Karlsruhe Institute of Technology

Advanced Data Structures

Lecture 02: Succinct Trees

Florian Kurpicz

PINGO

https://pingo.scc.kit.edu/306589

Recap: Rank Queries on Bit Vectors (1/2)

$\text{rank}_\alpha(i)$ # of αs before i

$\text{select}_\alpha(j)$ position of j-th α

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	1	0	0

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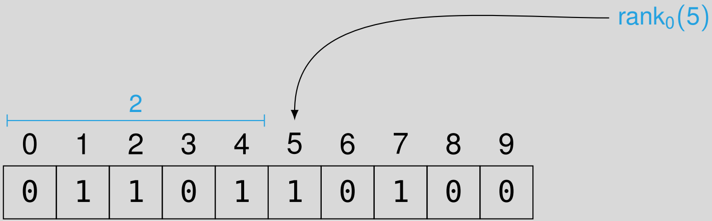

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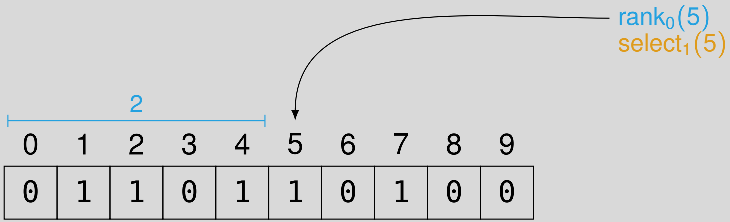

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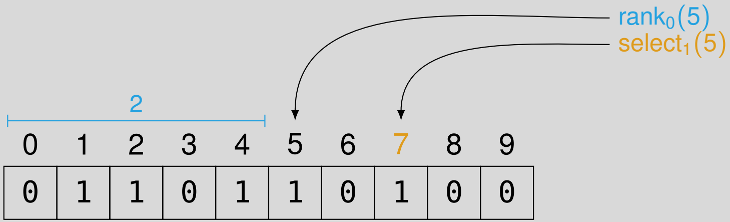

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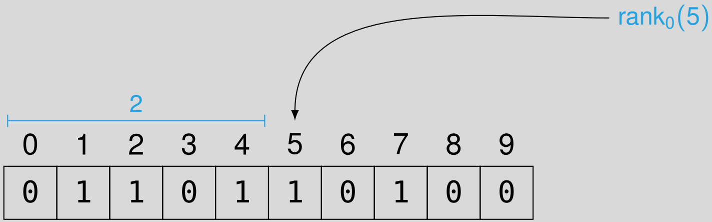

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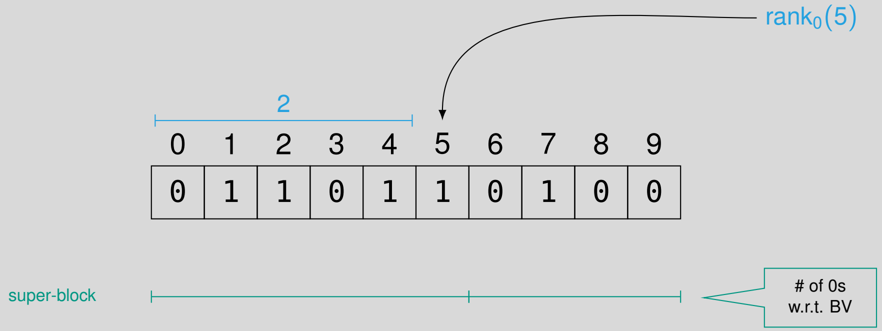

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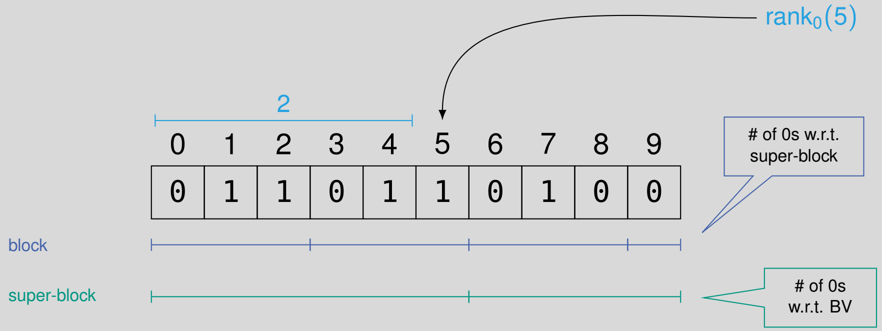

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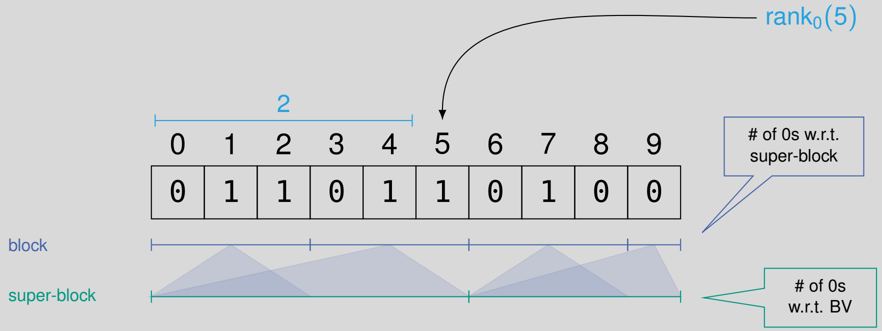

