Algorithmen II

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http://algo2.itii.kit.edu/AlgorithmenII_WS23.php
13 Onlinealgorithmen

- Information is revealed to the algorithm in parts
- Algorithm needs to process each part before receiving the next
- There is no information about the future (in particular, no probabilistic assumptions!)
- How well can an algorithm do compared to an algorithm that knows everything?
- Lack of knowledge vs. lack of processing power
Examples

- Renting Skis etc.
- Paging in a virtual memory system
- Routing in communication networks
- Scheduling machines in a factory, where orders arrive over time
- Google placing advertisements
Competitive analysis

- Idea: compare online algorithm ALG to offline algorithm OPT
- Worst-case performance measure

**Definition:**

\[ C_{ALG} = \sup_{\sigma} \frac{\text{ALG}(\sigma)}{\text{OPT}(\sigma)} \]

(We look for the input that results in worst relative performance)

- Goal:

find ALG with minimal \( C_{ALG} \)
A typical online problem: ski rental

- Renting skis costs 50 euros, buying them costs 300 euros
- You do not know in advance how often you will go skiing
- Should you rent skis or buy them?
A typical online problem: ski rental

- Renting skis costs 50 euros, buying them costs 300 euros
- You do not know in advance how often you will go skiing
- Should you rent skis or buy them?
- Suggested algorithm: buy skis on the sixth trip
- Two questions:
  - How good is this algorithm?
  - Can you do better?
Upper bound for ski rental

☐ You plan to buy skis on the sixth trip

☐ If you make five trips or less, you pay optimal cost
  (50 euros per trip)

☐ If you make at least six trips, you pay 550 euros

☐ In this case OPT pays at least 300 euros

☐ Conclusion: algorithm is $\frac{11}{6}$-competitive:
  it never pays more than $\frac{11}{6}$ times the optimal cost
Lower bound for ski rental

☐ Suppose you buy skis earlier, say on trip $x < 6$.
You pay $300 + 50(x - 1)$, OPT pays only $50x$

$$\frac{250 + 50x}{50x} = \frac{5}{x} + 1 \geq 2.$$  

☐ Suppose you buy skis later, on trip $y > 6$.
You pay $300 + 50(y - 1)$, OPT pays only $300$

$$\frac{250 + 50y}{300} = \frac{5 + y}{6} \geq 2.$$  

☐ Idea: do not pay the large cost (buy skis) until you would have paid the same amount in small costs (rent)
Paging

- Computers usually have a small amount of fast memory (cache)
- This can be used to store data (pages) that are often used
- Problem when the cache is full and a new page is requested
- Which page should be thrown out (evicted)?
Definitions

- $k =$ size of cache (number of pages)
- We assume that access to the cache is free, since accessing external memory costs much more
- Thus, a cache hit costs 0 and a miss (fault) costs 1
- The goal is to minimize the number of page faults
## Paging algorithms

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Which Page to Evict</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIFO</td>
<td>Newest</td>
</tr>
<tr>
<td>FIFO</td>
<td>Oldest</td>
</tr>
<tr>
<td>LFU</td>
<td>Requested least often</td>
</tr>
<tr>
<td>LRU</td>
<td>Requested least recently</td>
</tr>
<tr>
<td>FWF</td>
<td>All</td>
</tr>
<tr>
<td>LFD</td>
<td>(Re)Requested latest in the future</td>
</tr>
</tbody>
</table>

![Diagram showing LFD algorithm](image)
Longest Forward Distance is optimal

We show: any optimal offline algorithm can be changed to act like LFD without increasing the number of page faults.

Inductive claim: given an algorithm ALG, we can create $ALG_i$ such that

- $ALG$ and $ALG_i$ act identically on the first $i-1$ requests
- If request $i$ causes a fault (for both algorithms), $ALG_i$ evicts page with longest forward distance
- $ALG_i(\sigma) \leq ALG(\sigma)$

$ALG$

\[\sigma | i\]

$ALG_i$

\[\sigma | LFD | \text{no more faults}\]
Using the claim

- Start with a given request sequence $\sigma$ and an optimal offline algorithm $\text{ALG}$
- Use the claim for $i = 1$ on $\text{ALG}$ to get $\text{ALG}_1$, which evicts the LFD page on the first request (if needed)
- Use the claim for $i = 2$ on $\text{ALG}_1$ to get $\text{ALG}_2$
- ...$
- Final algorithm $\text{ALG}_{|\sigma|}$ is equal to $\text{OPT}$
Proof of the claim

not this time
Comparison of algorithms

- OPT is not online, since it looks forward
- Which is the best online algorithm?
- LIFO is not competitive: consider an input sequence
  \[ p_1, p_2, \ldots, p_{k-1}, p_k, p_{k+1}, p_k, p_{k+1}, \ldots \]
- LFU is also not competitive: consider
  \[ p_1^m, p_2^m, \ldots, p_{k-1}^m, (p_k, p_{k+1})^{m-1} \]
A general lower bound

- To illustrate the problem, we show a lower bound for any online paging algorithm ALG.
- There are $k + 1$ pages.
- At all times, ALG has $k$ pages in its cache.
- There is always one page missing: request this page at each step.
- OPT only faults once every $k$ steps.

$\Rightarrow$ lower bound of $k$ on the competitive ratio.
Resource augmentation

- We will compare an online algorithm ALG to an optimal offline algorithm which has a smaller cache.
- We hope to get more realistic results in this way.
- Size of offline cache $= h < k$.
- This problem is known as $(h, k)$-paging.

