Algorithmen / Algorithms II

Peter Sanders

Exercise:

Daniel Seemaier, Tobias Heuer

Institute of Theoretical Informatics

Web:

`http://algo2.iti.kit.edu/AlgorithmenII_WS20.php`

6 Randomised Algorithms

Using random (bits) to accelerate/simplify algorithms

Las Vegas: Guarantee a correct result – running time is a random variable

already known:

- quicksort
- hashing

Monte Carlo: The result is incorrect with a failure probability p

Repeating the algorithm k times decreases the failure-probability exponentially (p^k).

Further details in “Randomised Algorithms” by Thomas Worsch

6.1 Sorting – (Result-)Checking

Permutation-Property (sortedness: is trivial)

$\langle e_1, \ldots, e_n \rangle$ is a permutation of $\langle e'_1, \ldots, e'_n \rangle$ exactly when

$$q(z) := \prod_{i=1}^{n}(z - \mathrm{field}(\mathrm{key}(e_i))) - \prod_{i=1}^{n}(z - \mathrm{field}(\mathrm{key}(e'_i))) = 0,$$

Let $\mathbb{F}$ be a field, and map : Key $\rightarrow \mathbb{F}$ is injective.

Observation: q has at most n zeros (roots).

Evaluating q at random position $x \in \mathbb{F}$.

$$\mathbb{P}[q \neq 0 \wedge q(x) = 0] \leq \frac{n}{|\mathbb{F}|}$$

Linear time Monte Carlo algorithm

Question: Which field $\mathbb{F}$ do we use?

Sort Checking II – with Lorenz Hübschle-Schneider

Is the finite sequence E a permutation of another sequence E'?

Let h be a random hash function with destination range $0..U-1$,

$h(S) := \sum_{e \in S} h(e)$

Checker: return $h(E) = h(E')$

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Checker: return $h(E) = h(E')$

Correct if $E = E'$.

Case $E \neq E'$: We show $\mathbb{P}[h(E) = h(E')] \leq \frac{1}{U}$

Let e be an element, that appears $k\times$ in E and $k' \neq k\times$ in E'.

$$h(E) = h(E') \Leftrightarrow h(E \setminus e) + kh(e) = h(E' \setminus e) + k'h(e)$$

$$\Leftrightarrow h(e) = \frac{h(E' \setminus e) - h(E \setminus e)}{k - k'} =: x$$

$\mathbb{P}[h(e) = x] \leq \frac{1}{U}$ because x is independent of $h(e)$ ■

6.2 Hashing II

Perfect Hashing

Idea: given a set of inputs S make h injective on this set.

This needs $\Omega(n)$ bits of space !

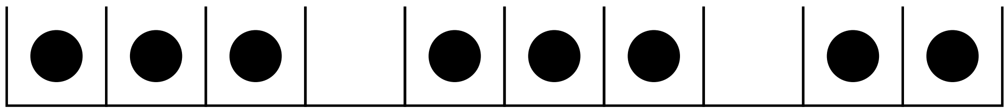
Here: Fast Space Efficient Hashing

Represent a set of n elements (with associated information) using space $(1+\varepsilon)n$.

Support operations insert, delete, lookup, (doall) efficiently.

Assume a truly random hash function h

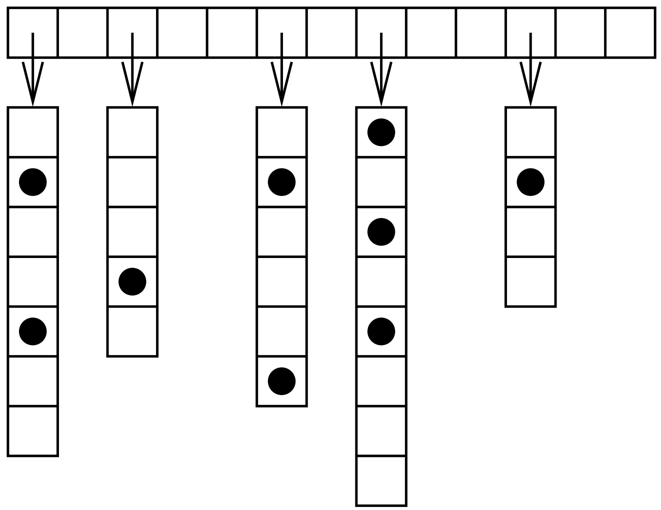
([Dietzfelbinger, Weidling 2005] shows that this is justified.)



