Lecture 11: Distance Oracles

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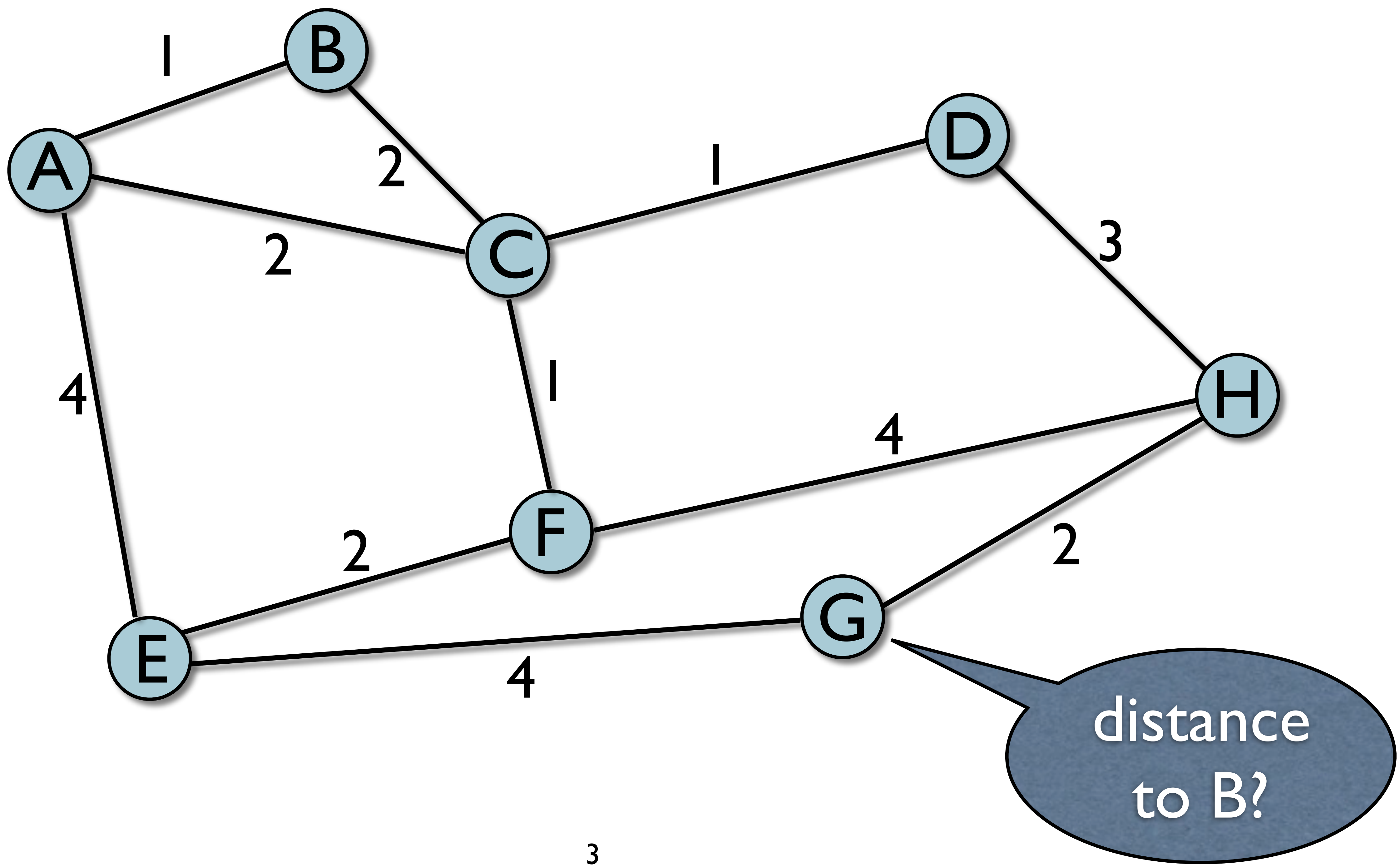
Distance Oracles in Graphs

- M. Thorup, U. Zwick: *Approximate Distance Oracles*. J. ACM **52**(1), 2005.