![Diagram]

```
ALG 1 ... k
OPT 1 h ...
```
Conservative algorithms

- An algorithm is **conservative** if it has at most $k$ page faults on any request sequence that contains at most $k$ distinct pages.

- The request sequence may be **arbitrarily long**.

- LRU and FIFO are conservative.

- LFU and LIFO are **not** conservative (recall that they are not competitive).
Competitive ratio

**Theorem:** Any conservative algorithm is $\frac{k}{k-h+1}$-competitive

**Proof:** divide request sequence $\sigma$ into phases.

- Phase 0 is the empty sequence
- Phase $i > 0$ is the maximal sequence following phase $i - 1$ that contains at most $k$ distinct pages

Phase partitioning does not depend on algorithm. A conservative algorithm has at most $k$ faults per phase.
Counting the faults of OPT

Consider some phase $i > 0$, denote its first request by $f$

Thus OPT has at least

$$k - (h - 1) = k - h + 1$$

faults on the grey requests
Conclusion

- In each phase, a conservative algorithm has $k$ faults.

- To each phase except the last one, we can assign (charge) $k - h + 1$ faults of OPT.

- Thus

$$ALG(\sigma) \leq \frac{k}{k-h+1} \cdot OPT(\sigma) + r$$

where $r \leq k$ is the number of page faults of ALG in the last phase.

- This proves the theorem.
Notes

- For $h = k/2$, we find that conservative algorithms are 2-competitive.

- The previous lower bound construction does not work for $h < k$.

- In practice, the “competitive ratio” of LRU is a small constant.

- Resource augmentation can give better (more realistic) results than pure competitive analysis.
New results (Panagiotou & Souza, STOC 2006)

- Restrict the adversary to get more “natural” input sequences
- Locality of reference: most consecutive requests to pages have short distance
- Typical memory access patterns: consecutive requests have either short or long distance compared to the cache size
Randomized algorithms

- Another way to avoid the lower bound of $k$ for paging is to use a randomized algorithm.

- Such an algorithm is allowed to use random bits in its decision making.

- Crucial is what the adversary knows about these random bits.
Three types of adversaries

- **Oblivious**: knows only the probability distribution that ALG uses, determines input in advance
- **Adaptive online**: knows random choices made so far, bases input on these choices
- **Adaptive offline**: knows random choices in advance (!)

Randomization does not help against adaptive offline adversary

We focus on the **oblivious** adversary
Marking Algorithm

- marks pages which are requested
- never evicts a marked page

When all pages are marked and there is a fault, unmark everything
(but mark the page which caused the fault)
(new phase)
Marking Algorithms

Only difference is eviction strategy

- LRU
- FWF
- RMARK: Evict an unmarked page chosen uniformly at random
Competitive ratio of RMARK

**Theorem:** RMARK is $2H_k$-competitive

where

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \leq \ln k + 1$$

is the $k$-the harmonic number
Analysis of RMARK

Consider a phase with \( m \) new pages

(that are not cached in the beginning of the phase)

Miss probability when \( j + 1 \)st old page becomes marked

\[
1 - \frac{\text{# old unmarked cached pages}}{\text{# old unmarked pages}} \leq 1 - \frac{k - m - j}{k - j} = \frac{m}{k - j}
\]

Overall expected number of faults (including new pages):

\[
m + \sum_{j=0}^{k-m-1} \frac{m}{k-j} = m + m \sum_{i=m+1}^{k} \frac{1}{i} = m(1 + H_k - H_m) \leq mH_k
\]
Lower bound for OPT

- There are $m_i$ new pages in phase $i$
- Thus, in phases $i-1$ and $i$ together, $k + m_i$ pages are requested
- OPT makes at least $m_i$ faults in phases $i$ and $i-1$ for any $i$
- Total number of OPT faults is at least $\frac{1}{2} \sum_i m_i$
Upper bound for RMARK

- Expected number of faults in phase $i$ is at most $m_i H_k$ for RMARK
- Total expected number of faults is at most $H_k \sum_i m_i$
- OPT has at least $\frac{1}{2} \sum_i m_i$ faults
- Conclusion: RMARK is $2H_k$-competitive
Randomized lower bound

**Theorem:** No randomized can be better than $H_k$-competitive against an oblivious adversary.

**Proof:** not here
Discussion

- \( H_k \ll k \)

- The upper bound for RMARK holds against an oblivious adversary (the input sequence is fixed in advance)

- No algorithm can be better than \( H_k \)-competitive

- Thus, RMARK is optimal apart from a factor of 2

- There is a (more complicated) algorithm that is \( H_k \) competitive
Why competitive analysis?

There are many models for “decision making in the absence of complete information”

- Competitive analysis leads to algorithms that would not otherwise be considered
- Probability distributions are rarely known precisely
- Assumptions about distributions must often be unrealistically crude to allow for mathematical tractability
- Competitive analysis gives a guarantee on the performance of an algorithm, which is essential in e.g. financial planning
Disadvantages of competitive analysis

- Results can be too pessimistic (adversary is too powerful)
  - Resource augmentation
  - Randomization
  - Restrictions on the input

- Unable to distinguish between some algorithms that perform differently in practice
  - Paging: LRU and FIFO
  - more refined models