Recap: Rank Queries on Bit Vectors (2/2)

Lemma: Binary Rank- and Select-Queries

Given a bit vector of size n, there exist data structures that can be computed in time $O(n)$ of size $o(n)$ bits that can answer rank and select queries on the bit vector in $O(1)$ time

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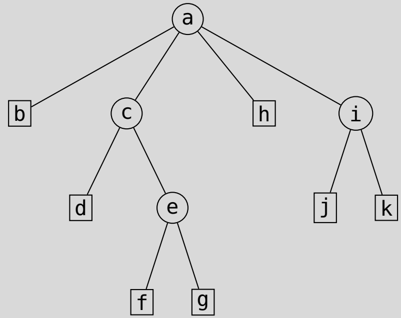
Given a bit vector of size n, there exist data structures that can be computed in time $O(n)$ of size $o(n)$ bits that can answer rank and select queries on the bit vector in $O(1)$ time

Word RAM

- unlimited memory
- words of size w ℹ $w = \Theta \log n$
- constant time load and store
- constant time bit operations on words

Plan for Today

- represent tree with n nodes using $2n$ bits
- make succinct tree fully-functional using additional $o(n)$ bits



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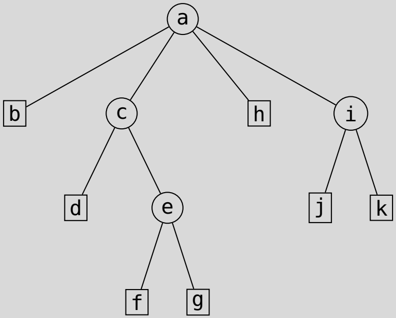
  - searching for keys
  - maintaining directories
  - representations of parsings
  - . . .



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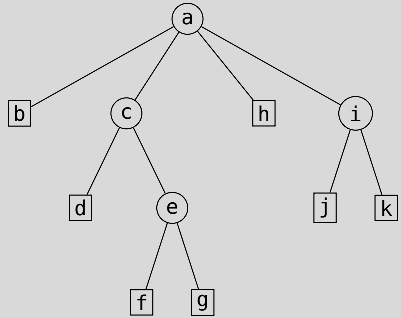

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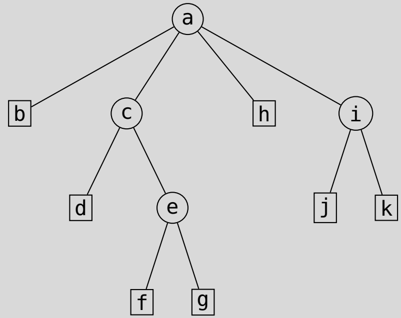
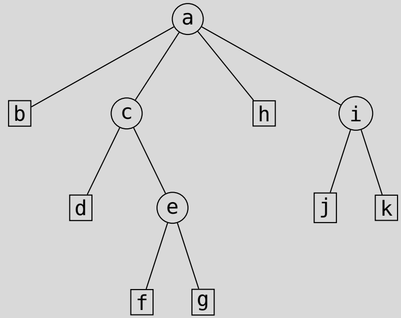


Handout

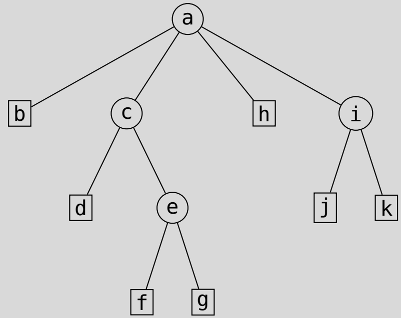
Preliminaries

- a tree is an acyclic connected graph $G = (V, E)$ with a root $r \in V$
- degree δ is the number of children
- leaves have degree 0
- depth of a node is the length of the path from the root to that node



Level Ordered Unary Degree Sequence (1/2) [Jac88]

- represent tree level-wise
- use ≤ 2 bits per node



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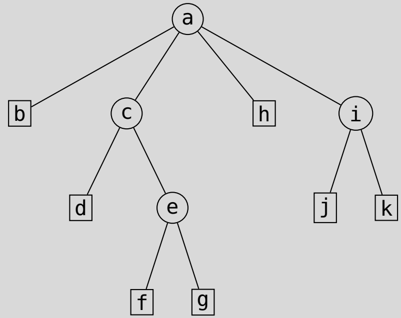
Definition: LOUDS

