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Algorithmen / Algorithms II

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Exercise:

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Web:

`http://algo2.iti.kit.edu/AlgorithmenII_WS20.php`

2 Advanced Data Structures

Here using the example of priority queues.

further examples:

- ☐ Monotone integer priority queues — chapter: shortest paths
- ☐ Perfect Hashing — chapter: randomized algorithms
- ☐ search trees with advanced operations — see book
- ☐ External priority queues — chapter: external algorithms
- ☐ Geometric data structures — chapter: geom. algorithms

2.1 Addressable Priority Queues – Operations

Procedure build$(\{e_1,\ldots,e_n\})$ $M := \{e_1,\ldots,e_n\}$

Function size **return** $|M|$

Procedure insert(e) $M := M \cup \{e\}$

Function min **return** $\min M$

Function deleteMin $e := \min M$; $M := M \setminus \{e\}$; **return** e

Function remove$(h : \text{Handle})$ $e := h$; $M := M \setminus \{e\}$; **return** e

Procedure decreaseKey$(h : \text{Handle}, k : \text{Key})$ **assert** key$(h) \geq k$; key$(h) := k$

Procedure merge(M') $M := M \cup M'$

Addressable Priority queues – Use Cases

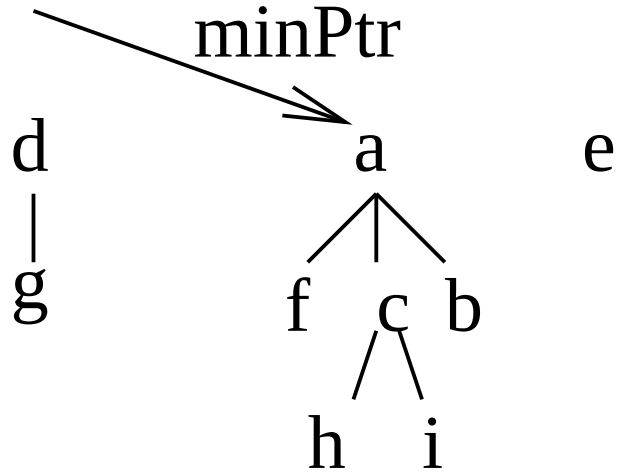
- Dijkstra's algorithm shortest paths
- Jarník-Prim algorithm for minimum spanning trees
- Here: hierarchy construction for route planning
- Here: graph partitioning
- Here: disk scheduling

In general:

Greedy algorithms, where priorities change (within limits).

Basic Data Structure

A forest of heap-ordered trees

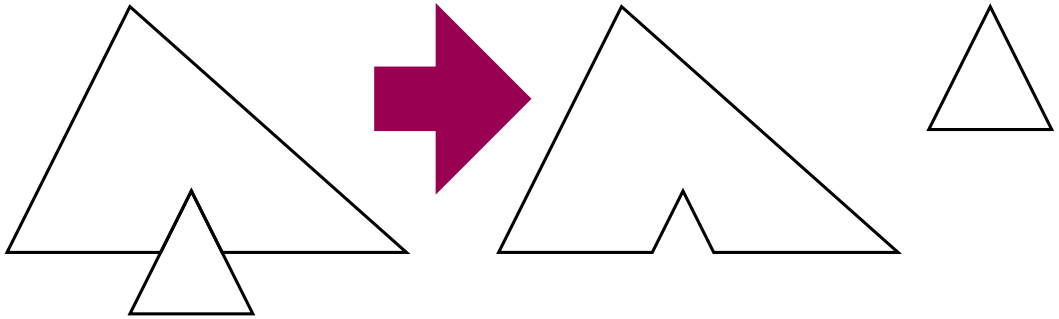


Generalization of binary heaps:

