Algorithmen / Algorithms II

Peter Sanders

Exercise:

Daniel Seemaier, Tobias Heuer

Institute of Theoretical Informatics

Web:

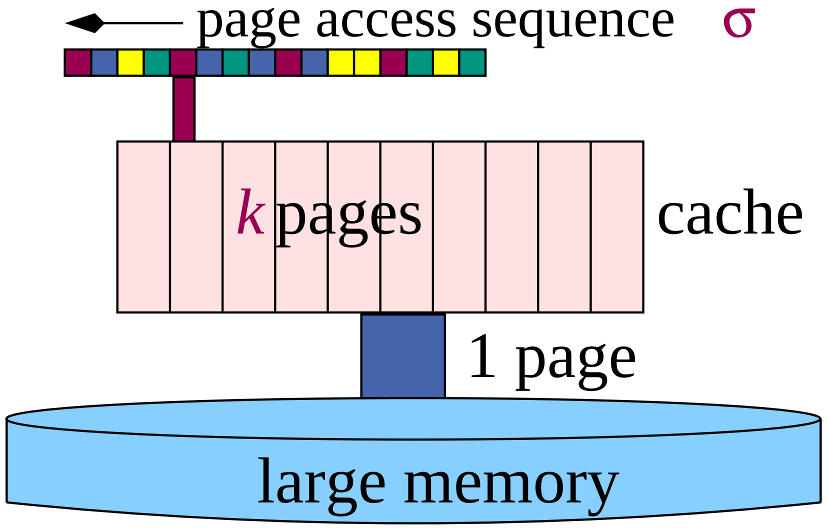
`http://algo2.iti.kit.edu/AlgorithmenII_WS20.php`

13 Online Algorithms [in part by Rob van Stee]

- Information is revealed to the algorithm in parts
- Algorithm needs to process each part before receiving the next
- There is **no information** about the future
  (in particular, no probabilistic assumptions!)
- How well can an algorithm do
  compared to an algorithm that knows everything?
- Lack of knowledge vs. lack of processing power

Examples

- Ski rental etc.
- Paging in a virtual memory system
- Routing in communication networks
- Scheduling machines in a factory, where orders arrive over time
- Google placing advertisements



Competitive analysis

- Idea: compare online algorithm ALG to offline algorithm OPT
- Worst-case performance measure
- Definition:

$$C_{ALG} = \sup_{\sigma} \frac{\mathrm{ALG}(\sigma)}{\mathrm{OPT}(\sigma)}$$

(we look for the input that results in worst relative performance)

- Goal:

  find ALG with minimal C_{ALG}

A Typical Online Problem: Ski Rental

- Renting skis costs 50 euros, buying them costs 300 euros
- You do not know in advance how often you will go skiing
- Should you rent skis or buy them?

A Typical Online Problem: Ski Rental

- Renting skis costs 50 euros, buying them costs 300 euros
- You do not know in advance how often you will go skiing
- Should you rent skis or buy them?
- Suggested algorithm: buy skis on the sixth trip
- Two questions:
  - How good is this algorithm?
  - Can you do better?

Upper Bound for Ski Rental

- You plan to buy skis on the sixth trip
- If you make five trips or less, you pay optimal cost (50 euros per trip)
- If you make at least six trips, you pay 550 euros
- In this case OPT pays at least 300 euros
- Conclusion: algorithm is $\frac{11}{6}$-competitive: it never pays more than $\frac{11}{6}$ times the optimal cost

Lower Bound for Ski Rental

- Suppose you buy skis earlier, say on trip $x \leq 5$.

  You pay $300+50(x-1)$, OPT pays only $50x$

$$\frac{250+50x}{50x} = \frac{5}{x}+1 \geq 2.$$

- Suppose you buy skis later, on trip $y \geq 7$.