Starting at the root, all nodes on the same depth

- are visited from left to right and
- for node v, $\delta(v)$ 1's followed by a 0 are

appended to the bit vector that contains an initial `10`



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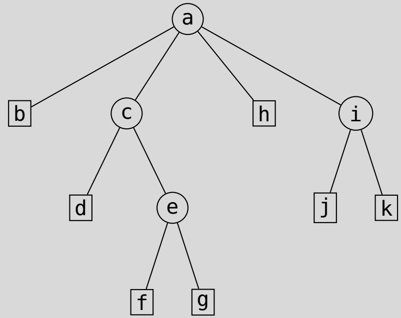
Lemma: Space Usage of LOUDS

Representing a tree with n nodes requires $2n + 1$ bits using LOUDS



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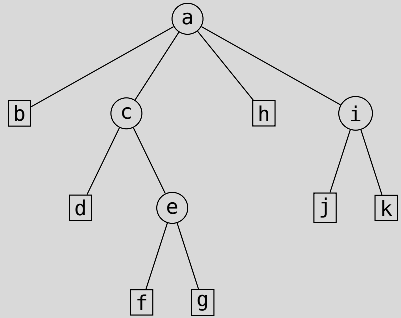
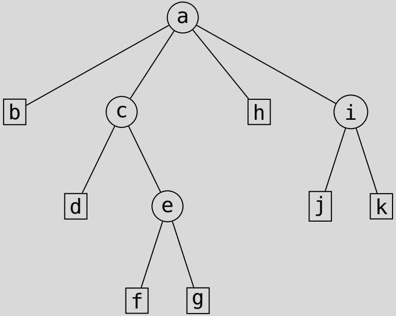


- write down the LOUDS representation of this example tree

Level Ordered Unary Degree Sequence (2/2)

```
1011110
```



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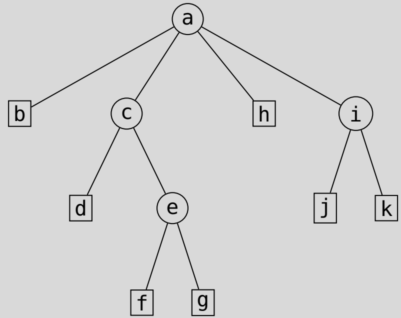
10111100110011 0



Level Ordered Unary Degree Sequence (2/2)

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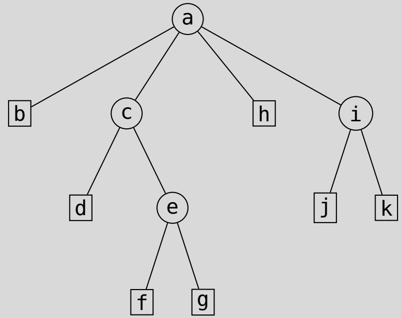

Level Ordered Unary Degree Sequence (2/2)

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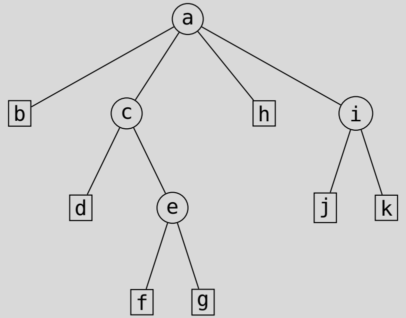
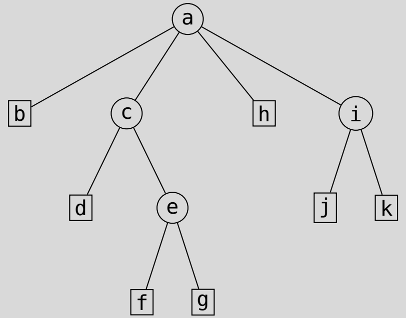


Level Ordered Unary Degree Sequence (2/2)

```
         ab  ch  id  ejkfg
      1011110011001100110000
```

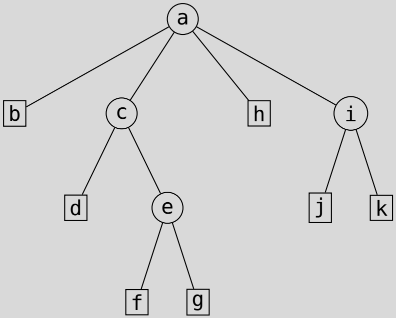
- node start at pertinent `0`



What is Fully-Functional?

Operations

- degree ⓘ is leaf
- *i*-th child
- parent
- subtree size



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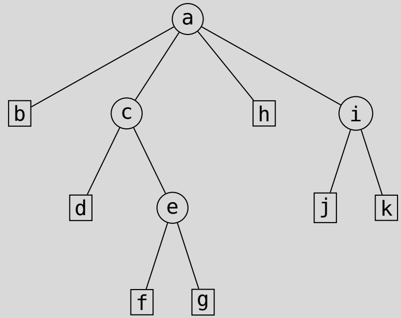
- depth
- lowest common ancestor
- rank (pre- or post-order)
- . . .

