Karlsruhe Institute of Technology

Advanced Data Structures

Lecture 08: Temporal Data Structures 2

Florian Kurpicz

Organization

Exams

- 10.08.2022 and 29.09.2022
- write to `blancani@kit.edu`
  - full name
  - Matrikelnummer
  - PO version
  - date
- online or in person ℹ depending on situation/personal preferences
- 18.07.2022 Q&A during last half of lecture

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Evaluation

- now

PINGO

`https://pingo.scc.kit.edu/329558`

Recap: Persistent Data Structures

- lecture based on: `http://courses.csail.mit.edu/6.851/spring12/lectures/L01`

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Persistence

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Definition: Partial Persistence

Only the latest version can be updated

Definition: Full Persistence

Any version can be updated

Definition: Confluent Persistence

Like full persistence, but two versions can be combined to a new version

Definition: Functional

Nodes cannot be modified, only new nodes can be created

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Persistence

- change in the past creates new branch
- similar to version control
- everything old/new remains the same

Retroactivity

- change in the past affects future
- make change in earlier version changes all later versions

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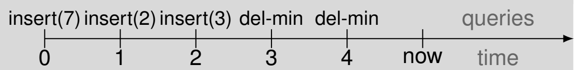
Retroactive Data Structures

Operations

- $\texttt{INSERT}(t, \mathit{operation})$: insert operation at time t
- $\texttt{DELETE}(t)$: delete operation at time t
- $\texttt{QUERY}(t, \mathit{query})$: ask *query* at time t

- for a priority queue updates are
  - insert
  - delete-min
- time is integer ℹ for simplicity otherwise use order-maintenance data structure



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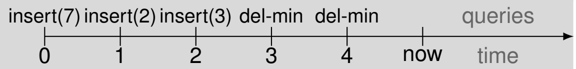

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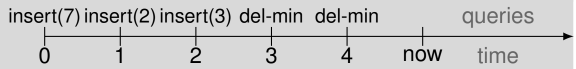

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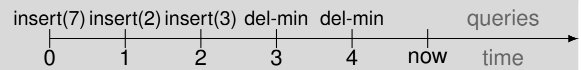

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`QUERY` is only allowed for $t = \infty$ ℹ now

Definition: Full Retroactivity

`QUERY` is allowed at any time t

Definition: Nonoblivious Retroactivity

`INSERT`, `DELETE`, and `QUERY` at any time t but also identify changed `QUERY` results

Easy Cases: Partial Retroactivity

- commutative operations
  - insert and delete-min are not commutative
  - insert and delete are commutative
- invertible updates
  - operation op^{-1} such that $op^{-}1(op(\cdot)) = \emptyset$
  - `DELETE` becomes `INSERT` inverse operation
- makes partial retroactivity easy
- $\texttt{INSERT}(t, \mathit{operation}) = \texttt{INSERT}(\infty, \mathit{operation})$
- $\texttt{DELETE}(t, op) = \texttt{INSERT}(\infty, op^{-1})$

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Partial Retroactivity

- hashing
- dynamic dictionaries
- array with updates only ℹ $A[i] += \mathit{value}$

Search Problems

Definition: Search Problem

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- range minimum queries

- nearest neighbor
- point location
- ...

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- point location
- . . .

- these types of problems are also “easy”

Decomposable Search Problems: Full Retroactivity

Lemma: Full Retroactivity for DSP

Every decomposable search problems can be made fully retroactive with a $O(\log m)$ overhead in space and time, where m is the number of operations

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Proof (Sketch)

- use balances search tree
- each leaf corresponds to an update
- node n corresponds to interval of time $[s_n, e_n]$
- if an object exists in the time interval $[s, e]$, then it appears in all node n if $[s_n, e_n] \subseteq [s, e]$ if non of n's ancestors' are $\subseteq [s, e]$
- each object occurs in $O(\log n)$ nodes

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- to query find leaf corresponding to t
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- $O(\log m)$ results which can be combined in $O(\log m)$ time

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- data structure is stored for each operation!
- $O(\log m)$ space overhead!

General Full Retroactivity

Lemma: Lower Bound

Rewinding m operations has a lower bound of $\Omega(m)$ overhead

- general case

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Proof (Sketch)

- two values X and Y
- initially $X = \emptyset$ and $Y = \emptyset$
- supported operations
  - $X = x$
  - $Y += \mathit{value}$
  - $Y = X \cdot Y$
  - $\mathit{query}\ Y$

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Proof (Sketch, cnt.)

