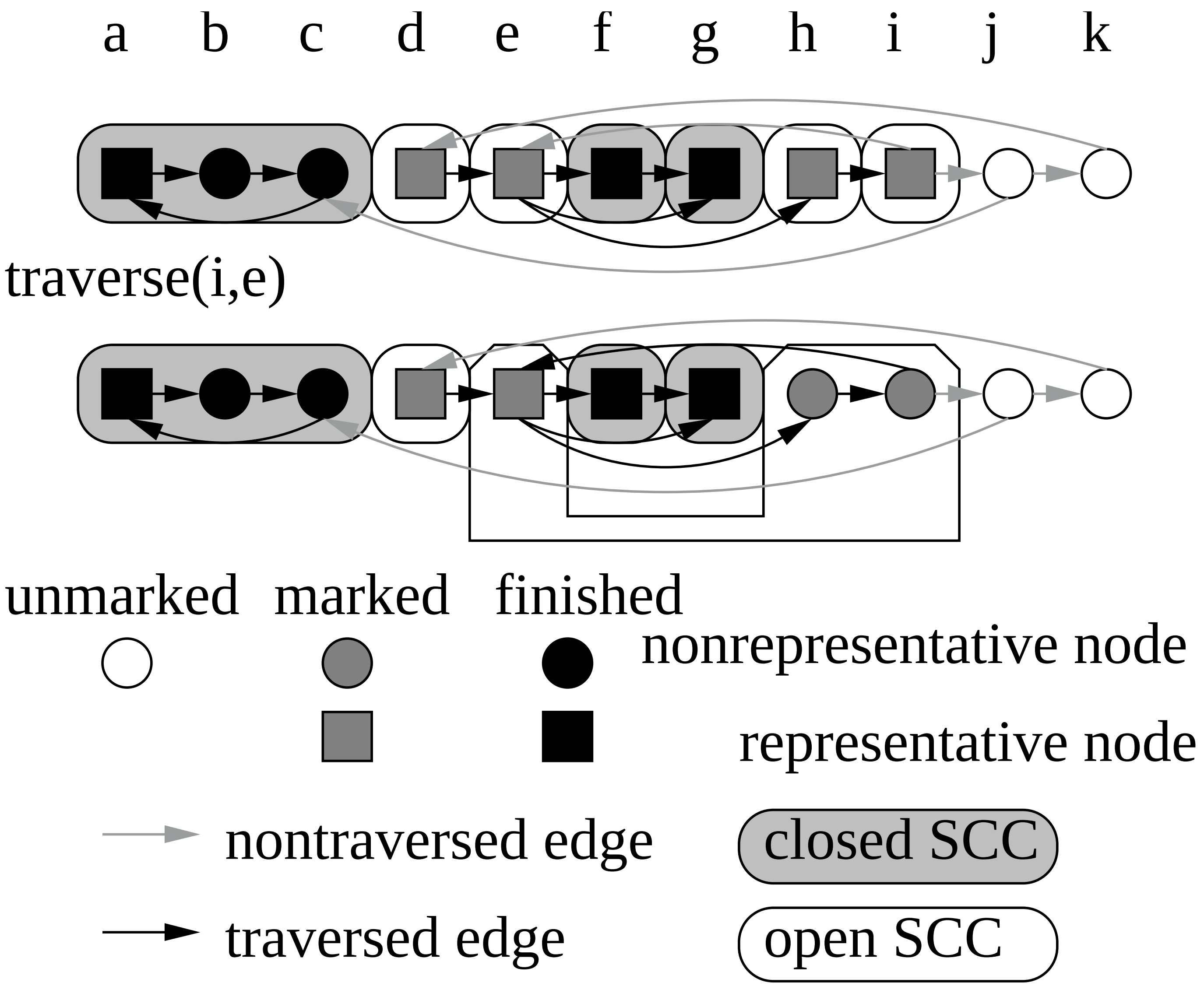


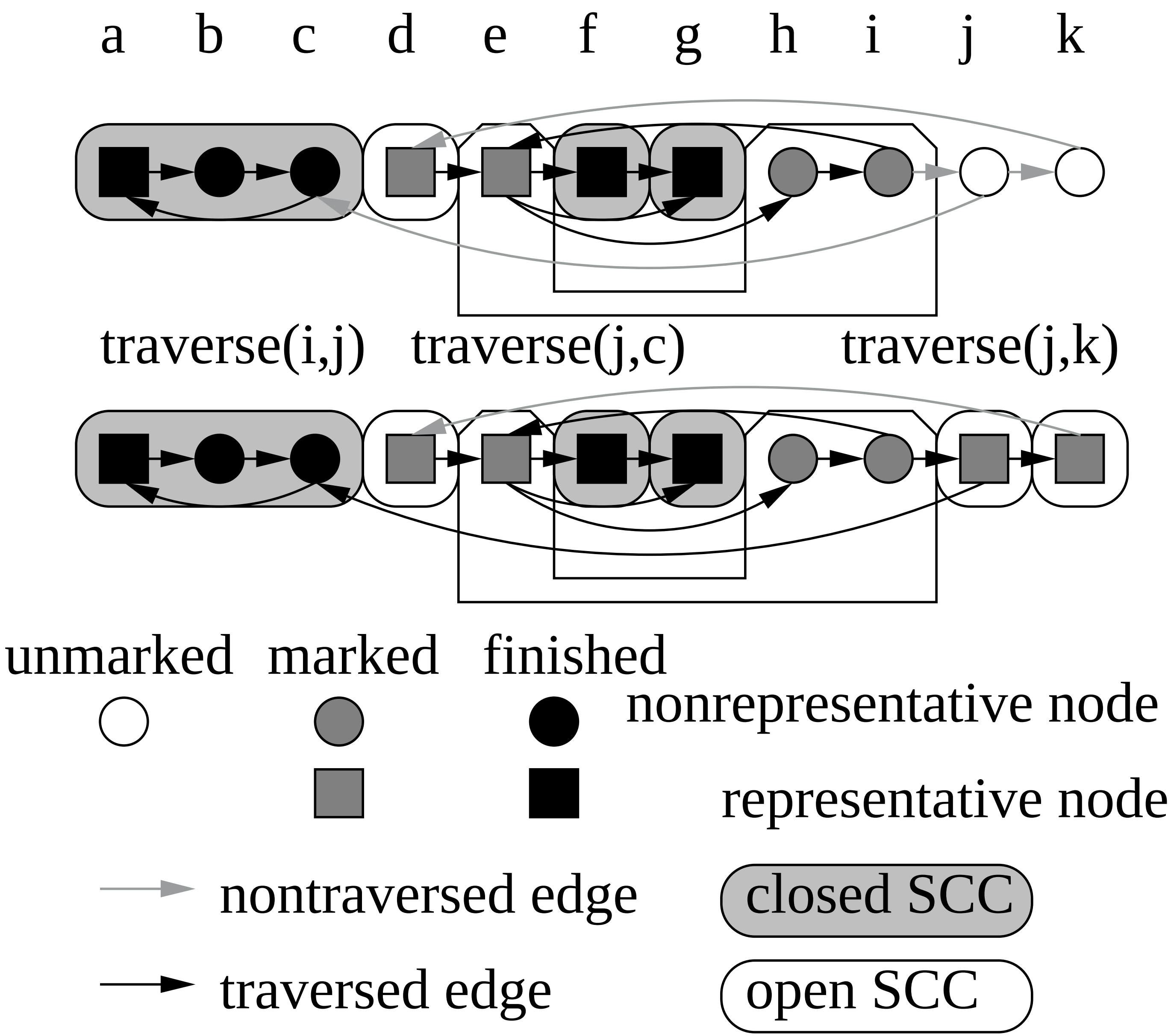

a b c d e f g h i j k
root(d) traverse(d,e) traverse(e,f) traverse(f,g)
unmarked
marked
finished
nonrepresentative node
representative node
nontraversed edge
traversed edge
closed SCC
open SCC

