Karlsruhe Institute of Technology

Advanced Data Structures

Lecture 01: Bit Vectors

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PINGO

https://pingo.scc.kit.edu/424928

Bit Vectors

Succinct Data Structures

- represent data structures space efficient
- close to their information theoretical minimum
- using every bit becomes necessary

Succinct Trees

- represent a tree with n nodes using only $2n$ bits
- navigation is possible with additional $o(n)$ bits

- storing a bit vector in practice is tricky
- `11011101` should require only a single byte

Efficient Bit Vectors in Practice (1/3)

`std::vector<char/int/...>`

- easy access
- very big: $1, 4, \ldots$ bytes per bit

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- requires 8 bytes per bit(?)
- store 64 bits in 8 bytes
- how to access bits

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- $i\%64$ is position in 64-bit word

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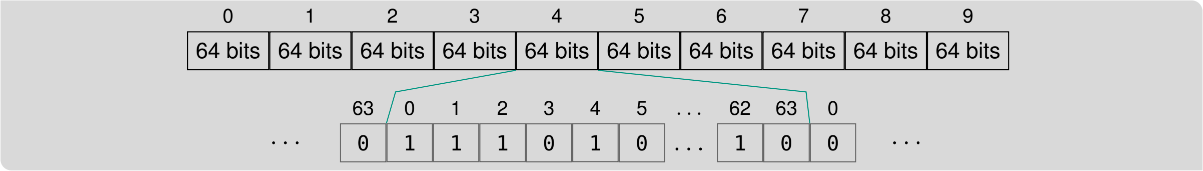
0 1 2 3 4 5 6 7 8 9
64 bits 64 bits 64 bits 64 bits 64 bits 64 bits 64 bits 64 bits 64 bits 64 bits
63 0 1 2 3 4 5 ... 62 63 0
... 0 1 1 1 0 1 0 ... 1 0 0 ...

Efficient Bit Vectors in Practice (2/3)

```cpp
// There is a bit vector
std::vector<uint64_t> bit_vector;

// access i-th bit
uint64_t block = bit_vector[i/64];
bool bit = (block >> ( 63 - (i % 64)) )  & 1ULL;
```

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shift bits right

0	1	2	3	4	5	...	62	63
1	1	1	0	1	0	...	1	0

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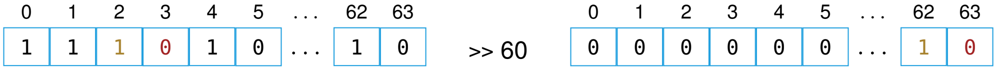

0	1	2	3	4	5	...	62	63
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>> 60

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# bits

logical and 1

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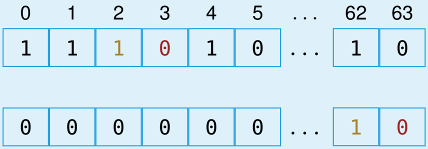
Efficient Bit Vectors in Practice (3/3)

`(block >> (63-(i%64))) & 1ULL;`

- fill bit vector from left to right



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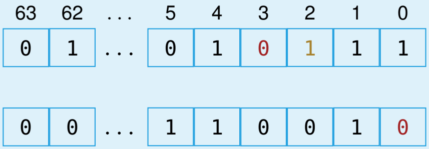
- fill blocks in bit vector right to left