- perform operations
  - $Y+ = a_n$
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  - $Y+ = a_{n=1}$
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  - $\dots$
  - $Y+ = a_0$
- what are we computing here? **PINGO**

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- $Y = a_n \cdot X^n + a_{n-1} X^{n-1} + \cdots + a_0$

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- evaluate polynomial at $X = x$ using `t=0,X=x`

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- $Y = a_n \cdot X^n + a_{n-1}X^{n-1} + \cdots + a_0$
- evaluate polynomial at $X = x$ using `t=0,X=x`
- this requires $\Omega(n)$ time [FHM01]

Priority Queues: Partial Retroactivity (1/6)

- priority queue with
  - insert
  - delete-min
- delete-min makes PQ non-commutative

Lemma: Partial Retroactive PQ

A priority queue can be partial retroactive with only $O(\log n)$ overhead per partially retroactive operation

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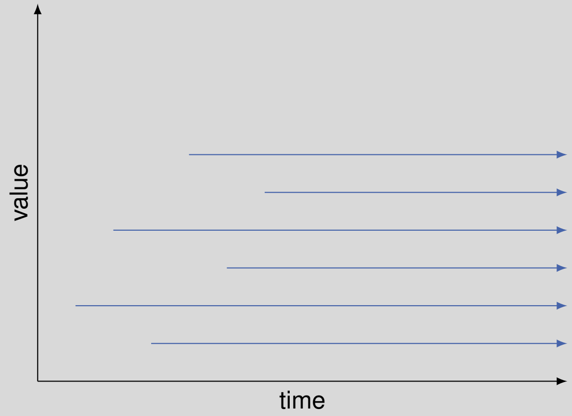

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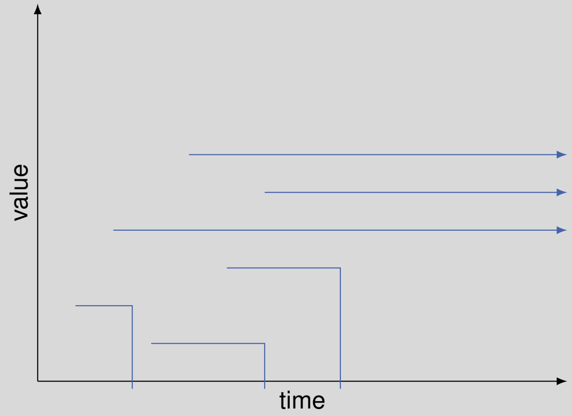

Priority Queues: Partial Retroactivity (2/6)

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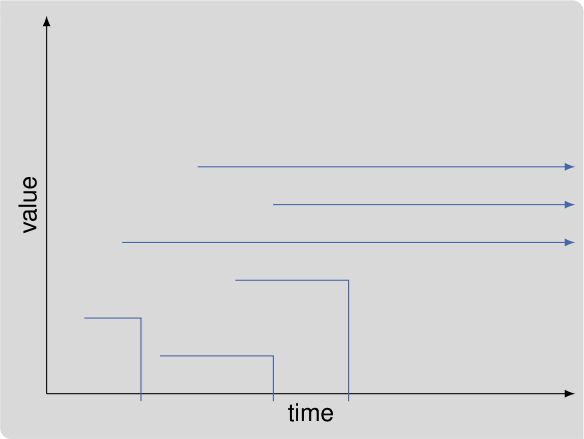

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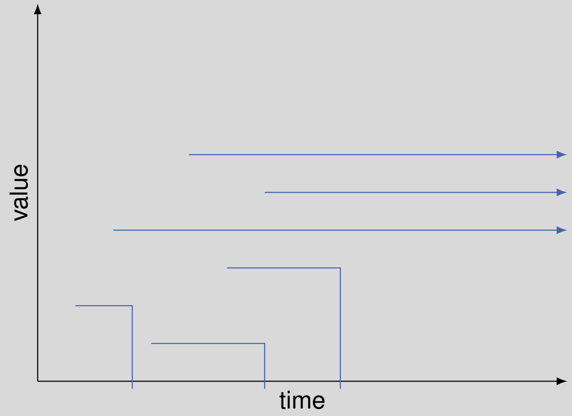

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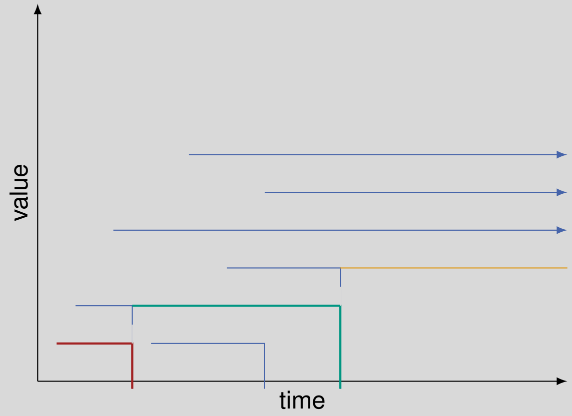