Related Work

Linear probing: $E[T_{\text{find}}] \approx \frac{1}{2\varepsilon^2}$

Uniform hashing: $E[T_{\text{find}}] \approx \frac{1}{\varepsilon}$



Dynamic Perfect Hashing,

[Dietzfelbinger et al. 94]

Worst case constant time

for lookup but ε is not small.

Approaching the Information Theoretic Lower Bound:

[Brodnik Munro 99,Raman Rao 02]

Space $(1+o(1))\times$lower bound without associated information

[Botelho Pagh Ziviani 2007] static case.

Simple, fast, ≈ 3bits/element [FiRe/FiPHa:Müller,S,Schulze,Zhou 14]

Cuckoo Hashing

[Pagh Rodler 01]

Table of size $(2+\varepsilon)n$.

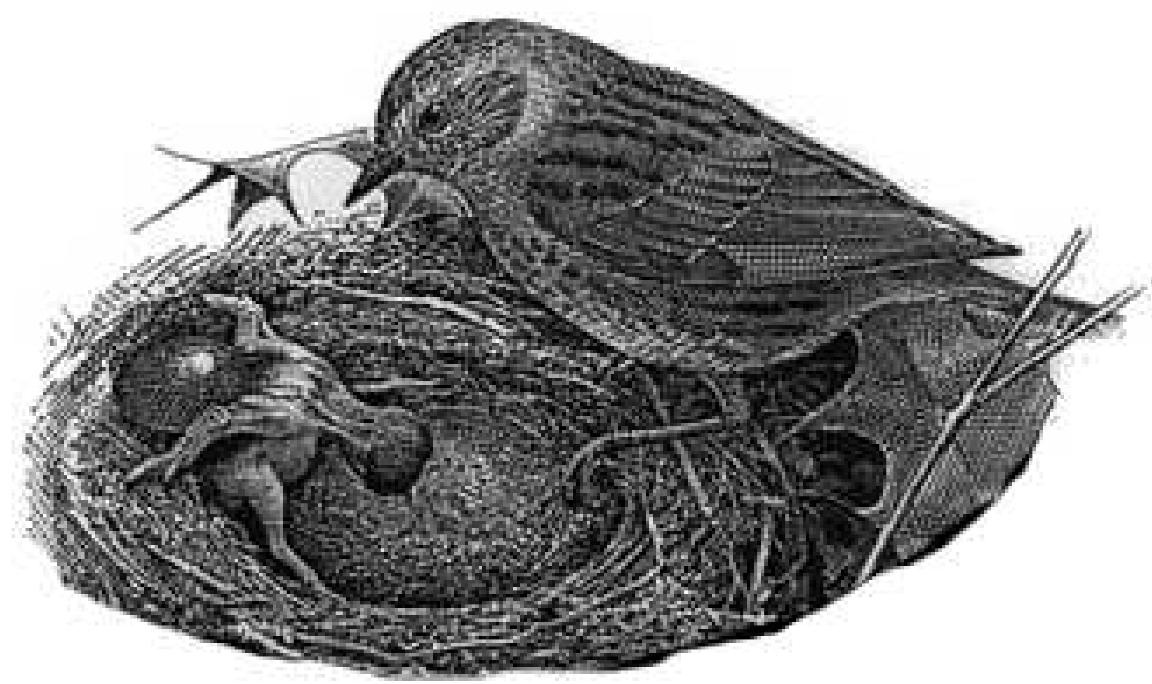
Two choices for each element.

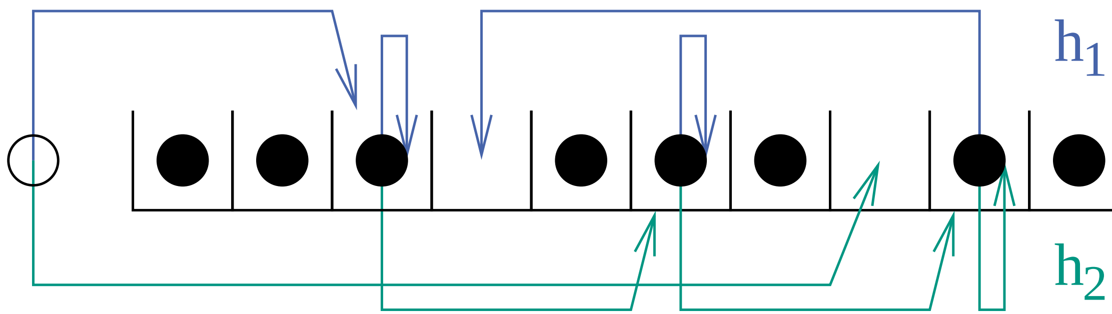
Insert moves elements;

rebuild if necessary.