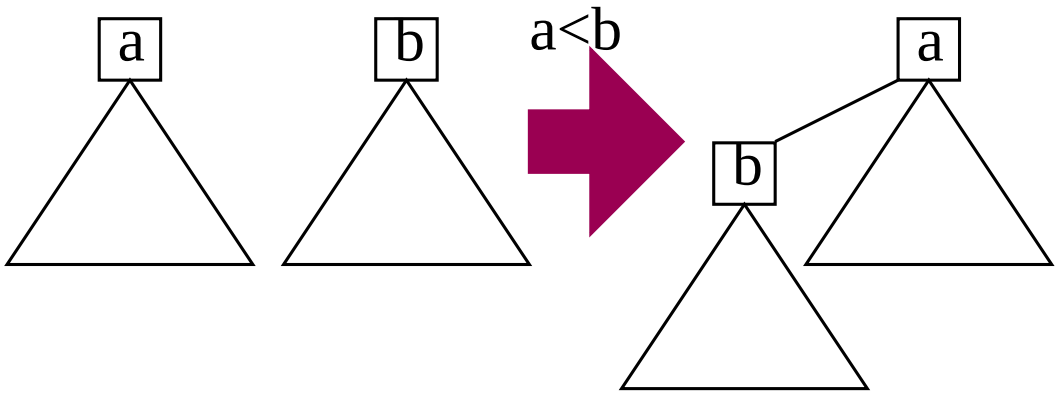
- Tree $\rightarrow$ forest
- Binary $\rightarrow$ arbitrary node-degrees

Processing Forests

Cut:



Link:



union(a,b): link$(\min(a,b),\max(a,b))$

Pairing Heaps

[Fredman Sedgewick Sleator Tarjan 1986]

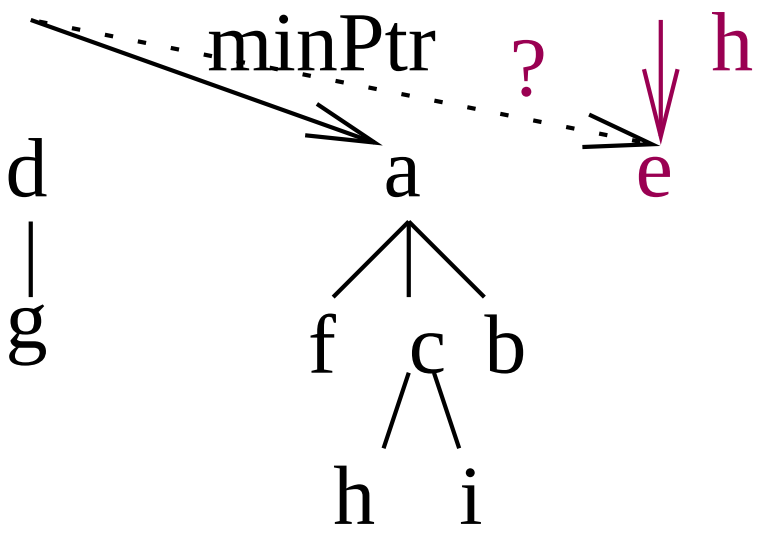
Procedure insertItem$(h$: Handle$)$

newTree(h)

Procedure newTree$(h$: Handle$)$

forest:= forest$\cup\{h\}$

if $*h <$ min **then** minPtr:= h



Attention: Simple implementation from the 1. English edition.

Further editions differ (e.g. the German edition).

Pairing Heaps

Procedure decreaseKey(h : Handle, k : Key)

key(h):= k

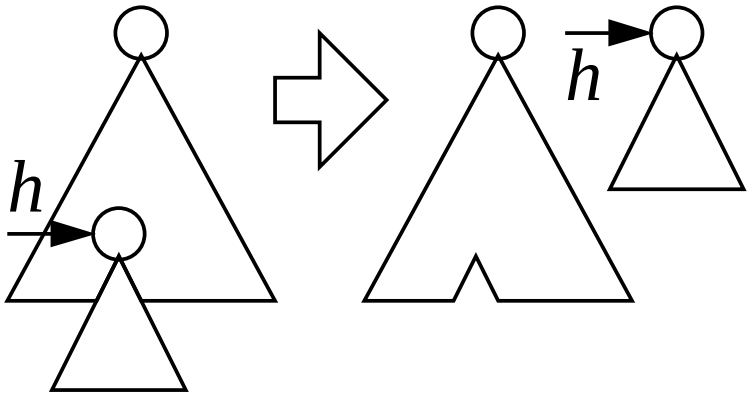
if h is not a root **then** cut(h)