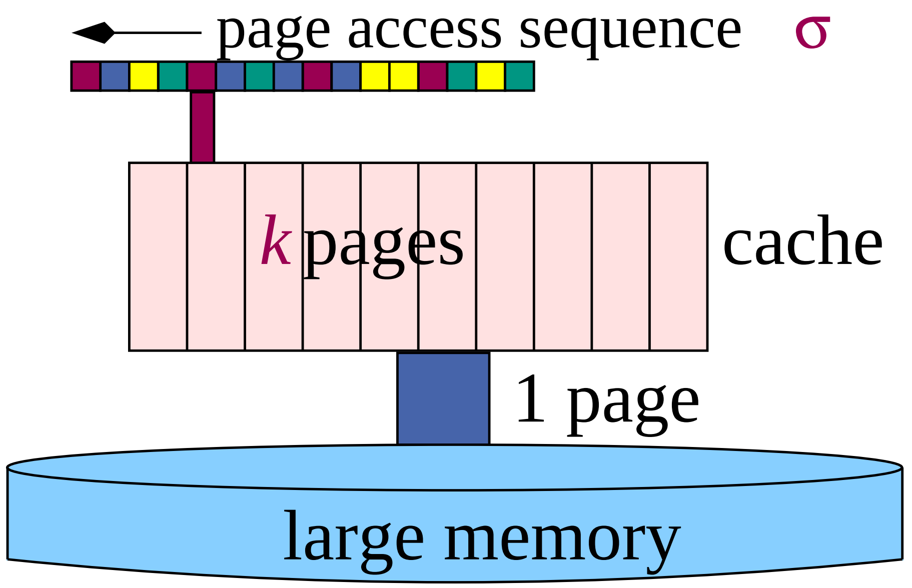
  You pay $300+50(y-1)$, OPT pays only 300

$$\frac{250+50y}{300} = \frac{5+y}{6} \geq 2.$$

- Idea: do not pay the large cost (buy skis) until you would have paid the same amount in small costs (rent)

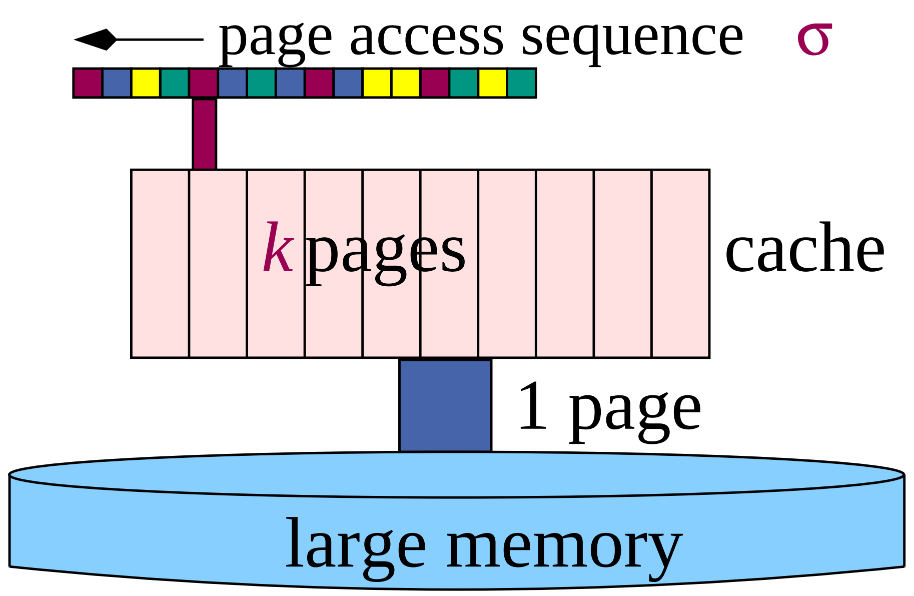
Paging

- Computers usually have a small amount of fast memory (cache)
- This can be used to store data (pages) that are often used
- Problem when the cache is full and a new page is requested
- Which page should be thrown out (evicted)?



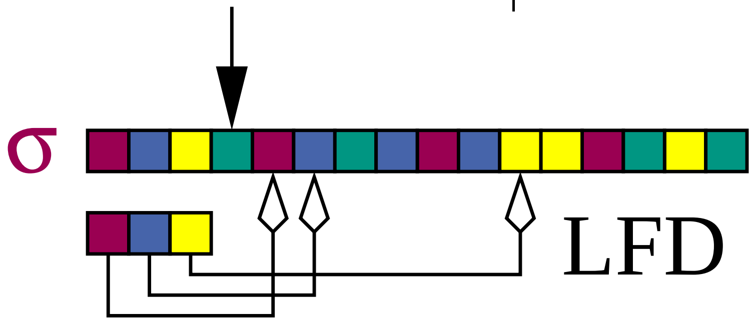
Definitions

- k = size of cache (number of pages)
- We assume that access to the cache is free, since accessing main memory costs much more
- Thus, a cache hit costs 0 and a miss (fault) costs 1
- The goal is to minimize the number of page faults