Distance Oracles



Basic Definitions

- $G=(V, E)$: weighted undirected **graph**
  - $|V|=n$, $|E| = m$
  - $e \in E$: $\omega(e) \geq 0$ **edge weights**
  - $\delta(u,v)$: **distance** from u to v (length of shortest path)
  - $\delta(A,v)$: distance from v to **nearest** $a \in A \subseteq V$
- $d(u,v)$ **stretch**-t approximation to $\delta(u,v)$

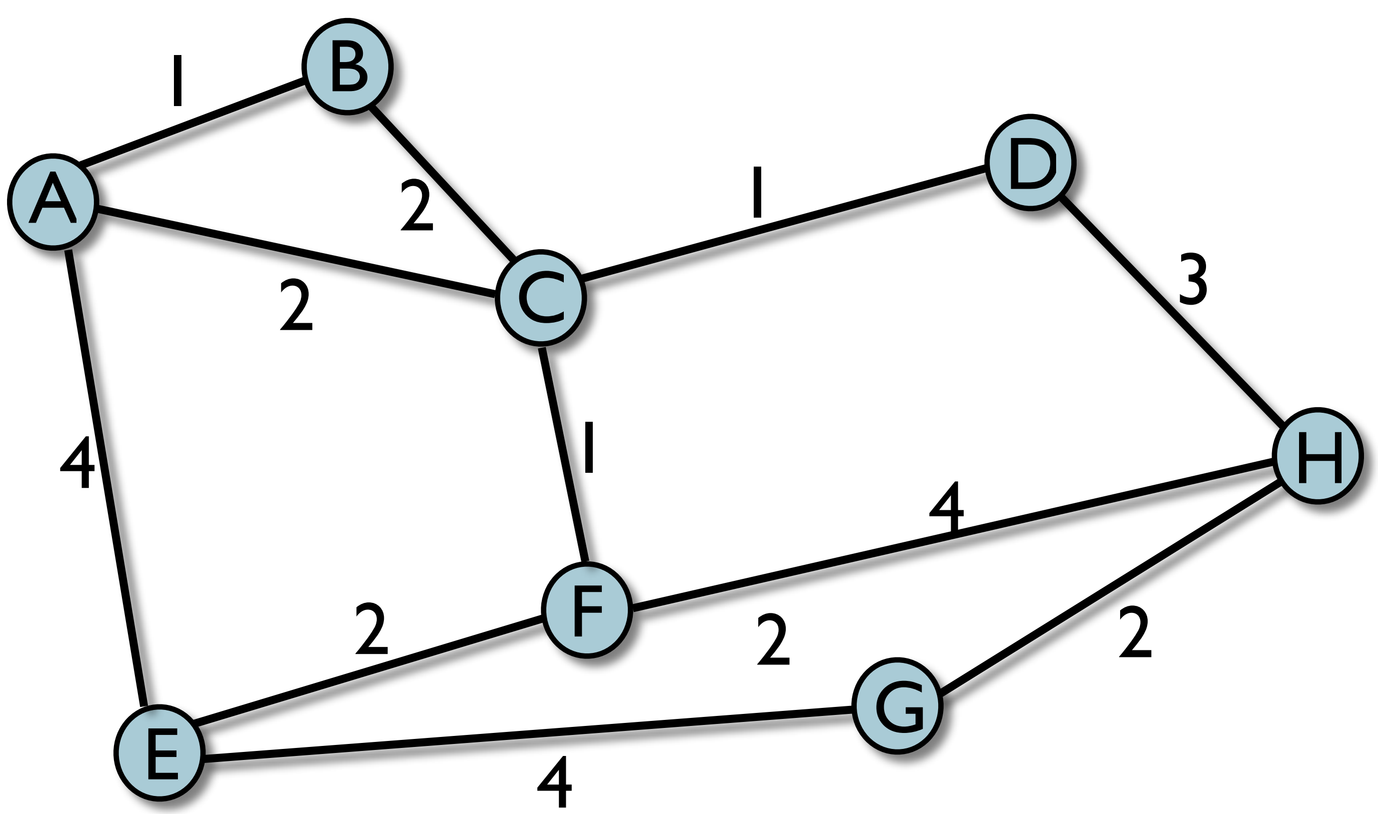
$$\Leftrightarrow \delta(u,v) \leq d(u,v) \leq t \cdot \delta(u,v)$$

Main Result

- k: arbitrary parameter
- Preprocess G in $O(kn^{1/k}(n \lg n + m))$ time
  - DS of size $O(kn^{1+1/k})$ (**words**)
  - distance **queries** $\text{dist}_k(u,v)$ in $O(k)$ time
  - stretch $\leq 2k-1$
  - report **path** of length $\leq \text{dist}_k(u,v)$
    in O(1) time per edge
- e.g. $k=2$
  - $O(n^{3/2})$ space, O(1) query time, stretch 3