else update minPtr

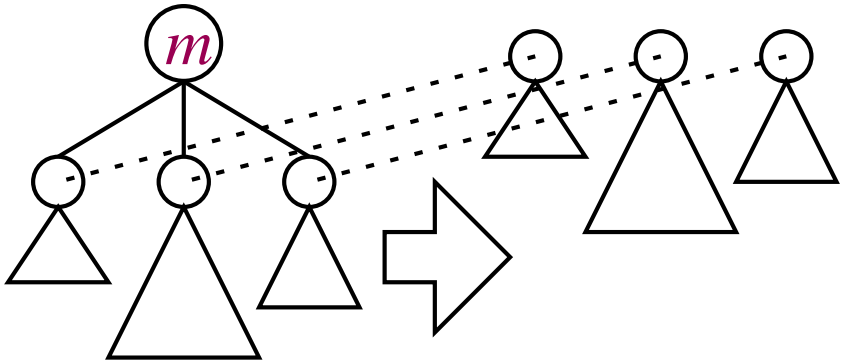
Procedure cut(h : Handle)

remove the subtree rooted at h

newTree(h)



Pairing Heaps



Function deleteMin : Handle

m:= minPtr

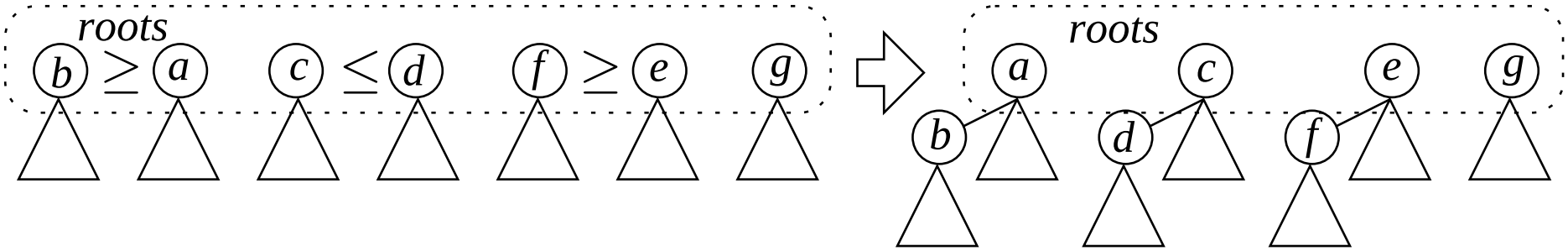
forest:= forest $\setminus \{m\}$

foreach child h of m **do** newTree(h)

perform pair-wise union operations on the roots in forest

update minPtr

return m



Pairing Heaps

Procedure merge(o : AdressablePQ)