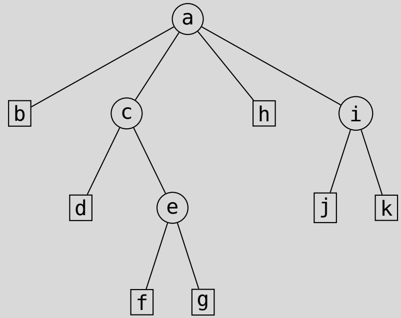


Making LOUDS Fully-Functional

```
        ab   ch   id   ejkfg
1011110011001100110000
```

- degree of p: $p - \mathit{select}_0(\mathit{rank}_0(p)) - 1$

- explanation on the board



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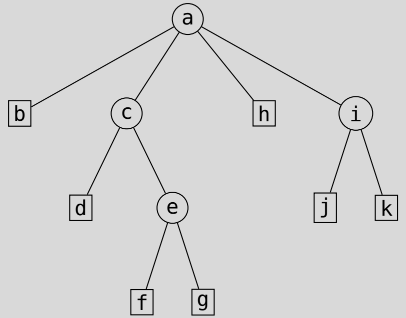

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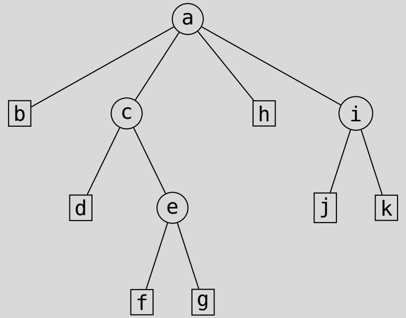

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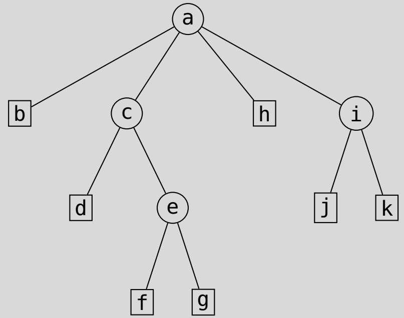
- subtree size **PINGO**



From Bit Vectors to Parentheses

- instead of `0` and `1`
- use `(` and `)`

- requires the same space
- can add relation between parentheses

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- how can we answer *excess* queries **PINGO**

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- here, a little bit simpler

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- $excess(i) = rank_{\text{“(”}}(i) - rank_{\text{“)”}}(i)$
- $fwd\_search(i, d) = \min\{j > i \colon excess(j) - excess(i-1) = d\}$
- $bwd\_search(i, d) = \max\{j < i \colon excess(i) - excess(j-1) = d\}$

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- $findclose(i) = fwd\_search(i, 0)$
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- can be answered with a min-max-tree

Range Min-Max Trees (1/2)

Definition: Range Min-Max Tree

Given a bit vector B of length n and a block size b, store for each consecutive block (from s to e) of BV

- total excess in block:
  $excess(e) - excess(s-1)$
- minimum left-to-right excess in block:
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and build a binary tree over these blocks, where each node stores the same total information for blocks in all its leaves

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Lemma: Range Min-Max Tree Space

A range min-max tree with block size b for a bit vector of size n requires $n + O((n/b)\log n)$ bits of space

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fwdsearch in a Range Min-Max Tree

- scan block
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- scanning last block requires $O(c + b/c)$ time

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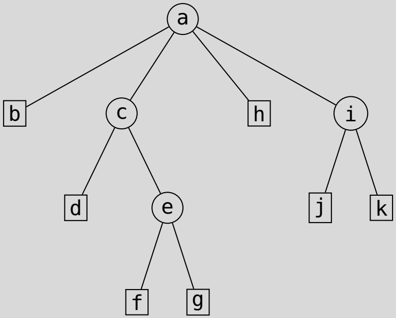
Improvements

- two level approach
- build range min-max trees for chunks of size $\Theta(\log^3 n)$
- $O(\log \log n)$ query time inside a chunk
- can result in total query time of $O(\log \log n)$

Balanced Parentheses (1/2) [MR01]

- represent tree as depth-first traversal
- using balanced parentheses



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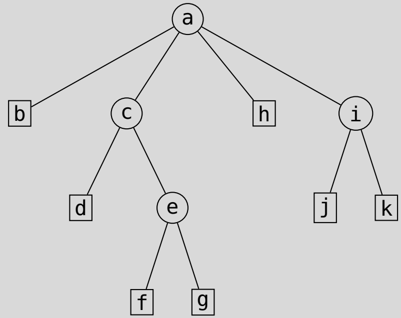

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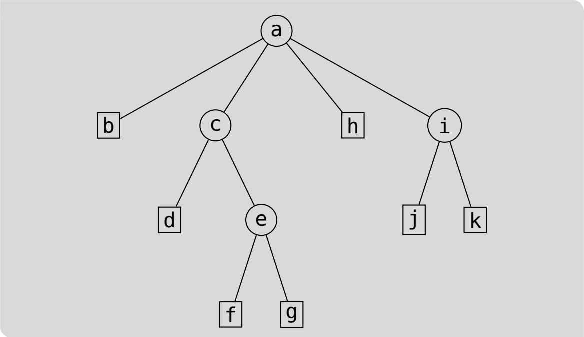
to the bit vector