Part I: Metric Spaces

- Assume metric as distance matrix



⇒

	A	B	C	D	E	F	G	H
A	0	1	2	3	4	3	8	6
B		0	2	3	5	3	8	6
C			0	1	3	1	6	4
D				0	4	2	5	3
E					0	2	4	6
F						0	6	4
G							0	2
H								0

Random Samples

- construct **randomly**

  $V = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_{k-1} \supseteq A_k = \emptyset$

- **Rule:**
  "Place $v \in A_{i-1}$ in A_i, independently, with probability $n^{-1/k}$."

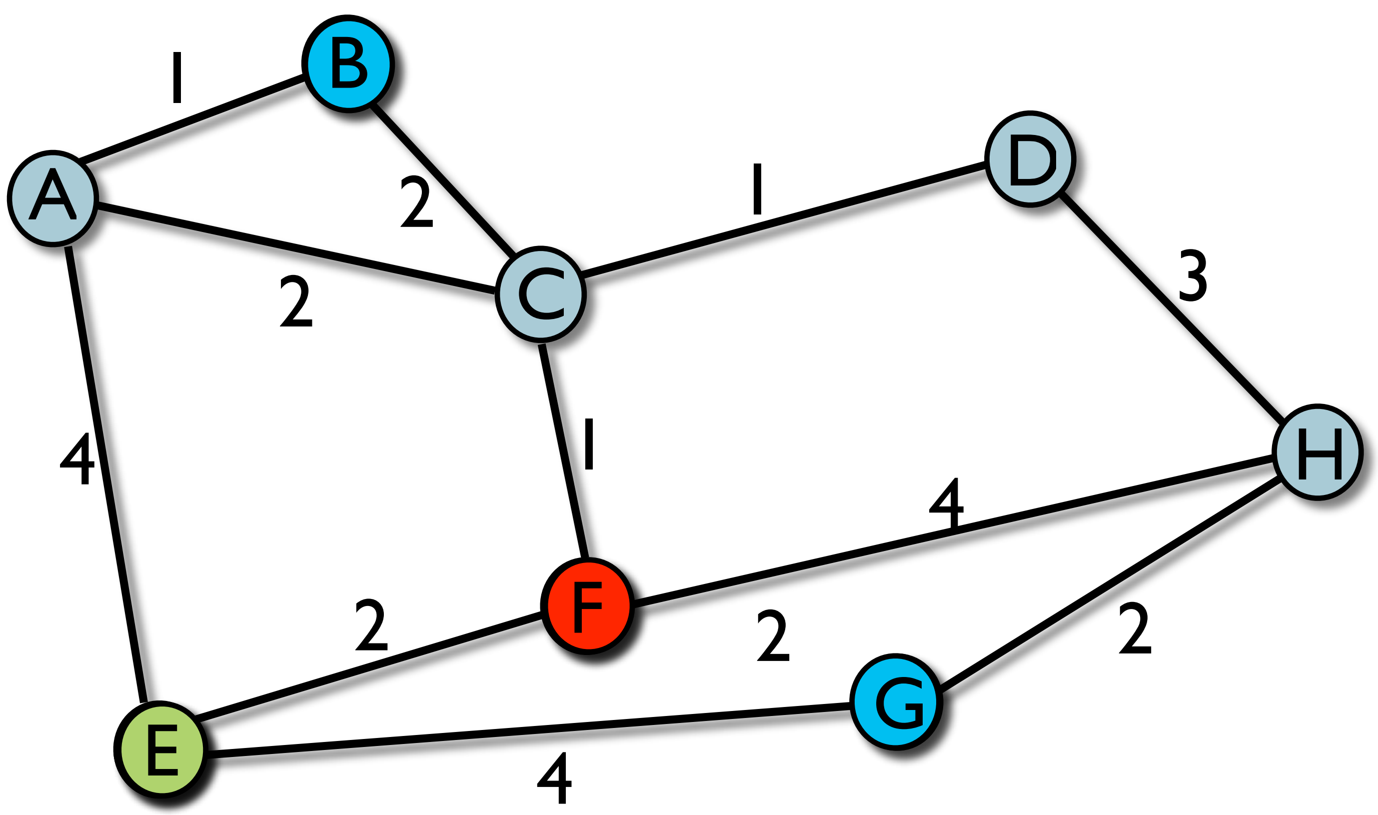
$\Longrightarrow$ $\text{Exp}[|A_i|] = |V| \times \text{Prob}[v \in A_j \ \forall \ 1 \leq j \leq i]$