if *minPtr $>$ $*$(o.minPtr) **then** minPtr:= o.minPtr

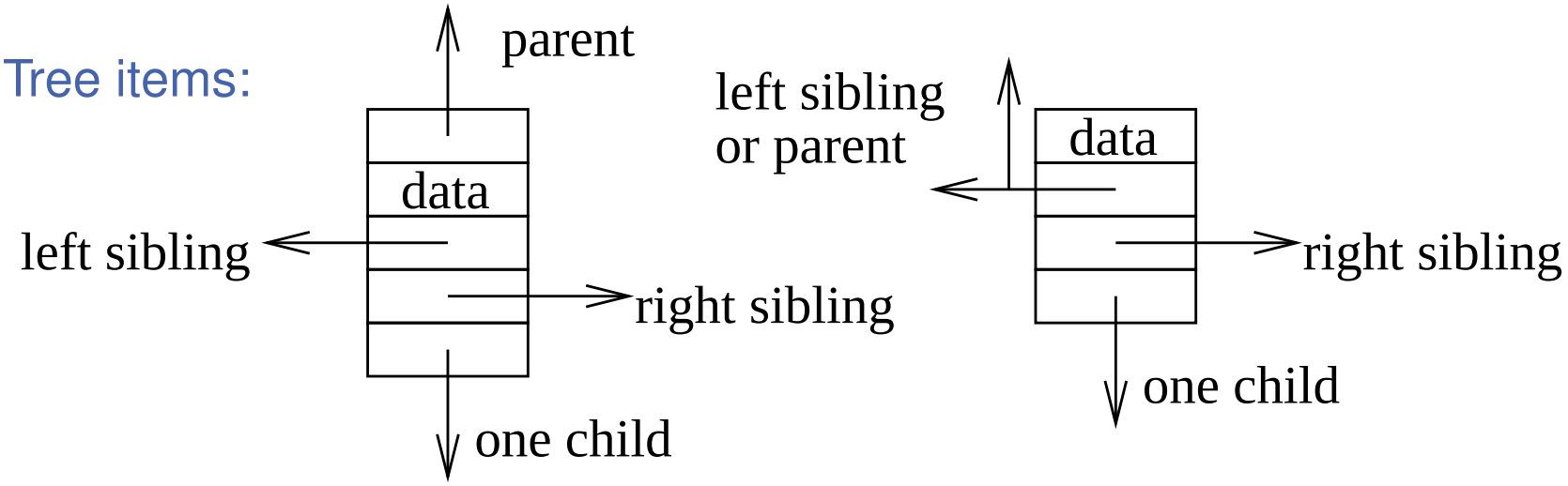
forest:= forest $\cup$ o.forest

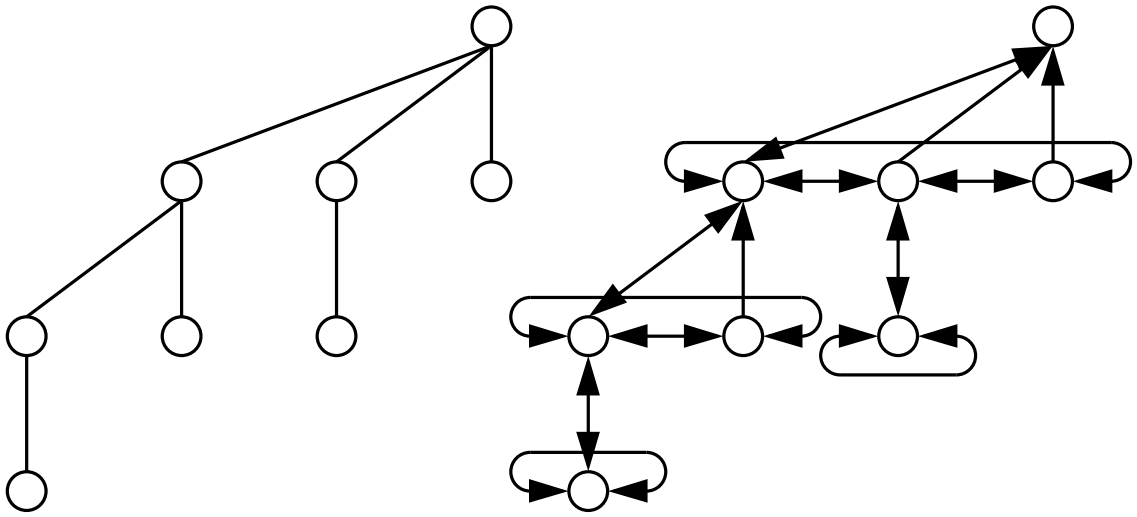
o.forest:= $\emptyset$

Pairing Heaps – Representation

Roots: in a doubly linked list

Tree items:





Pairing Heaps – Analysis

insert, merge: $O(1)$

deleteMin, remove: $O(\log n)$ amortized

decreaseKey: unknown! $O(\log\log n) \leq T \leq O(\log n)$ amortized,

but fast in practice.

Proofs: not here.

Fibonacci Heaps [Fredman Tarjan 1987]

Rank: Save the number of (immediate) children.

Union-by-rank: Only call union on roots with the same rank.

Mark: Mark nodes that have lost a child.

Cascading cuts: Cut at marked nodes

(i.e. nodes that have lost two children)

Theorem: Amortised complexity $O(\log n)$ for deleteMin and remove $O(1)$ for all other operations.