Lemma: Space Usage of BP

Representing a tree with n nodes requires $2n$ bits using *BP*



Balanced Parentheses (1/2) [MR01]

- represent tree as depth-first traversal
- using balanced parentheses

Definition: BP

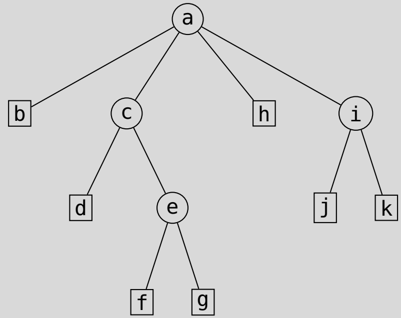
Starting at the root, traverse the tree in depth-first order and append a

- left parenthesis if a node is visited the first time
- right parenthesis if a node is visited the last time

to the bit vector

Lemma: Space Usage of BP

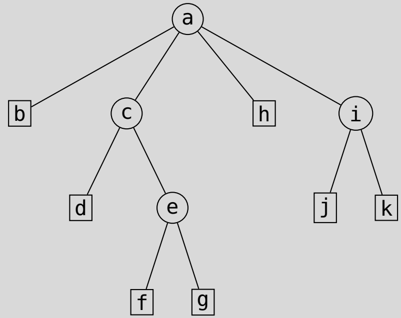
Representing a tree with n nodes requires $2n$ bits using *BP*



- write down the BP representation of this example tree

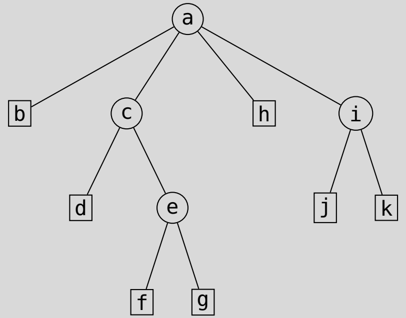
Balanced Parentheses (2/2)

a
(



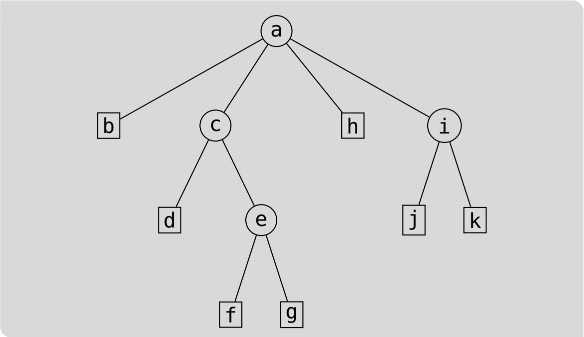
Balanced Parentheses (2/2)

```
ab
(()
```



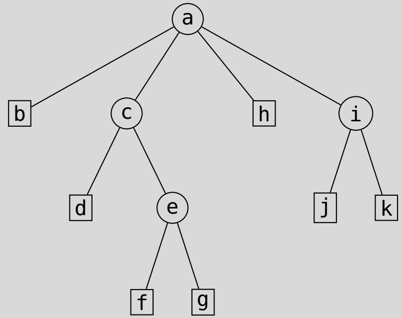
Balanced Parentheses (2/2)

```
ab cd
(()(()
```



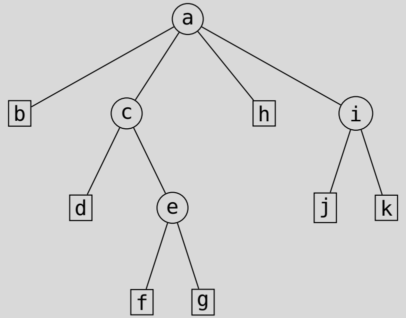
Balanced Parentheses (2/2)

```
ab cd ef g
(()(()(()())))
```



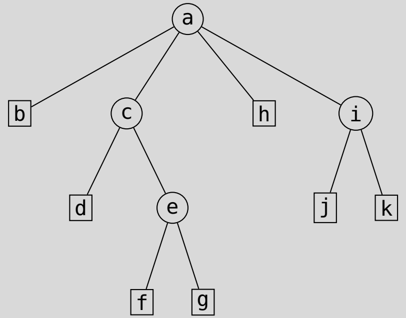
Balanced Parentheses (2/2)

```
ab cd ef g   h
(()((()(()())))()
```



Balanced Parentheses (2/2)

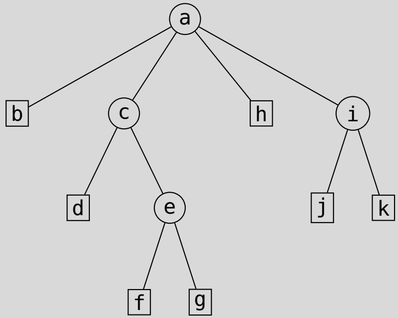
```
 ab cd ef g   h ij k
(()(()(()()))()(()()))
```