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- assembler code:

```
mov  ecx,  edi
not  ecx
shr  rsi,  cl
mov  eax,  esi
and  eax,  1
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Rank Queries on Bit Vectors (1/2)

$\text{rank}_\alpha(i)$ # of αs before i

$\text{select}_\alpha(j)$ position of j-th α

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	1	0	0

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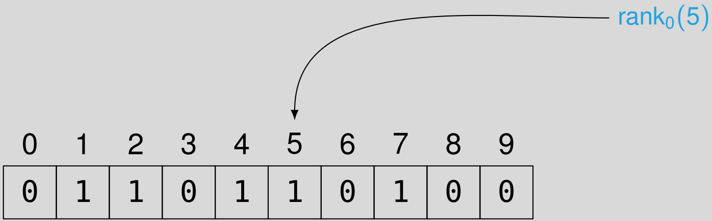

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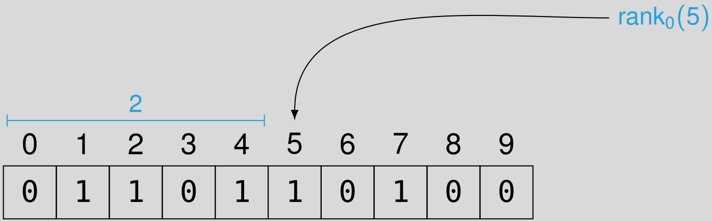

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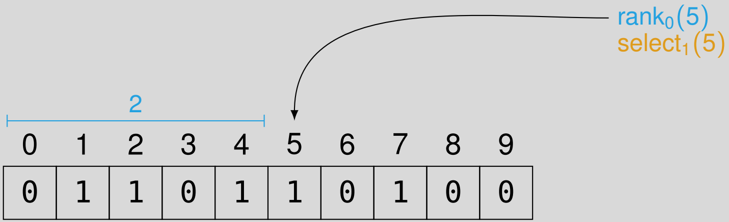

$\text{rank}_0(5)$
$\text{select}_1(5)$

0	1	2	3	4	5	6	7	8	9
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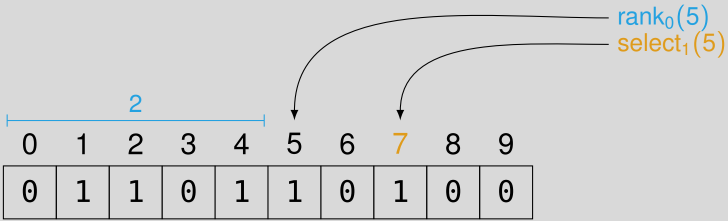

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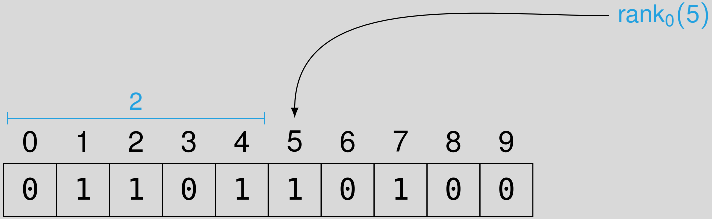
PINGO-Frage



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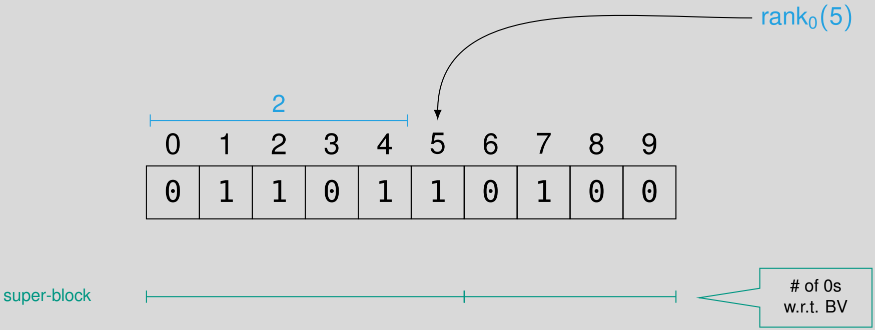

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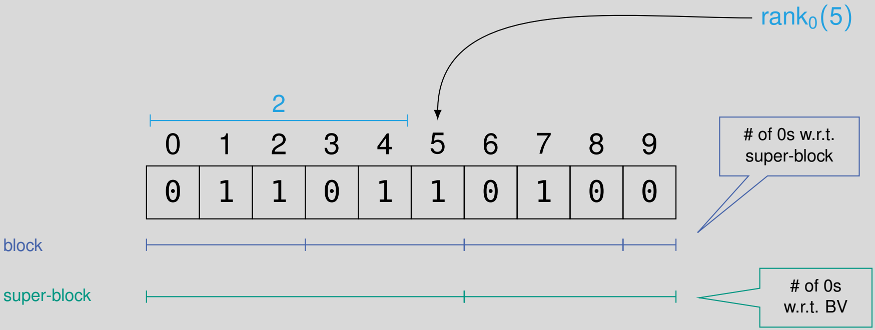

# of 0s w.r.t. super-block

of 0s w.r.t. BV

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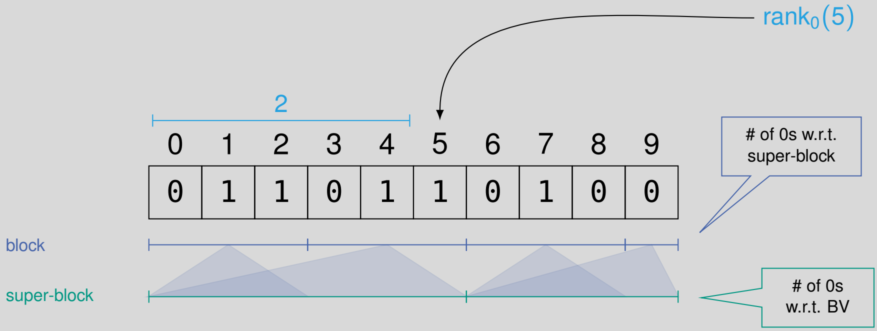

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- blocks of size $s = \lfloor \frac{\lg n}{2} \rfloor$
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- for all length-s bit vectors, for every position i store number of 0s up to i
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- query in $O(1)$ time
- $rank_0(i) = i - rank_1(i)$

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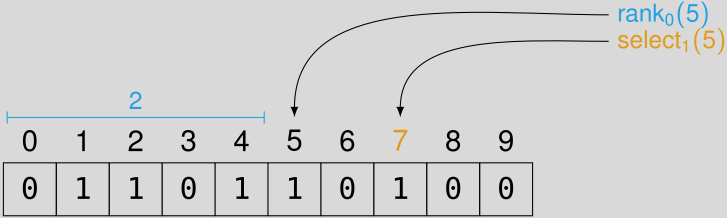

Select in $o(n)$ Space and $O(1)$ Time

- $select_0$ in a bit vector of size n that contains k zeros
- **PINGO-Frage**

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  - scan bit vector: $O(n)$ time and no space overhead