Paging Algorithms

algorithm		which page to evict
LIFO	Last In First Out	newest
FIFO	First In First Out	oldest
LFU	Least Frequently used	requested least often
LRU	Least Recently Used	requested least recently
FWF	Flush When Full	all
LFD	Longest Forward Distance	(re)requested latest in the future

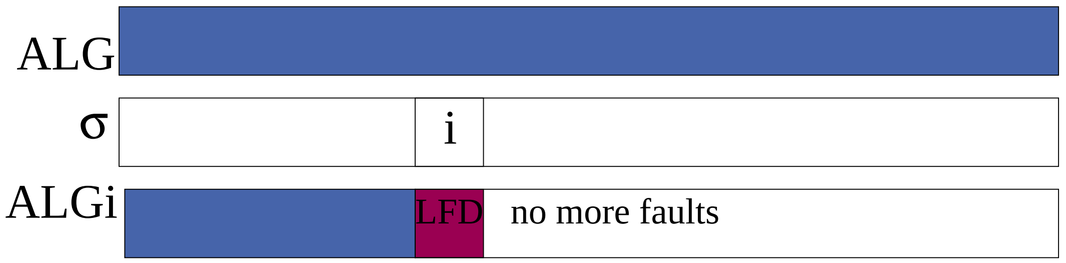


Longest Forward Distance is Optimal

We show: any optimal offline algorithm can be changed to act like LFD without increasing the number of page faults.

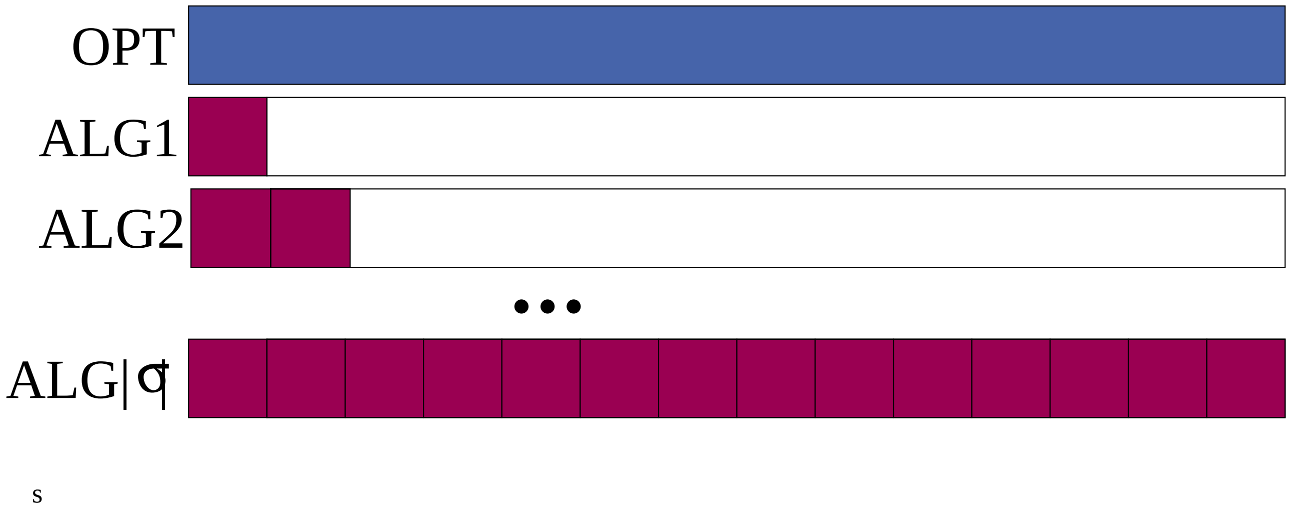
Inductive claim: given an algorithm ALG, we can create ALG$_i$ such that