Balanced Parentheses (2/2)

```
ab cd ef g   h ij k
(()(()(()()))()(()()))
```

- node starts at first parenthesis
- subtree structure is encoded in parentheses

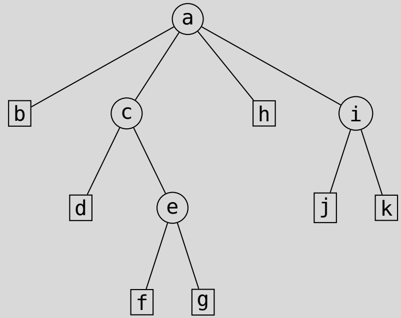


Making BP Fully-Functional

```
      ab cd ef g   h ij k
     (()(()(()()))()(()()))
```

- subtree size of p: $(\mathit{findclose}(p) - p + 1)/2$

- explanation on the board



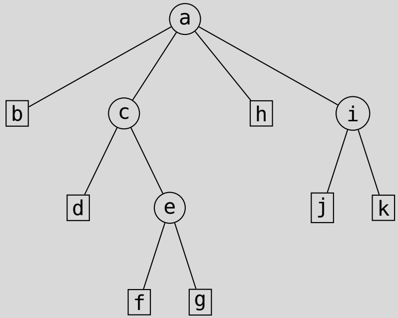
Making BP Fully-Functional

Karlsruhe Institute of Technology

```
      ab cd ef g   h ij k
      (()(()(()()))()(()()))
```

- subtree size of p: $(\mathit{findclose}(p) - p + 1)/2$
- parent of p: $\mathit{enclose}(p)$

- explanation on the board



Making BP Fully-Functional

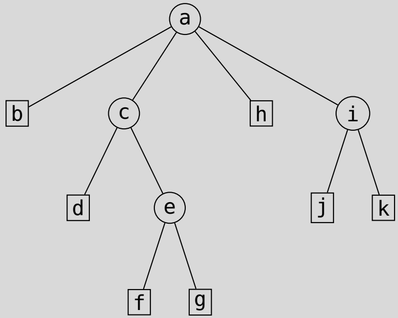
```
   ab cd ef g   h ij k
(()(()(()()))()(()()))
```

- subtree size of p: $(\mathit{findclose}(p) - p + 1)/2$
- parent of p: $\mathit{enclose}(p)$

- explanation on the board

Complicated Constant Time [NS14]

- degree
- i-th child



Advantages and Disadvantages of Both Approaches

- LOUDS cannot answer subtree size
- BP cannot easily answer i-th child and degree

- all other operations can be done easily

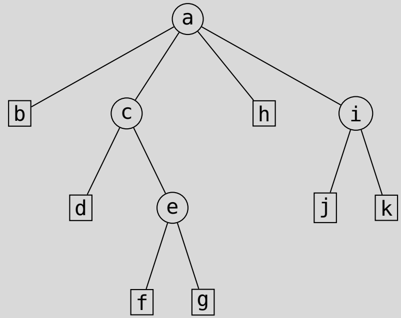
Depth First Unary Degree Sequence (1/2) [Ben+05]

Definition: DFUDS

Starting at the root, traverse tree in depth-first order and append

- for node v, $\delta(v)$ left parentheses and
- a right parenthesis if v is visited the first time

to the bit vector that initially contains a left parenthesis ℹ to make them balanced



Depth First Unary Degree Sequence (1/2) [Ben+05]

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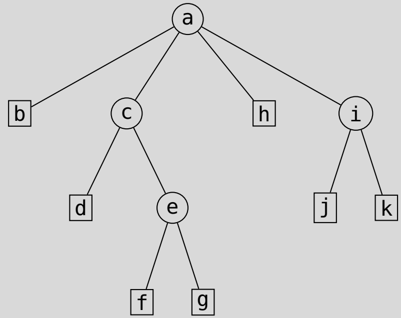
Lemma: Space Usage of DFUDS

Representing a tree with n nodes requires $2n$ bits using DFUDS



Depth First Unary Degree Sequence (1/2) [Ben+05]

Karlsruhe Institute of Technology

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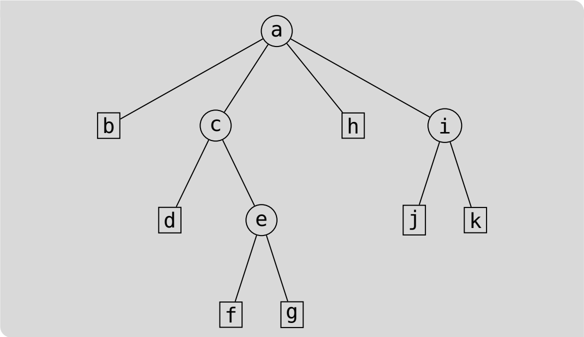
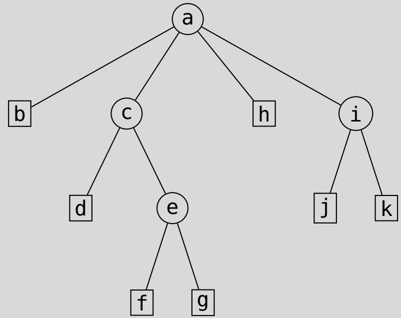