$= n \times n^{-1/k} \cdot n^{-1/k} \cdot \ldots \cdot n^{-1/k}$ (i times)

$= n^{1-i/k}$

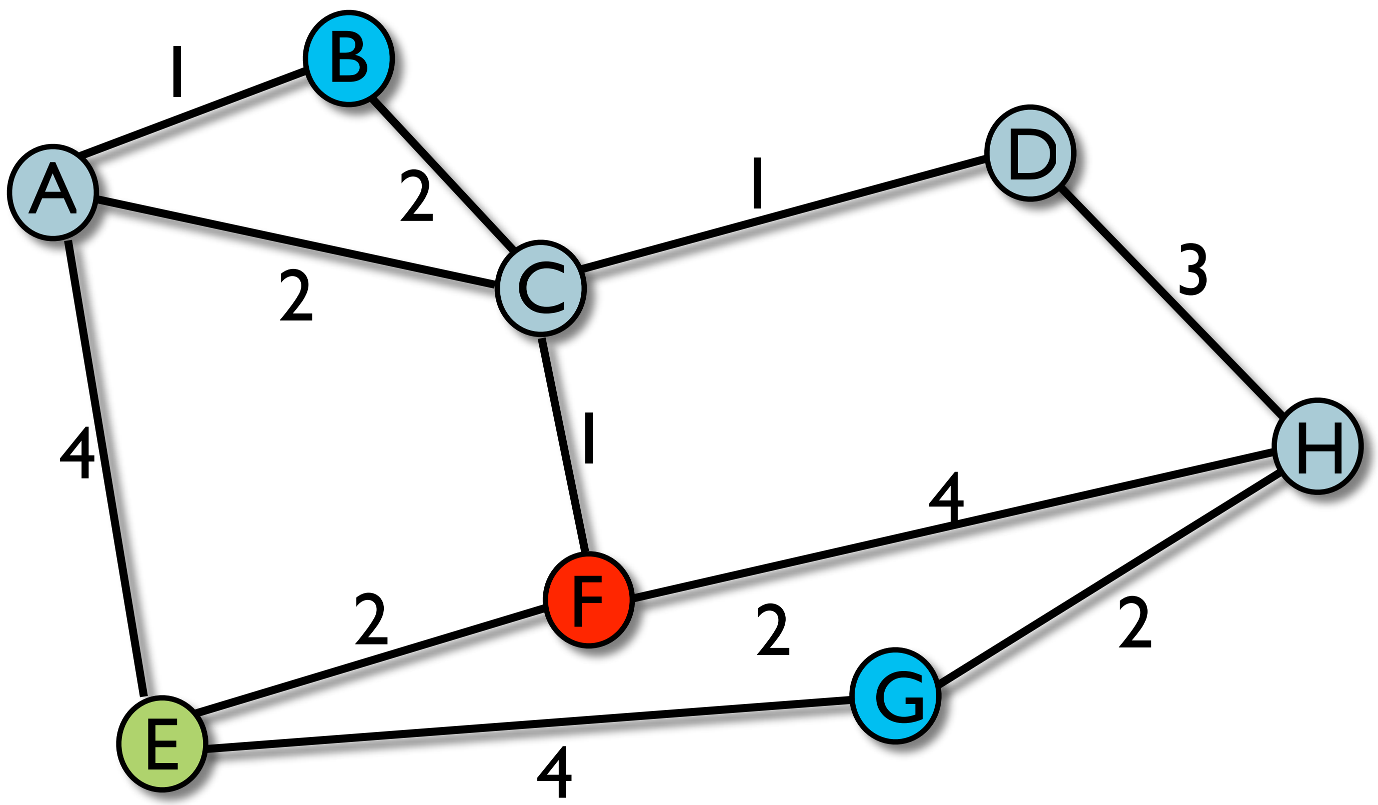
- **compute and store** $\forall v, \forall i$:
  - $\delta(A_i,v)$ and $p_i(v)$ with $\delta(p_i(v),v)=\delta(A_i,v)$ ("witness")