- ALG and ALG$_i$ act identically on the first $i-1$ requests
- If request i causes a fault (for both algorithms), ALG$_i$ evicts page with longest forward distance
- ALG$_i(\sigma) \leq$ ALG(σ)



Using the Claim

- Start with a given request sequence σ and an optimal offline algorithm ALG
- Use the claim for $i = 1$ on ALG to get ALG$_1$, which evicts the LFD page on the first request (if needed)
- Use the claim for $i = 2$ on ALG$_1$ to get ALG$_2$
- ...
- Final algorithm ALG$_{|\sigma|}$ is equal to OPT



Proof of the Claim

not this time

Comparison of Algorithms

- OPT is not online, since it looks forward
- Which is the best online algorithm?
- LIFO is not competitive: consider an input sequence

$$p_1, p_2, \ldots, p_{k-1}, p_k, p_{k+1}, p_k, p_{k+1}, \cdots$$

- LFU is also not competitive: consider

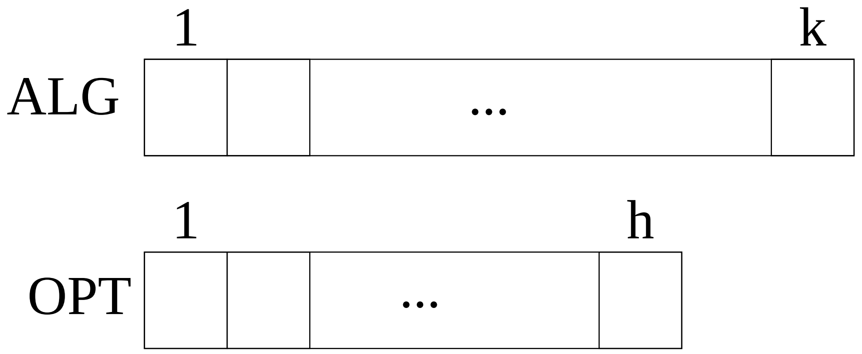
$$p_1^m, p_2^m, \ldots, p_{k-1}^m, (p_k, p_{k+1})^{m-1}$$

A General Lower Bound

- To illustrate the problem, we show a lower bound for any online paging algorithm ALG
- There are $k+1$ pages
- At all times, ALG has k pages in its cache
- There is always one page missing: request this page at each step
- OPT only faults once every k steps
  $\Rightarrow$ lower bound of k on the competitive ratio

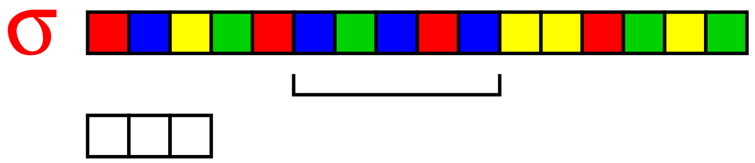
Resource Augmentation

- We will compare an online algorithm ALG to an optimal offline algorithm which has a smaller cache
- We hope to get more realistic results in this way
- Size of offline cache = $h < k$
- This problem is known as (h, k)-paging



Conservative Algorithms

- An algorithm is conservative if it has at most k page faults on any request sequence that contains at most k distinct pages
- The request sequence may be arbitrarily long
- LRU and FIFO are conservative
- LFU and LIFO are not conservative (recall that they are not competitive)



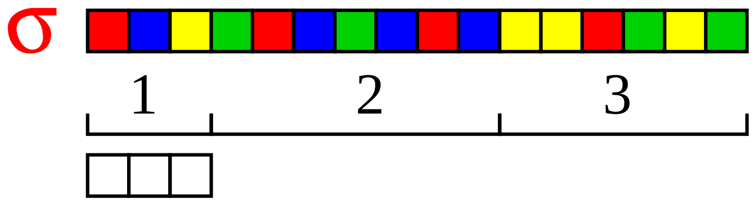
Competitive Ratio

Theorem: Any conservative algorithm is $\frac{k}{k-h+1}$-competitive

Proof: divide request sequence σ into **phases**.