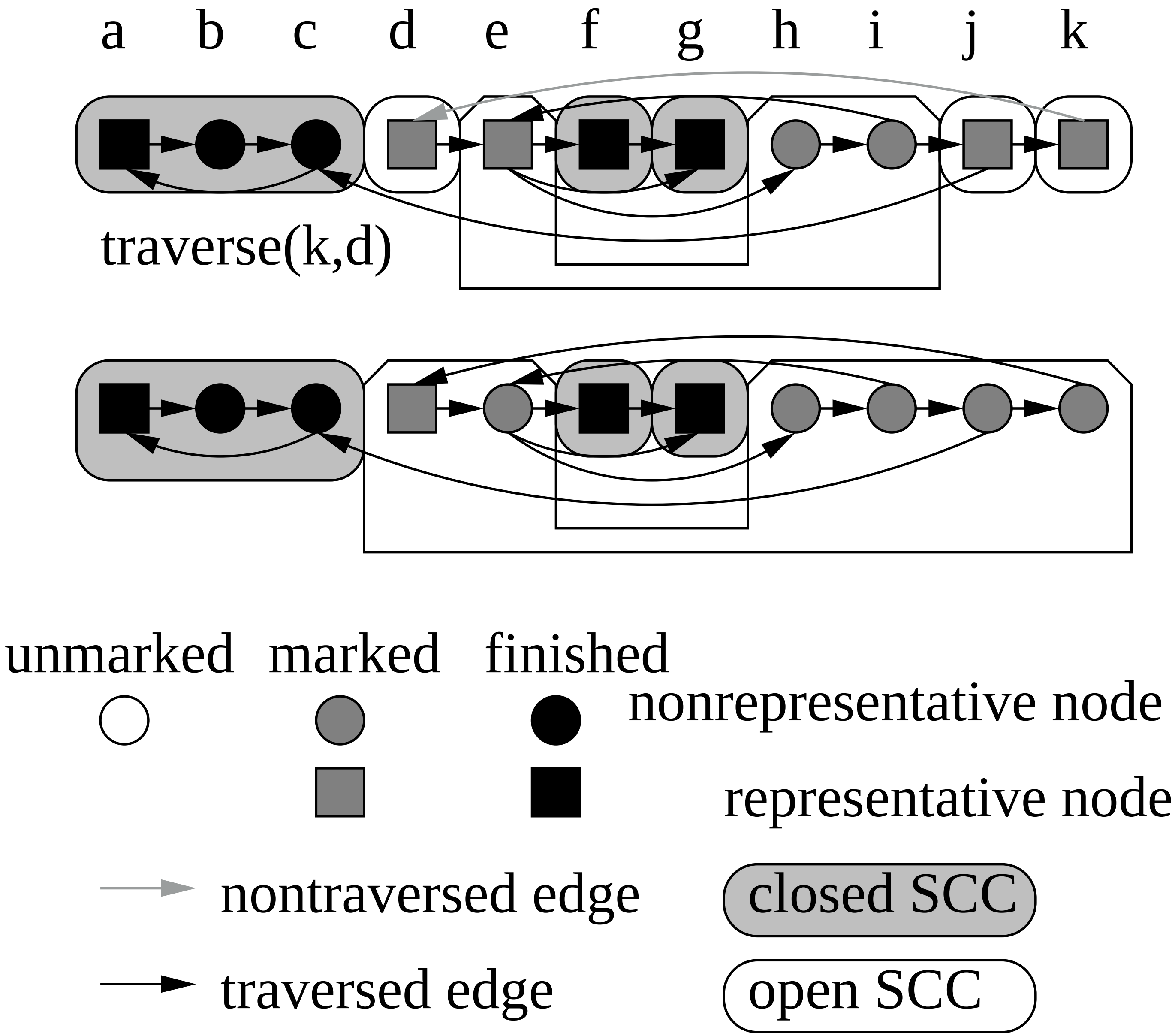

a b c d e f g h i j k
backtrack(f,g) backtrack(e,f)