(i.e. *total time* $= O(o + d \log n)$ if

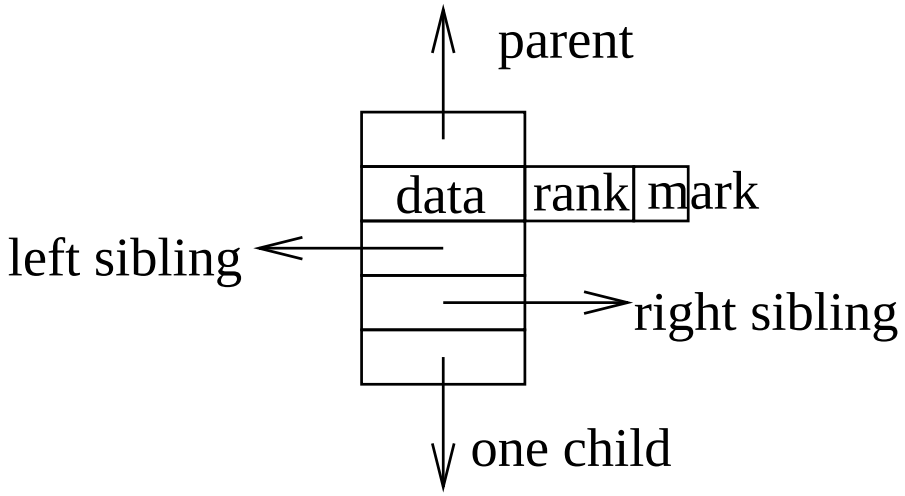
$d =$#deleteMin, $o =$#otherOps, $n = \max |M|$)

Fibonacci Heaps – Representation

Roots: in a doubly linked list

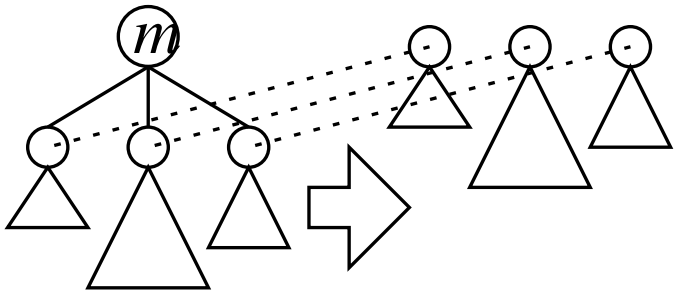
(and a temporary array for deleteMin)

Rree items:



insert, merge: as before, in time $O(1)$

deleteMin with Union-by-Rank



Function deleteMin : Handle

m:= minPtr

forest:= forest $\setminus \{m\}$

foreach child h of m **do** newTree(h)

while $\exists a, b \in$ forest : rank$(a) =$ rank(b) **do**

union(a, b) // increments rank of surviving root

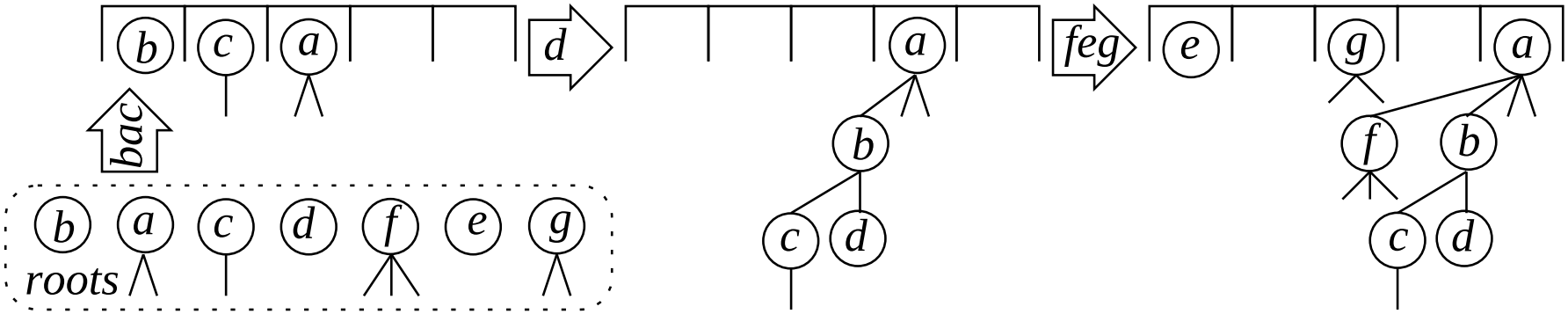
update minPtr

return m

Fast Union-by-Rank

An array that is addressed by the rank.

Execute link until a free entry is found.



Analysis: Time $O(\#\text{unions} + |\text{forest}|)$

Amortised Analysis for deleteMin

$$\text{maxRank} := \max_{a \in \text{forest}} \text{rank}(a) \text{ (after)}$$

Lemma: $T_{\text{deleteMin}} = O(\text{maxRank})$

Proof: Using the accounting method. One token per Root

$\text{rank}(\text{minPtr}) \leq \text{maxRank}$

$\rightsquigarrow$ costs of $O(\text{maxRank})$ for newTrees and a new token.