  - store k solutions in $S[1..k]$ and $select_0(i) = S[i]$ ℹ if $k \in O(n/lgn)$ this suffice

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- better: k/b variable-sized super-blocks B_i, such that super-block contains $b = \lg^2 n$ zeros
- $select_0(i) =$ $\sum_{j=0}^{\lfloor i/b \rfloor - 1} |B_j| + select_0(B_{\lfloor i/b \rfloor}, j - (\lfloor i/b \rfloor b))$

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- $O((k \lg n)/b) = o(n)$ bits of space

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- select on block depends on size of block
- $|B_{\lfloor i/b \rfloor}| \geq \lg^4 n$: store answers naively
  - requires $O(b \lg n) = O(\lg^3 n)$ bits of space
  - there are at most $O(n/\lg^4 n)$ such blocks
  - total $O(n/\lg n) = o(n)$ bits of space

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  - total $O(n/\lg n) = o(n)$ bits of space
- $|B_{\lfloor i/b \rfloor}| < \lg^4 n$: divide super-block into blocks
  - same idea: variable-sized blocks containing $b' = \sqrt{\lg n}$ zeros
  - (prefix) sum $O((k \lg \lg n)/b') = o(n)$ bits
  - if size $\geq \lg n$ store all answers
  - if size $< \lg n$ store lookup table

Rank- and Select-Queries on Bit Vectors

Lemma: Binary Rank- and Select-Queries

Given a bit vector of size n, there exist data structures that can be computed in time $O(n)$ of size $o(n)$ bits that can answer rank and select queries on the bit vector in $O(1)$ time

Conclusion and Outlook

This Lecture

- bit vectors
- rank and select on bit vectors

Advanced Data Structures

BV

Conclusion and Outlook

This Lecture

- bit vectors
- rank and select on bit vectors

- efficient bit vectors in practice

Advanced Data Structures

BV

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Next Lecture

- succinct trees using bit vectors
- navigation in succinct trees

Advanced Data Structures

BV