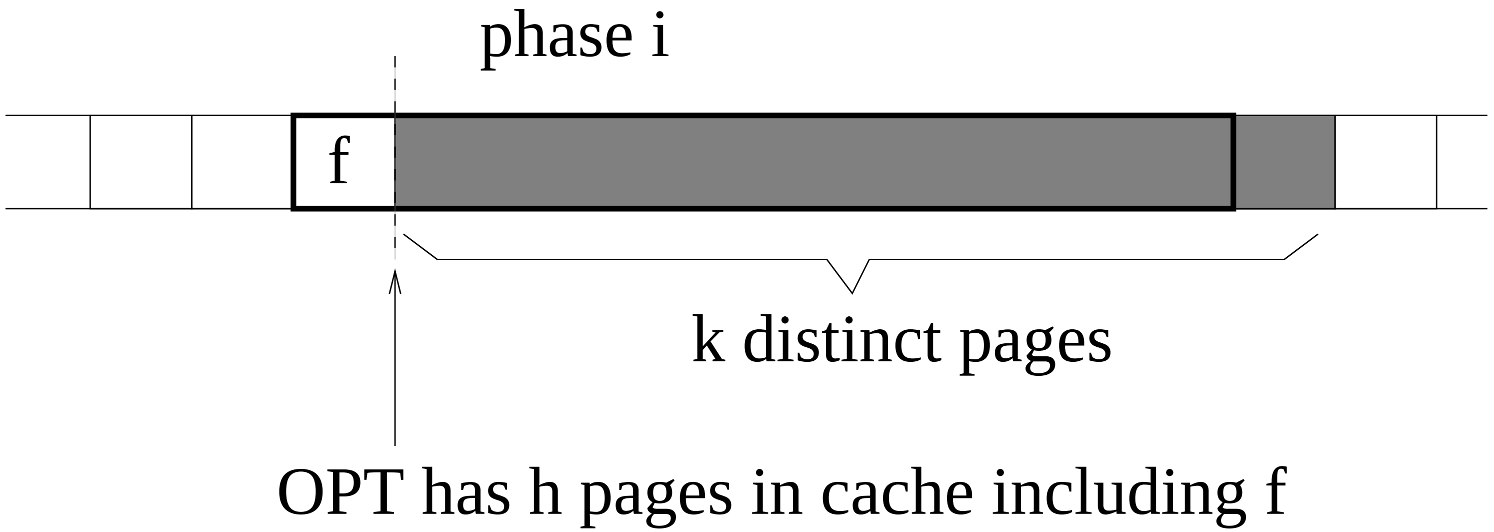
- Phase 0 is the empty sequence
- Phase $i > 0$ is the maximal sequence following phase $i-1$ that contains at most k distinct pages

Phase partitioning does not depend on algorithm. A conservative algorithm has at most k faults per phase.



Counting the Faults of OPT

Consider some phase $i > 0$, denote its first request by f



Thus OPT has at least $k-(h-1) = k-h+1$ faults on the grey requests

Conclusion

- In each phase, a conservative algorithm has k faults
- To each phase except the last one, we can assign (charge) $k-h+1$ faults of OPT
- Thus
$$\mathrm{ALG}(\sigma) \leq \frac{k}{k-h+1} \cdot \mathrm{OPT}(\sigma) + r$$
where $r \leq k$ is the number of page faults of ALG in the last phase
- This proves the theorem

Notes

- ☐ For $h = k/2$, we find that conservative algorithms are 2-competitive
- ☐ The previous lower bound construction does not work for $h < k$
- ☐ In practice, the “competitive ratio” of LRU is a small constant
- ☐ Resource augmentation can give better (more realistic) results than pure competitive analysis

New Results (Panagiotou & Souza, STOC 2006)

- Restrict the adversary to get more “natural” input sequences
- Locality of reference: most consecutive requests to pages have short distance
- Typical memory access patterns: consecutive requests have either short or long distance compared to the cache size

Randomized Algorithms

- Another way to avoid the lower bound of k for paging is to use a randomized algorithm
- Such an algorithm is allowed to use random bits in its decision making
- Crucial is what the adversary knows about these random bits

Three Types of Adversaries

- ☐ Oblivious: knows only the probability distribution that ALG uses, determines input in advance
- ☐ Adaptive online: knows random choices made so far, bases input on these choices
- ☐ Adaptive offline: knows random choices in advance (!)