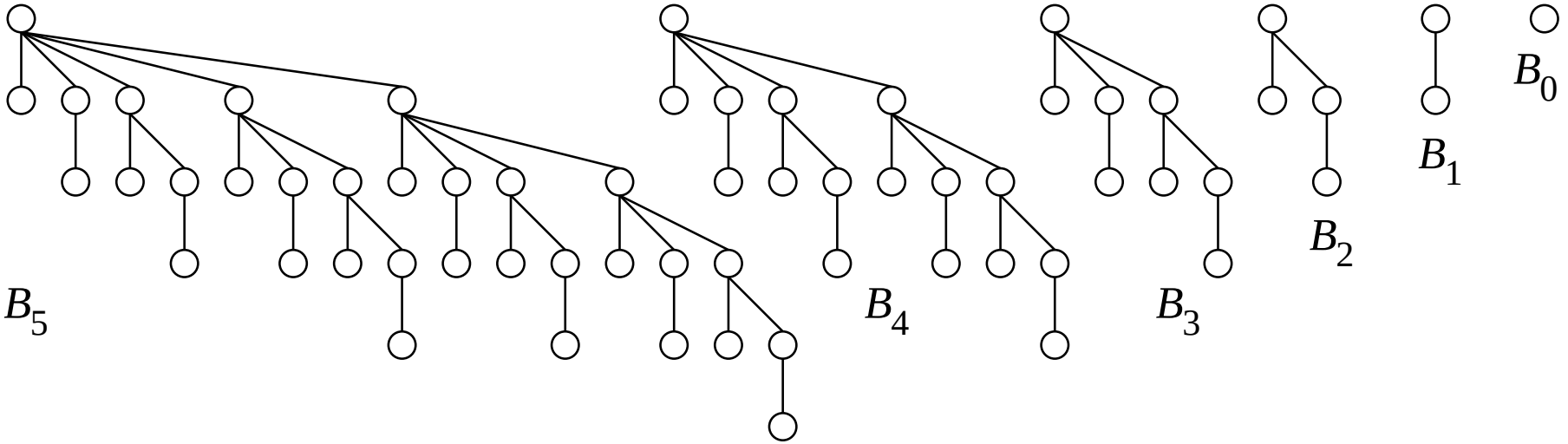
Union-by-rank: token pays for

- union operations (a token becomes free) and
- iterating through roots (old and new).

At the end there are $\leq$ maxRank roots.

Why is maxRank logarithmic? – Binomial Trees

$2^k + 1 \times$ insert, $1 \times$ deleteMin $\rightsquigarrow$ rank k



[Vuillemin 1978] PQ (only) with binomial trees, $T_{\text{decreaseKey}} = \mathrm{O}(\log n)$.

Problem: Cuts can lead to high ranking trees.

Cascading Cuts

Procedure decreaseKey(h : Handle, k : Key)

key(h):= k

cascadingCut(h)

Procedure cascadingCut(h)

if h is not a root **then**

p:= parent(h)

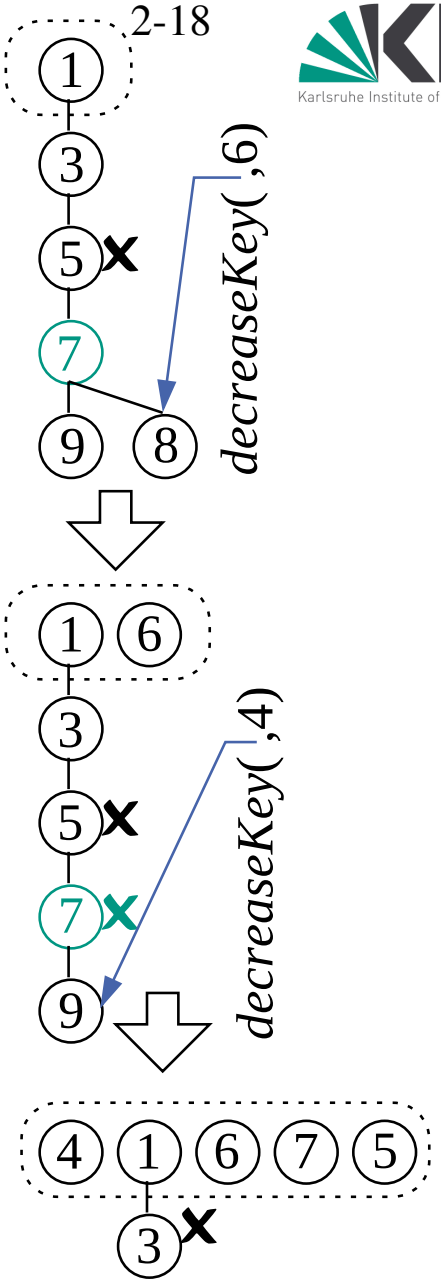
unmark h

cut(h)