Very fast lookup and delete.

Expected constant insertion time.





Cuckoo Hashing – Rebuilds

When needed ?

Graph model.

Node: table cells

Undirected edge: element $x \rightsquigarrow$

edge $\{h_1(x), h_2(x)\}$

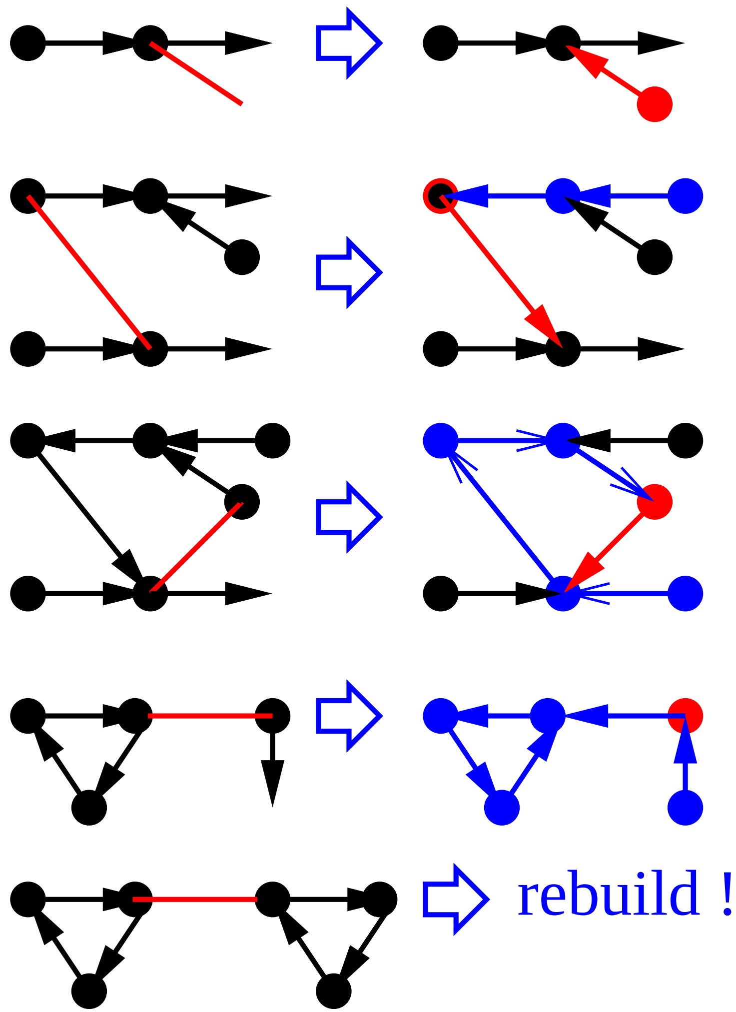
Directed: $(h_2(x), h_1(x))$ means

element x is stored at cell $h_2(x)$

Lemma: insert(x) succeeds iff

the component containing $h_1(x), h_2(x)$

contains no more edges than nodes.



Cuckoo Hashing – Rebuilds

Lemma: insert(x) succeeds iff

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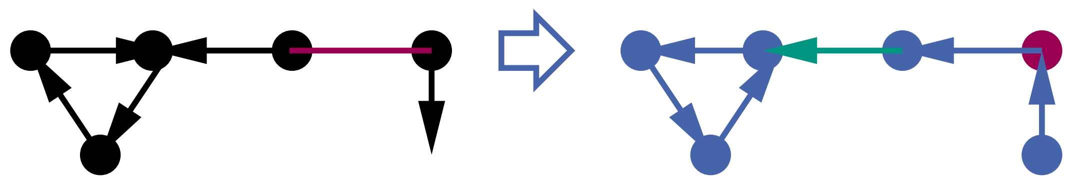
Proof outline: (if-part)

$h_1(x)$ **in tree:** flip path to root

$h_1(x)$ **in pseudotree** p**,** $h_2(x)$ **in tree** t**:**

flip cycle and path to root in t



Cuckoo Hashing – How Many Rebuilds?

Theorem: For truly random hash functions,
$\mathbf{Pr}[\text{rebuild necessary}] = \mathrm{O}(1/n)$

Proof: via random graph theory

Random Graph Theory

[Erdős, Rényi 1959]

$\mathscr{G}(n,m)$:= sample space of all graphs with n nodes, m edges.