Randomization does not help against adaptive offline adversary

We focus on the oblivious adversary

Marking Algorithm

- marks pages which are requested
- never evicts a marked page
- When all pages are marked and there is a fault, unmark everything
  (but mark the page which caused the fault)
  (new phase)

Marking Algorithms

Only difference is eviction strategy

- LRU
- FWF
- RMARK: Evict an unmarked page choosen uniformly at random

Competitive Ratio of RMARK

Theorem: RMARK is $2H_k$-competitive

where

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \leq \ln k + 1$$

is the k-the harmonic number

Analysis of RMARK

Consider a phase with m new pages

(that are not cached in the beginning of the phase)

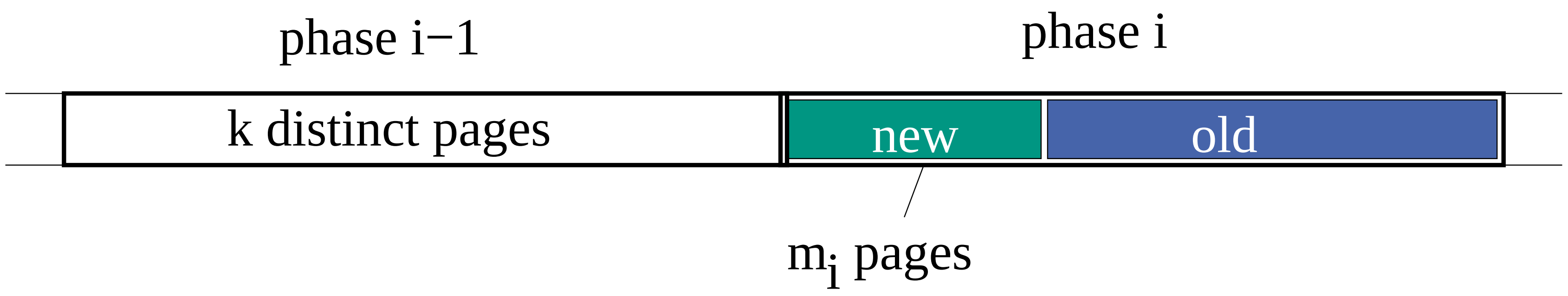
Miss probability when $j+1$st old page becomes marked

$$1-\frac{\text{\# old unmarked cached pages}}{\text{\# old unmarked pages}}\leq 1-\frac{k-m-j}{k-j}=\frac{m}{k-j}$$

Overall expected number of faults (including new pages):

$$m+\sum_{j=0}^{k-m-1}\frac{m}{k-j}=m+m\sum_{i=m+1}^{k}\frac{1}{i}=m(1+H_k-H_m)\leq mH_k$$

Lower Bound for OPT



- There are m_i new pages in phase i
- Thus, in phases $i-1$ and i together, $k+m_i$ pages are requested
- OPT makes at least m_i faults in phases i and $i-1$ for any i
- Total number of OPT faults is at least $\frac{1}{2}\sum_i m_i$

Upper Bound for RMARK

- Expected number of faults in phase i is at most $m_i H_k$ for RMARK
- Total expected number of faults is at most $H_k \sum_i m_i$
- OPT has at least $\frac{1}{2}\sum_i m_i$ faults
- Conclusion: RMARK is $2H_k$-competitive

Randomized Lower Bound

Theorem: No randomized can be better than H_k-competitive against an oblivious adversary.

Proof: not here

Discussion

- $H_k \ll k$
- The upper bound for RMARK holds against an oblivious adversary (the input sequence is fixed in advance)
- No algorithm can be better than H_k-competitive
- Thus, RMARK is optimal apart from a factor of 2
- There is a (more complicated) algorithm that is H_k competitive

Why Competitive Analysis?

There are many models for “decision making in the absence of complete information”

- Competitive analysis leads to algorithms that would not otherwise be considered
- Probability distributions are rarely known precisely
- Assumptions about distributions must often be unrealistically crude to allow for mathematical tractability
- Competitive analysis gives a guarantee on the performance of an algorithm, which is essential in e.g. financial planning

Disadvantages of Competitive Analysis

- Results can be too pessimistic (adversary is too powerful)
  - Resource augmentation
  - Randomization
  - Restrictions on the input
- Unable to distinguish between some algorithms that perform differently in practice
  - Paging: LRU and FIFO
  - more refined models