if p is marked **then**

cascadingCut(p)

else mark p



We will show: cascading cuts keep maxRank logarithmic

Lemma: decreaseKey has amortised complexity $O(1)$.

Accounting Method: ($\approx$ 1 Token per cut or union)

1 token per root

2 tokens for every marked node

Looking at decreaseKey with k consecutive marked predecessors:

2k token becomes free (nodes become unmarked)

2 token needed for new marks

k+1 tokens needed for the new roots

k+1 tokens pay for the cuts

Thus, there remains a cost of 4 tokens $+O(1)$ time for decreaseKey

Here is where Mr. Fibonacci comes in.

$$F_i := \begin{cases} 0 & \text{für i=0} \\ 1 & \text{für i=1} \\ F_{i-2} + F_{i-1} & \text{else} \end{cases}$$

Known: $F_{i+1} \geq ((1+\sqrt{5})/2)^i \geq 1.618^i$ for all $i \geq 0$.

We show:

A subtree with root v and $\text{rank}(v) = i$ contains $\geq F_{i+2}$ elements.

$\Rightarrow$

logarithmic time for deleteMin.

Proof:

Looking at the moment when the j-th child w_j of v was added:

w_j and v had the same rank $\geq j-1$ (v already had $j-1$ children)

rank(w_j) was reduced by at most one (cascading cuts)

$\Rightarrow$ rank$(w_j) \geq j-2$ and rank$(v) \geq j-1$

S_i:= lower bound for the # of nodes whose root has rank i:

$S_0 = 1$

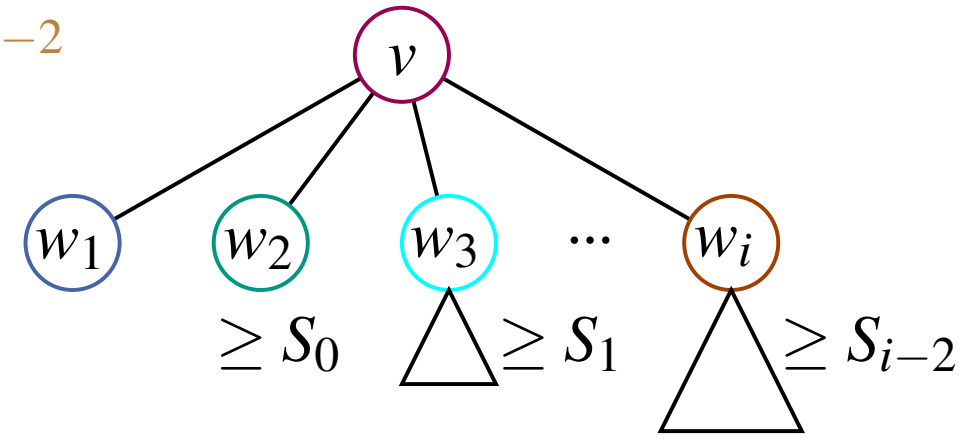
$S_1 = 2$

$S_i \geq 1 + 1 + S_0 + S_1 + \cdots + S_{i-2}$

for $i \geq 2$

This recurrence

has the solution $S_i \geq F_{i+2}$



Addressable Priority Queues – More

- Lower bound $\Omega(\log n)$ for deleteMin (comparison based) Proof: exercise
- Worst case Bounds: not here
- Monotone PQs with integer keys (stay tuned)

Open Problems:

Analysis of pairing heaps (simplification of Fibonacci Heaps)

Recap Data Structures

- In this lecture, we focused on the example of priority queues (see shortest path algorithms and external algorithms).
- Heap concept can take you far.
- Sibling-pointers can be used to represent arbitrary trees with a constant number of pointers per item.
- Fibonacci heaps – a non-trivial example for amortised analysis