Example



- $A_0 = \{A,B,C,D,E,F,G,H\}$
- $A_1 = \{B,E,F,G\}$
- $A_2 = \{E,F\}$
- $A_3 = \{E\}$
- $A_4 = \varnothing$

Example

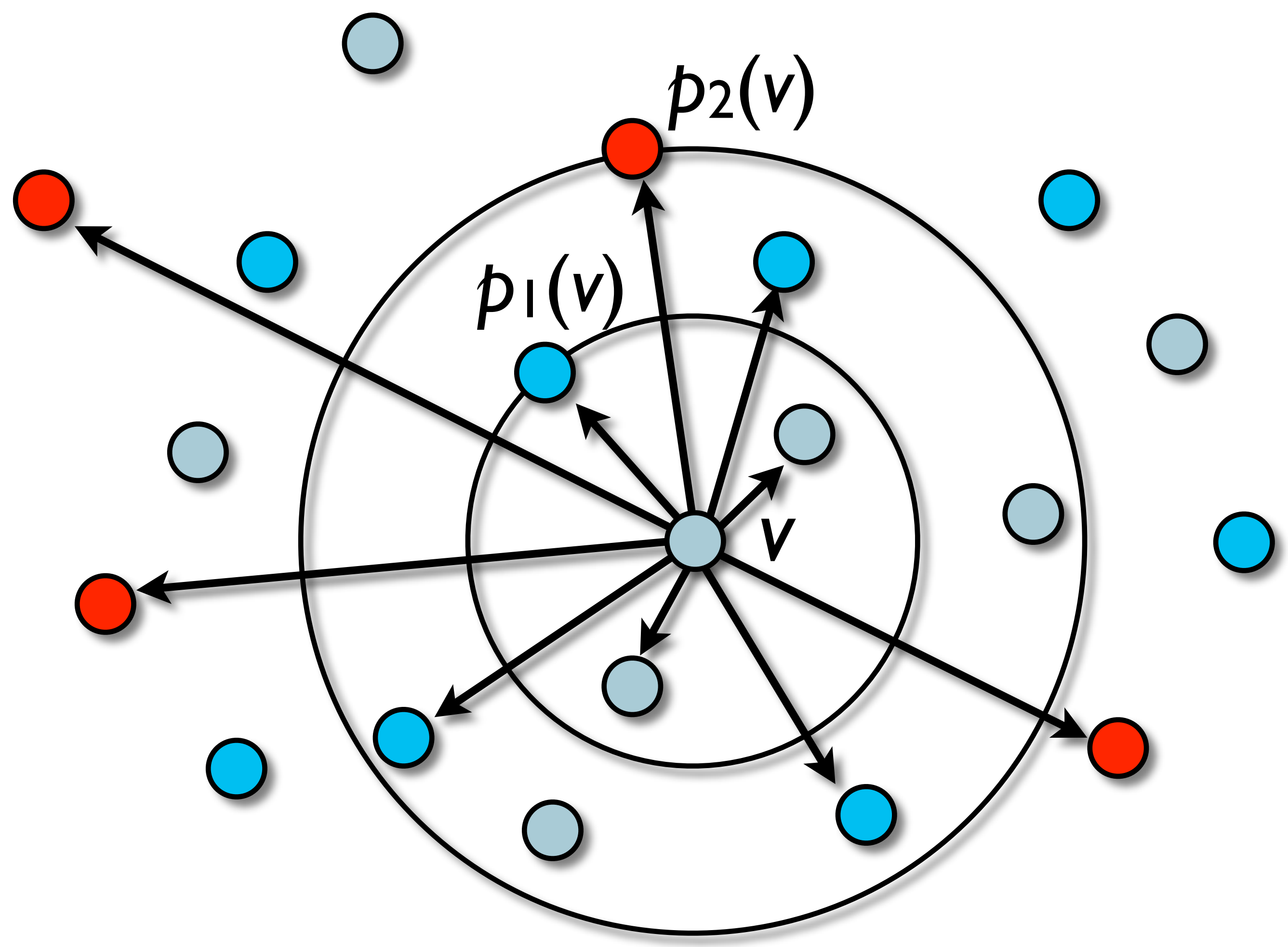


	$\delta(A_i,v)$					$p_i(v)$				
	0	1	2	3	4	0	1	2	3	4
A	0	1	3	4	∞	A	B	F	E	$\perp$
B	0	0	3	5	∞	B	B	F	E	$\perp$
C	0	1	1	3	∞	C	F	F	E	$\perp$
D	0	2	2	4	∞	D	F	F	E	$\perp$
E	0	0	0	0	∞	E	E	E	E	$\perp$
F	0	0	0	2	∞	F	F	F	E	$\perp$
G	0	0	4	4	∞	G	G	E	E	$\perp$
H	0	2	4	6	∞	H	G	F	E	$\perp$