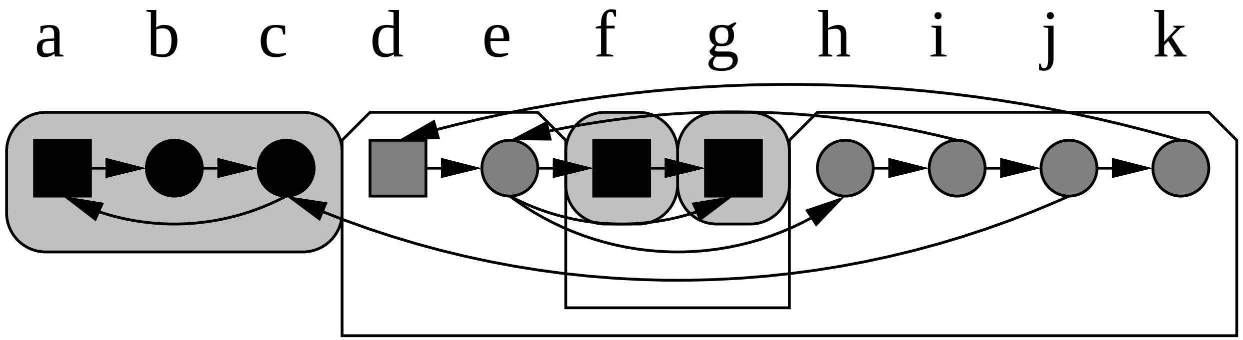

unmarked marked finished
nonrepresentative node
representative node
nontraversed edge
traversed edge
closed SCC
open SCC


a b c d e f g h i j k
traverse(e,g) traverse(e,h) traverse(h,i)
unmarked
marked
finished
nonrepresentative node
representative node
nontraversed edge
traversed edge
closed SCC
open SCC

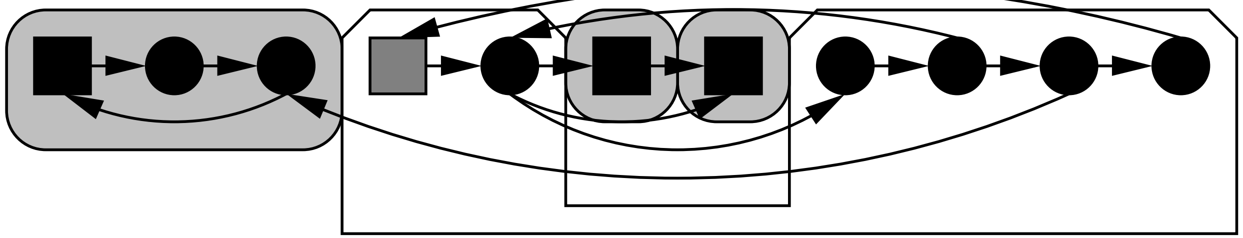

a
b
c
d
e
f
g
h
i
j
k
traverse(i,e)
unmarked
marked
finished
nonrepresentative node
representative node
nontraversed edge
traversed edge
closed SCC
open SCC