A random graph from $\mathscr{G}(n,m)$ has certain properties with high probability, here $\geq 1 - \mathrm{O}(1/n)$.

Famous: The evolution of component sizes with increasing m:

$< (1-\varepsilon)n/2$: Trees and pseudotrees of size $\mathrm{O}(\log n)$

$> (1+\varepsilon)n/2$: A "giant" component of size $\Theta(n)$ (sudden emergence)

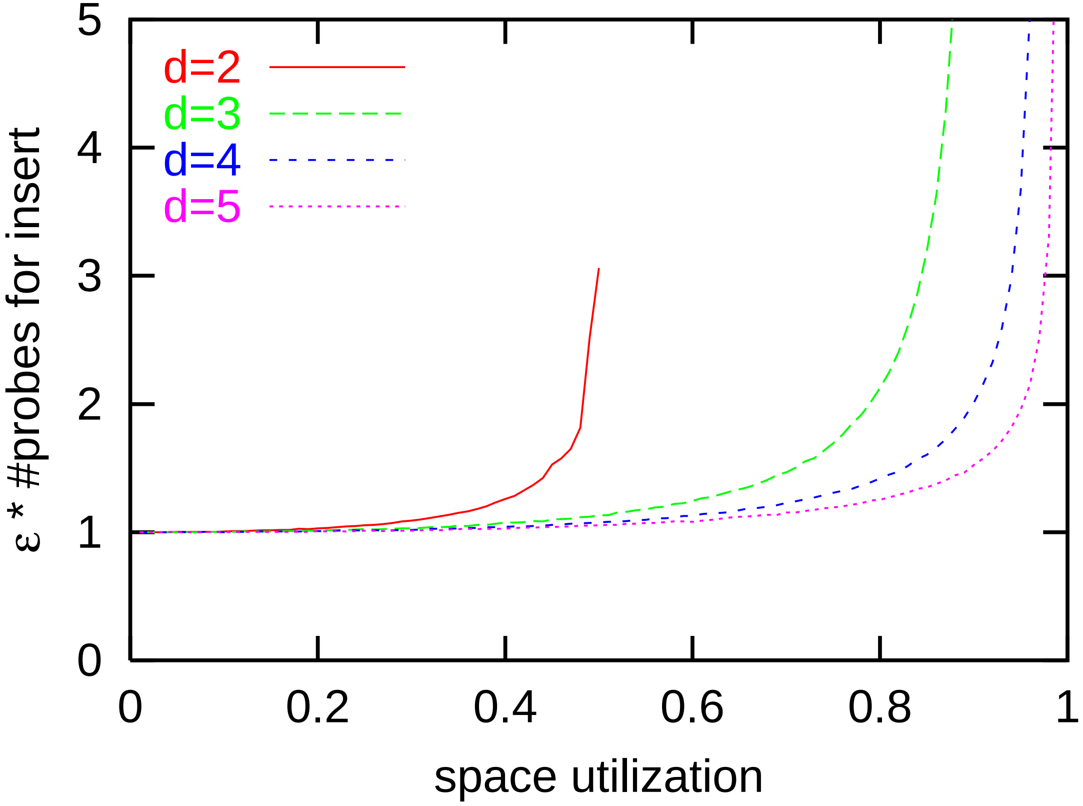
$> (1+\varepsilon)n \ln n/2$: One single component

Space Efficient Cuckoo Hashing

d**-ary:** [Fotakis, Pagh, Sanders, Spirakis 2003] d possible places.

Insertion: BFS, random walk, . . .

expected time: $\mathrm{O}\left(\frac{1}{\varepsilon}\right)$?



Space Efficient Cuckoo Hashing

d-ary: [Fotakis, Pagh, Sanders, Spirakis 2003] d possible places.

blocked: [Dietzfelbinger, Weidling 2005] cells house d elements.

Cache efficient !

blocked, d-ary, dynamic growing:

[Maier, Sanders 2017]

Recap – Randomised Algorithms

- Easy, efficient algorithms
- In many cases the best known procedures
- Sometimes deterministic solutions are (provably) impossible
- Often examples for non-trivial analysis
- Sometimes esoteric theory leads to tools that are relevant in practice, e.g. random graph evolution
- Las Vegas versus Monte Carlo
- Bridge to algebra, e.g. Sort-Checker

Lookout – Randomised Algorithms

- External minimum spanning trees
- More quicksort (strings, parallel)
- Smallest enclosing circle
- Online paging