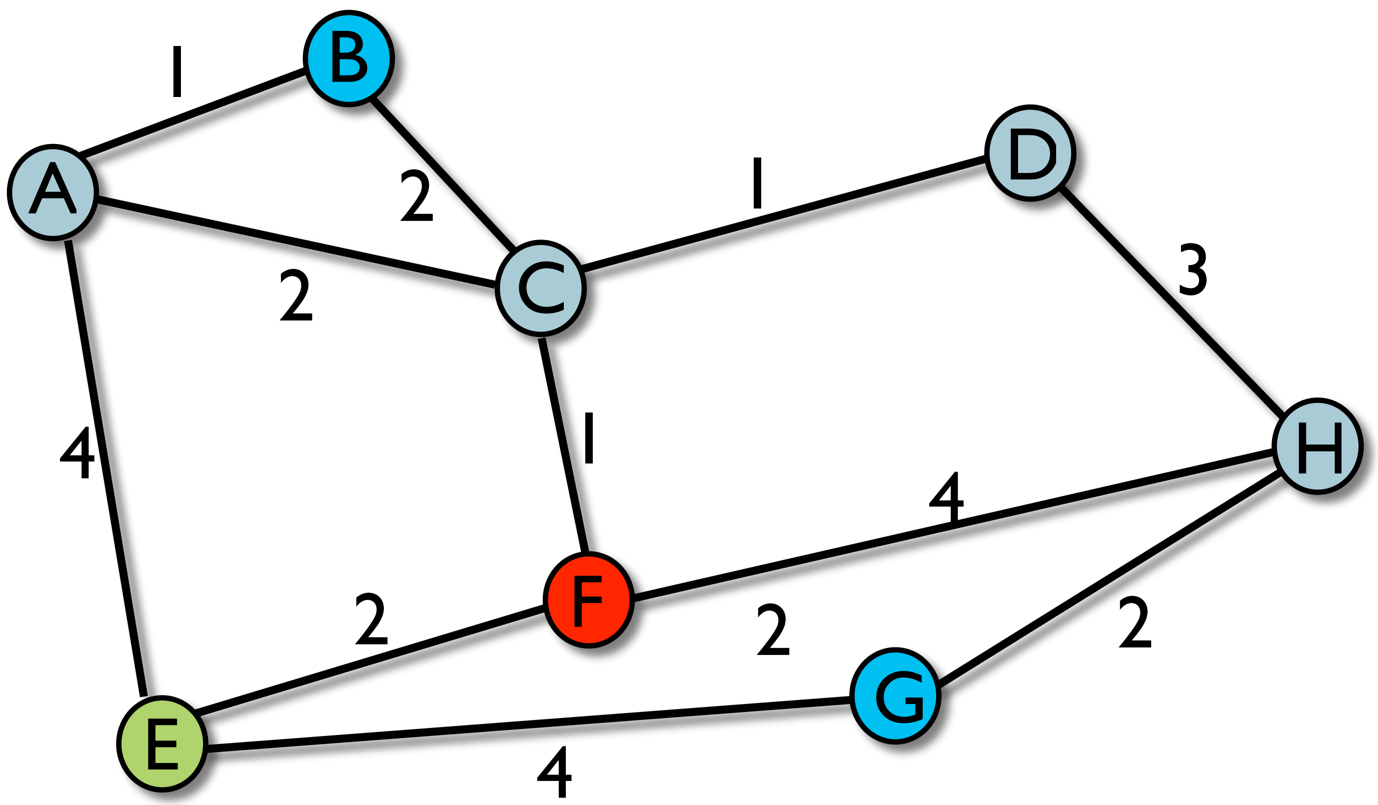
size $O(kn)$

Bunches

- **bunch** $B(v)$ of $v \in V$:
  - $w \in B(v) \Leftrightarrow \exists i : w \in A_i \setminus A_{i+1}$ and $\delta(w,v) < \delta(A_{i+1},v)$