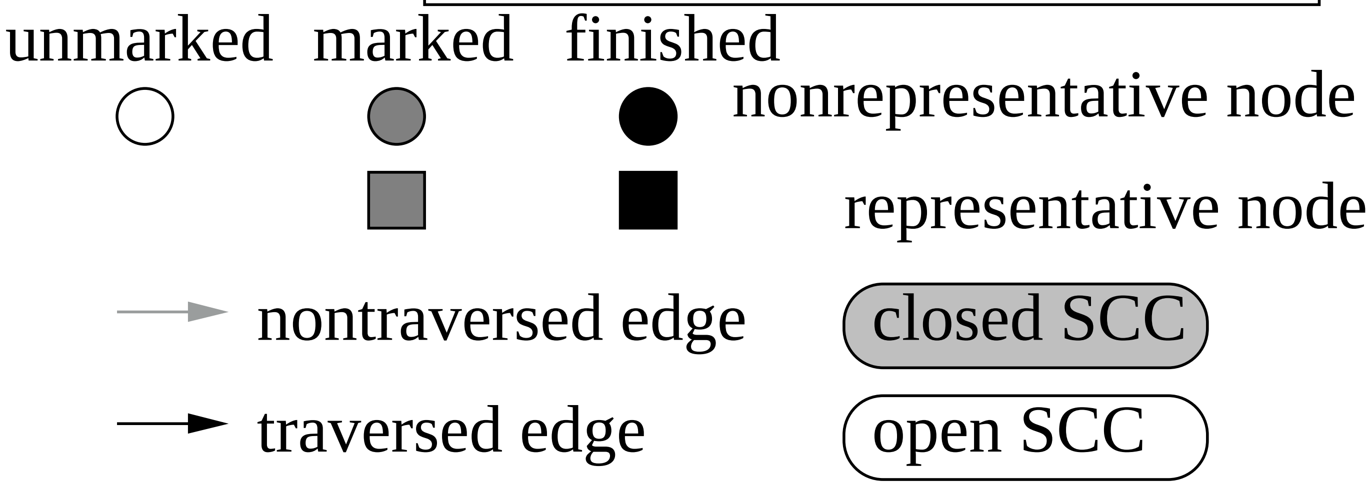

a b c d e f g h i j k
traverse(i,j)
traverse(j,c)
traverse(j,k)
unmarked
marked
finished
nonrepresentative node
representative node
nontraversed edge
traversed edge
closed SCC
open SCC

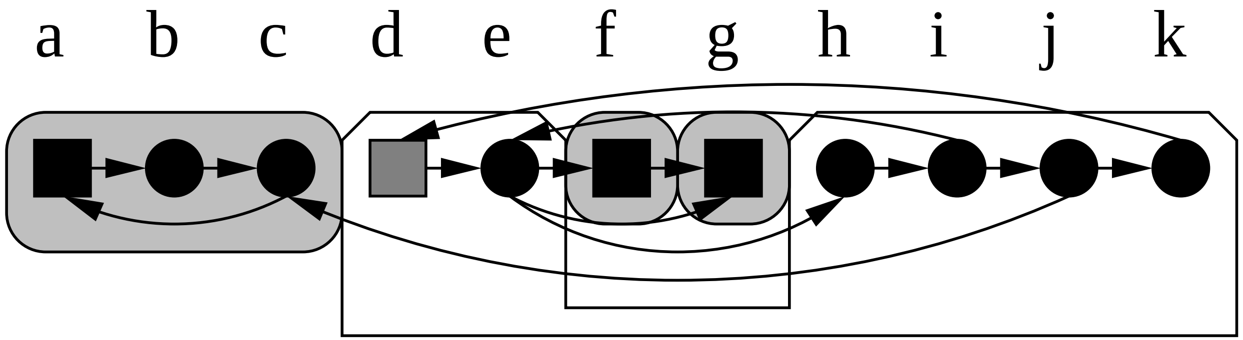

a
b
c
d
e
f
g
h
i
j
k
traverse(k,d)
unmarked
marked
finished
nonrepresentative node
representative node
nontraversed edge
traversed edge
closed SCC
open SCC



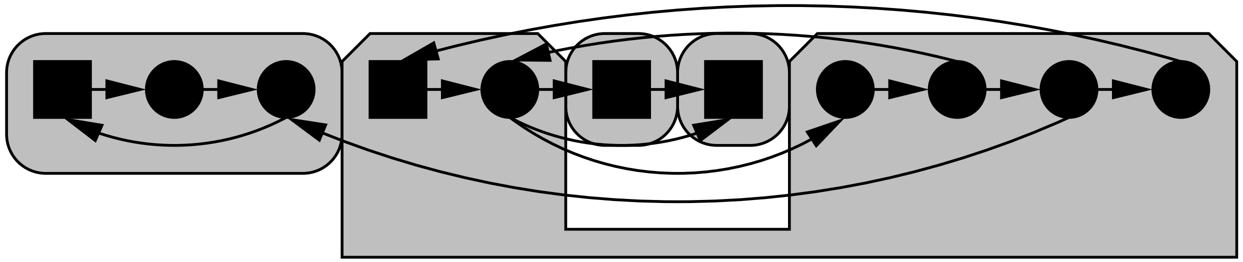
backtrack(j,k) backtrack(i,j) backtrack(h,i)
backtrack(e,h) backtrack(d,e)