- write down the DFUDS representation of this example tree

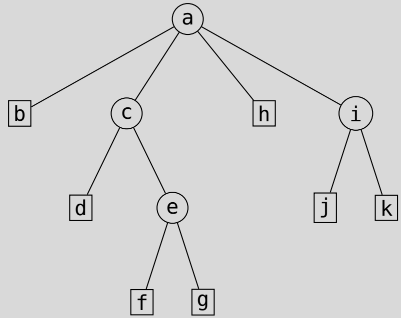
Depth First Unary Degree Sequence (2/2)

a
(((((



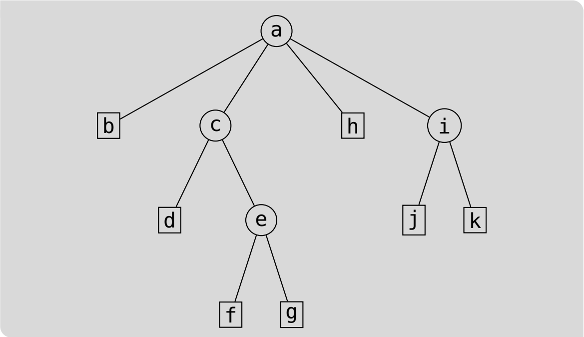
Depth First Unary Degree Sequence (2/2)

```
 a    b
((((()) 
```



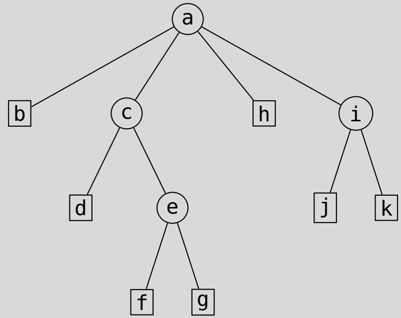
Depth First Unary Degree Sequence (2/2)

```
a    bc
((((())(()
```



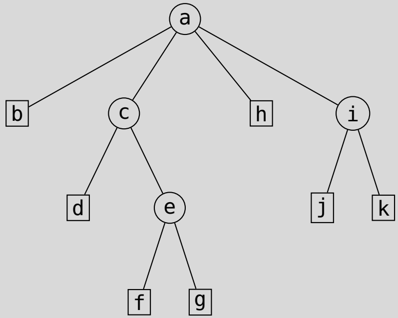
Depth First Unary Degree Sequence (2/2)

```
 a    bc  d
((((())(())
```



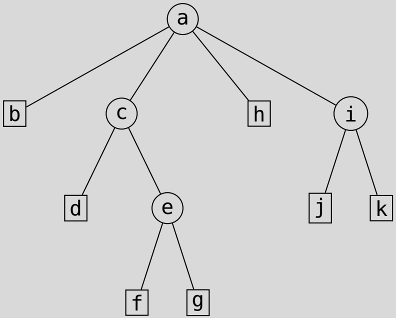
Depth First Unary Degree Sequence (2/2)

```
 a    bc  de  fg
((((())(())(()))
```



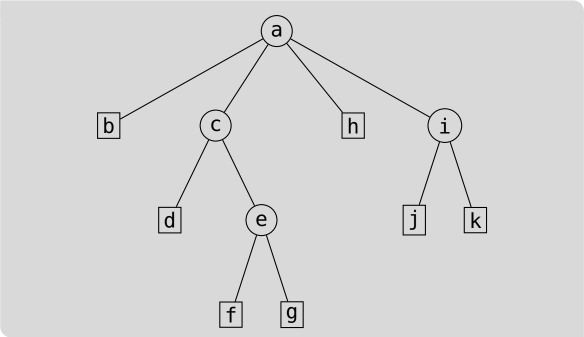
Depth First Unary Degree Sequence (2/2)

```
   a    bc  de  fgh
((((())(())(())))
```



Depth First Unary Degree Sequence (2/2)

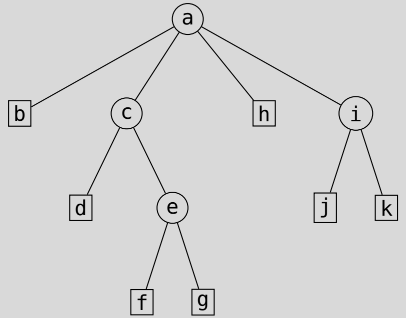
```
 a    bc  de  fghi  jk
((((())(())(())))(()))
```



Depth First Unary Degree Sequence (2/2)

```
 a    bc  de  fghi  jk
((((())(())(())))(()))
```