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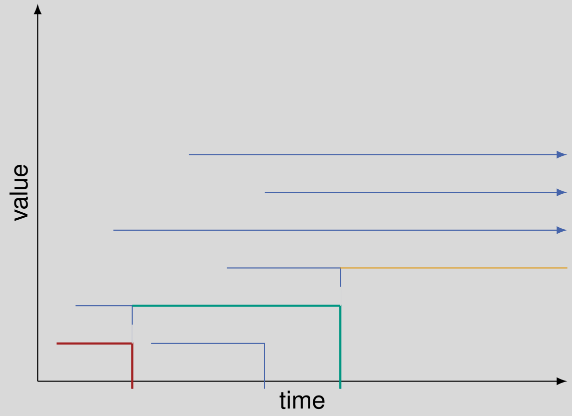

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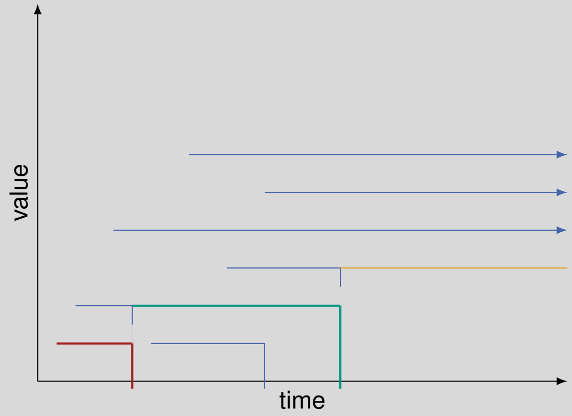
- insert deleted minimum right after deletion



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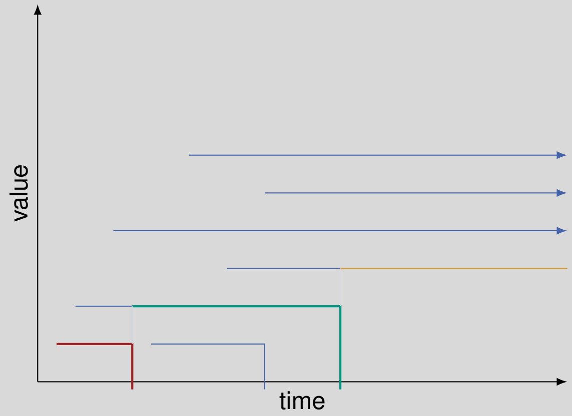
- let Q_t be elements in PQ at time t

- what values are in Q_∞? partial retroactivity
- what value inserts $\texttt{INSERT}(t, \text{insert}(v))$ in Q_∞
- values is $\max\{v, v' : v' \text{ deleted at time} \geq t\}$
- maintaining deleted elements is hard can change a lot



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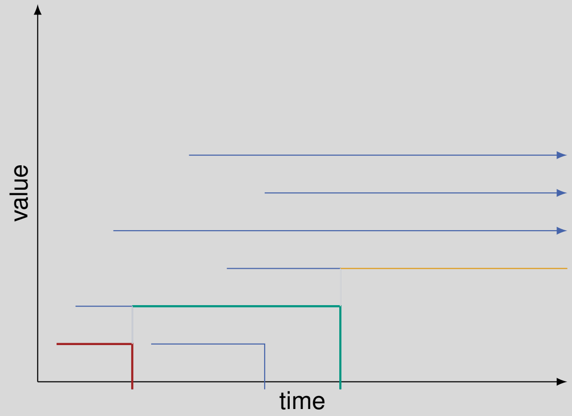

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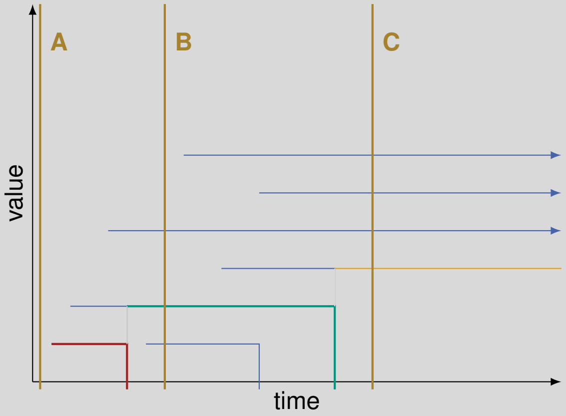