Example



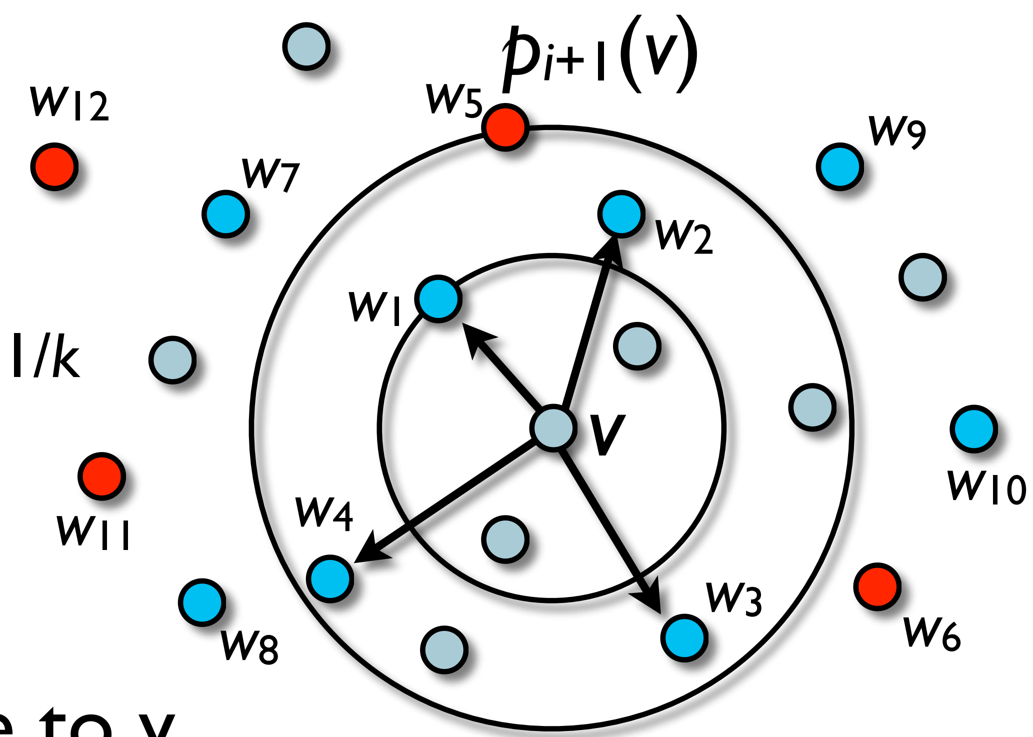
	$\delta(A_i,v)$					$p_i(v)$				
	0	1	2	3	4	0	1	2	3	4
A	0	1	3	4	∞	A	B	F	E	⊥
B	0	0	3	5	∞	B	B	F	E	⊥
C	0	1	1	3	∞	C	F	F	E	⊥
D	0	2	2	4	∞	D	F	F	E	⊥
E	0	0	0	0	∞	E	E	E	E	⊥
F	0	0	0	2	∞	F	F	F	E	⊥
G	0	0	4	4	∞	G	G	E	E	⊥
H	0	2	4	6	∞	H	G	F	E	⊥

$B(\text{A})=\{\text{A, B, F, E}\}$

$1=\delta(\text{A},\text{B}) < \delta(A_2,\text{A})=3$

Bunches

- Store bunches in **perfect hash table**
  - can tell in O(1) time if $w \in B(v)$...
  - ...and if so, what is $\delta(w,v)$
- **Note**: $\mathrm{Exp}[|B(v)|] \leq kn^{1/k}$
  - show $\mathrm{Exp}[|B(v) \cap (A_i \setminus A_{i+1})|] \leq n^{1/k}$
  - Trivial for $i=k-1$. For $i<k-1$:
    - $w_1, w_2, \ldots, w_x$ ordered by distance to v
    - $w_j \in B(v) \Rightarrow \delta(w_j,v) < \delta(A_{i+1},v) \Rightarrow w_1,\ldots,w_{j-1} \notin A_{i+1}$
    - $\mathrm{Prob}[w_j \in B(v)] \leq (1-n^{-1/k})^j \Rightarrow \ldots \Rightarrow \mathrm{Exp}[|B(v)|] \leq kn^{1/k}$



Query Algorithm

function $dist_k(u,v)$:

  $w \leftarrow u; i \leftarrow 0;$

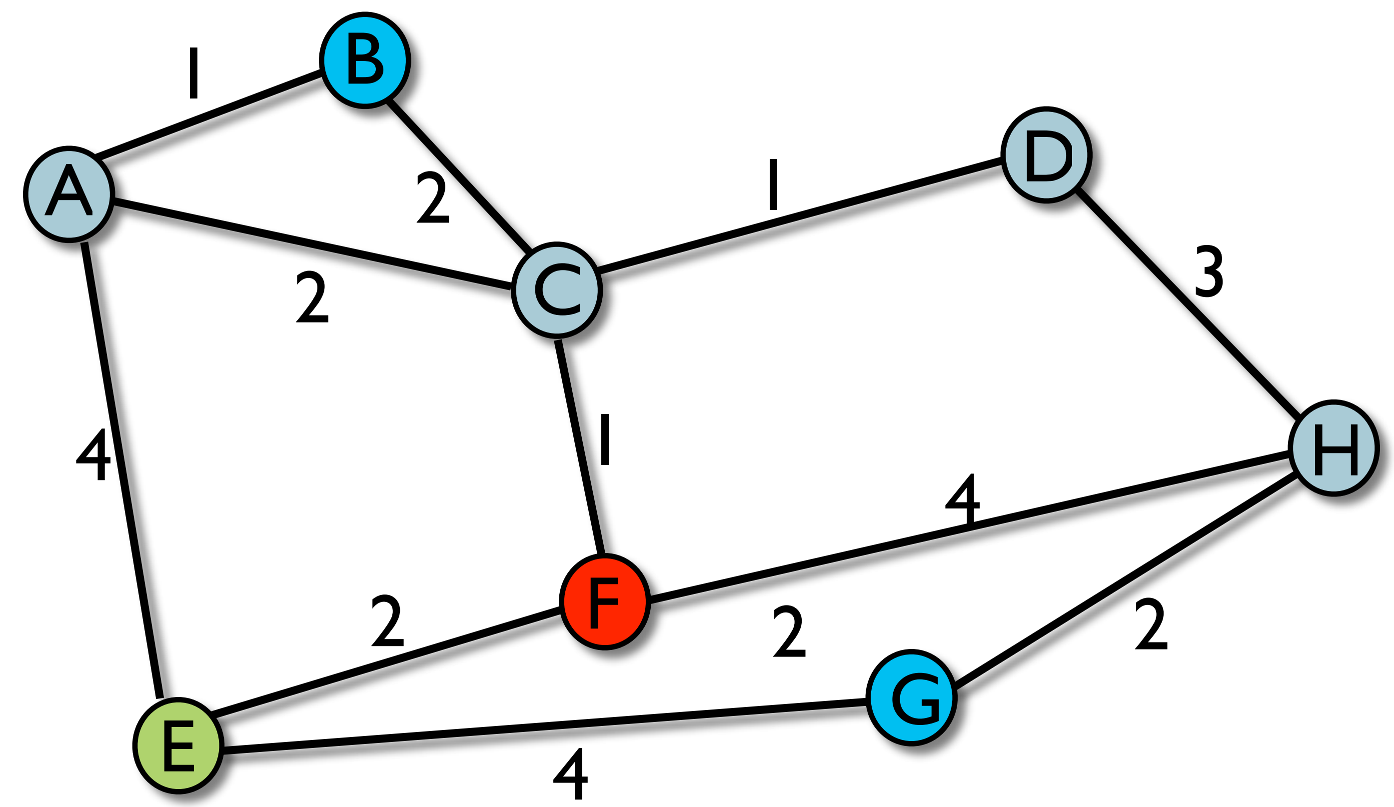
  while $(w \notin B(v))$

    $i \leftarrow i+1$

    $w \leftarrow p_i(v)$

    $(u,v) \leftarrow (v,u)$

  return $\delta(w,u) + \delta(w,v)$



- **prove** $dist_k(u,v) \leq (2k\text{-}1) \cdot \delta(v,u)$
  - every iteration increases $\delta(w,u)$ by $\leq \delta(v,u) = \Delta$
  - $\Rightarrow \delta(w,u) \leq (k\text{-}1) \cdot \Delta \Rightarrow dist_k(u,v) = \delta(w,u) + \delta(w,v) \Rightarrow$ ✓