- node starts at first parenthesis
- subtree structure is encoded

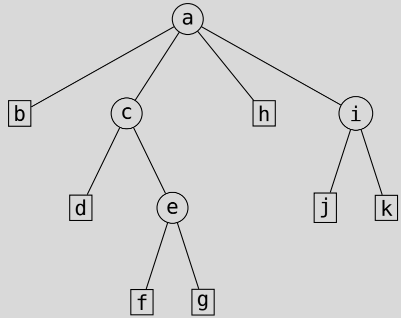


Making DFUDS Fully-Functional

```
  a    bc  de  fghi  jk
((((()))(()))((())))((()))
```

- degree of p: $\mathit{select}_{\text{“)”}}(\mathit{rank}_{\text{“)”}}(p)+1)-p$

- explanation on the board

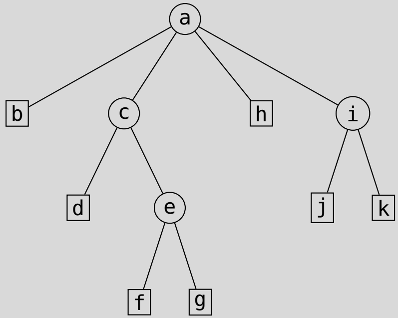


Making DFUDS Fully-Functional

```
    a    bc  de  fghi  jk
((((())(())(())))(()))
```

- degree of p: $select_{\text{“)”}}(rank_{\text{“)”}}(p) + 1) - p$
- i-th child of p:
  $findclose(select_{\text{“)”}}(rank_{\text{“)”}}(p) + 1) - i) + 1$

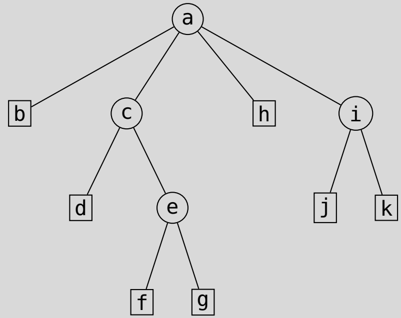
- explanation on the board



Making DFUDS Fully-Functional

```
   a    bc  de  fghi  jk
((((()(())(()))(()))
```

- degree of p: $\mathit{select}_{\text{“)”}}(\mathit{rank}_{\text{“)”}}(p)+1)-p$
- i-th child of p:
  $\mathit{findclose}(\mathit{select}_{\text{“)”}}(\mathit{rank}_{\text{“)”}}(p)+1)-i)+1$
- parent of p: $\mathit{select}_{\text{“)”}}(\mathit{rank}_{\text{“)”}}(\mathit{findopen}(p-1)))+1$



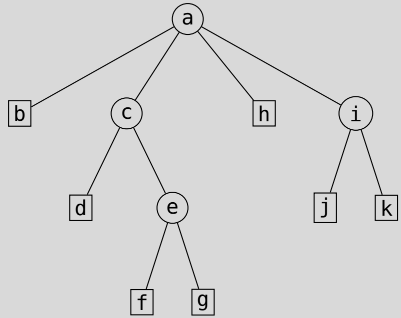
- explanation on the board

Making DFUDS Fully-Functional

```
      a     bc   de   fghi   jk
((((()) (()) (()))) (()))
```

- degree of p: $\mathit{select}_{\text{``)''}}(\mathit{rank}_{\text{``)''}}(p)+1)-p$
- i-th child of p:
  $\mathit{findclose}(\mathit{select}_{\text{``)''}}(\mathit{rank}_{\text{``)''}}(p)+1)-i)+1$
- parent of p: $\mathit{select}_{\text{``)''}}(\mathit{rank}_{\text{``)''}}(\mathit{findopen}(p-1)))+1$
- subtree size of p:
  $(\mathit{findclose}(\mathit{enclose}(p))-p)/2+1$

- explanation on the board



Conclusion and Outlook

This Lecture

- three succinct tree representations
- different advantages and disadvantages

Advanced Data Structures

BV — succ. trees

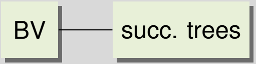
Conclusion and Outlook

This Lecture

- three succinct tree representations
- different advantages and disadvantages

- min-max-trees

Advanced Data Structures



Conclusion and Outlook

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This Lecture

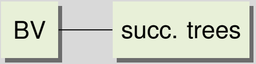
- three succinct tree representations
- different advantages and disadvantages

- min-max-trees

Next Lecture

- succinct graphs

Advanced Data Structures



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[MR01] J. Ian Munro and Venkatesh Raman. “Succinct Representation of Balanced Parentheses and Static Trees”. In: *SIAM J. Comput.* 31.3 (2001), pages 762–776. DOI: 10.1137/S0097539799364092.

[NS14] Gonzalo Navarro and Kunihiko Sadakane. “Fully Functional Static and Dynamic Succinct Trees”. In: *ACM Trans. Algorithms* 10.3 (2014), 16:1–16:39. DOI: 10.1145/2601073.