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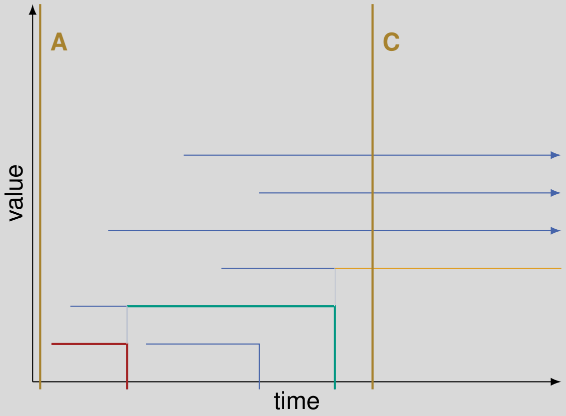

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Priority Queues: Partial Retroactivity (4/6)

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Lemma: Deletions after Bridges

If time t' is closest bridge preceding time t, then

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Proof (Sketch)

- $\max\{v' \notin Q_\infty : v' \text{ inserted at time} \geq t'\} \in \{v' : v' \text{ deleted at time} \geq t\}$
  - if maximum value is deleted between t' and t
  - then this time is a bridge
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Proof (Sketch, cnt.)

- $\max\{v' : v' \text{ deleted at time} \geq t\} \in \{v' \notin Q_\infty : v' \text{ inserted at time} \geq t'\}$
  - if v' is deleted at some time $\geq t$
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- keep track of inserted values
- use balanced binary search trees for $O(\log n)$ overhead

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- BBST where leaves are inserts ordered by time augmented with
  - for each node x store $\max\{v' \notin Q_\infty : v' \text{ inserted in subtree of } x\}$

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- BBST where leaves are inserts ordered by time augmented with
  - for each node x store $\max\{v' \notin Q_\infty : v' \text{ inserted in subtree of } x\}$
- BBST where leaves are all updates ordered by time augmented with
  - leaves store 0 for inserts with $v \in Q_\infty$, 1 for inserts with $v \notin Q_\infty$ and -1 for delete-mins
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Priority Queues: Partial Retroactivity (5/6)

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- use balanced binary search trees for $O(\log n)$ overhead

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Karlsruhe Institute of Technology

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- update all BBSTs in $O(\log n)$ time

Priority Queues: Partial Retroactivity (6/6)

Lemma: Partial Retroactive PQ

A priority queue can be partial retroactive with only $O(\log n)$ overhead per partially retroactive operation

- requires three BBSTs
- updates need to update all BBSTs

Nonoblivious Retroactivity

- priority queue with
  - insert
  - delete
  - min

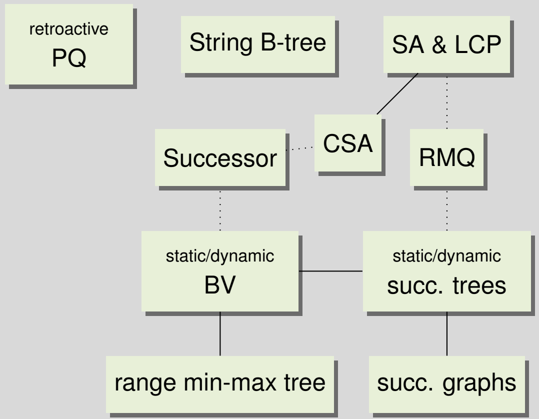
- identify queries that are now incorrect
- using ray shooting

Conclusion and Outlook

This Lecture

- retroactive data structures

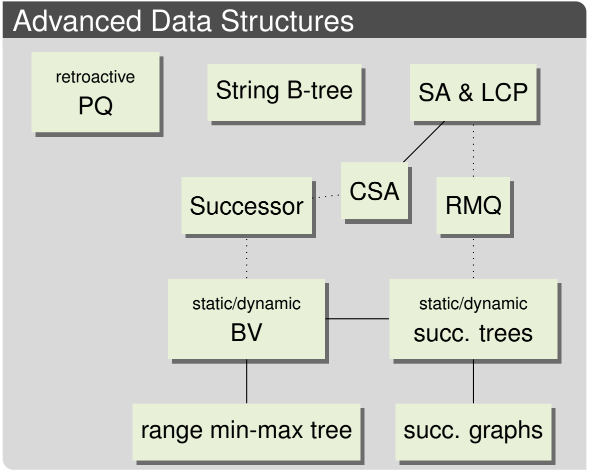
Advanced Data Structures



Conclusion and Outlook

This Lecture
- retroactive data structures

Next Lecture
- geometric data structures



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