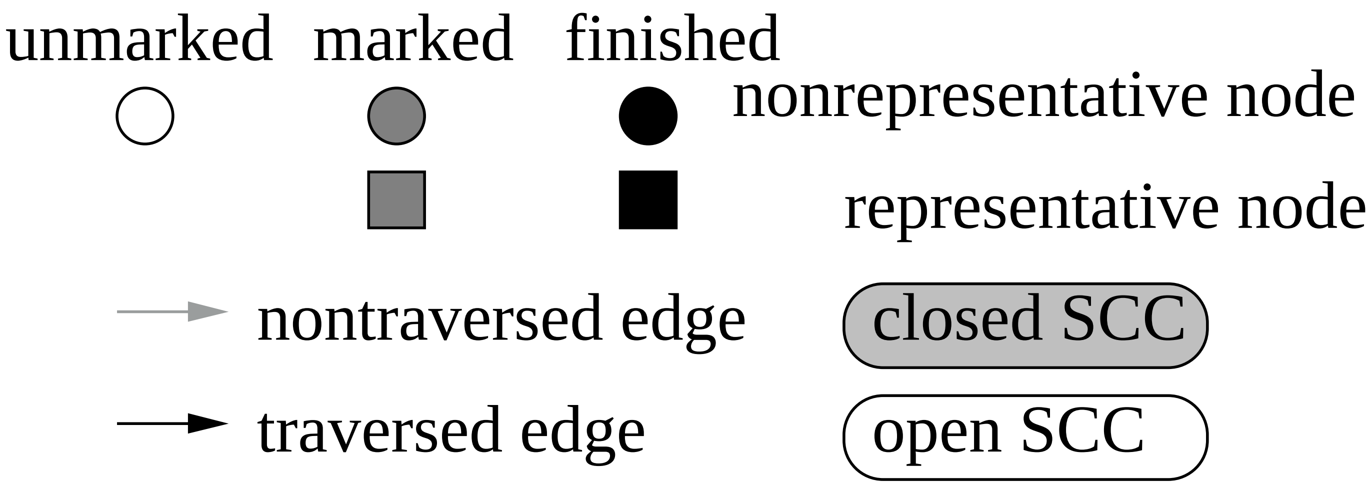






backtrack(d,d)





Zusammenfassung: SCC Berechnung

- Einfache Instantiierung des DFS-Musters
- Nichttrivialer Korrektheitsbeweis
- Laufzeit $\mathrm{O}(m+n)$: (Jeweils max. n push/pop Operationen)
- Ein einziger Durchlauf

Implementierungsdetails:

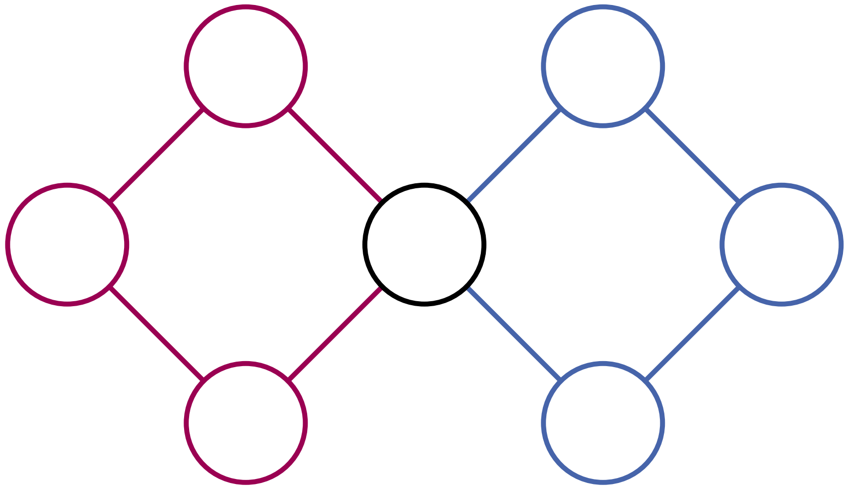
Mehlhorn, Näher, Sanders

Engineering DFS-Based Graph Algorithms

arxiv.org/abs/1703.10023

2-zusammenhängende Komponenten (ungerichtet)

Bei entfernen eines Knotens bleibt die Komponente zusammenhängend.
(Partitionierung der Kanten)



Geht in Zeit $\mathrm{O}(m+n)$ mit Algorithmus ähnlich zu SCC-Algorithmus

Karlsruhe Institute of Technology

Mehr DFS-basierte Linearzeitalgorithmen

- 3-zusammenhängende Komponenten
- Planaritätstest
- Einbettung planarer Graphen