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Text Indexing

Lecture 09: LZ Compressed Indeces

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PINGO

https://pingo.scc.kit.edu/309703

Recap: FM-Index and *r*-Index

- based on backwards-search
- used to answer *rank*-queries on *BWT*

FM-Index

- build wavelet tree directly on *BWT*
- wavelet tree can be H_0 compressed
- blind to repetitions

r-Index

- many arrays with *r* entries
- build wavelet tree on one of these arrays
- size in numbers of *BWT* runs *r*

Function *BackwardsSearch*$(P[1..n], C, rank)$**:**

  $s = 1, e = n$
  for $i = m, \ldots, 1$ **do**
    $s = C[P[i]] + rank_{P[i]}(s - 1) + 1$
    $e = C[P[i]] + rank_{P[i]}(e)$
    if $s > e$ **then**
      return $\emptyset$
  return $[s, e]$

Different Types of Compression

Statistical Coding

- based on frequencies of characters
- results in size $|T| \cdot H_k(T)$
  k-th order empirical entropy
- good if frequencies are skewed
- blind to repetitions
  $|\underbrace{T \ldots T}_{\ell}| \cdot H_k(\underbrace{T \ldots T}_{\ell}) \approx \ell |T| \cdot H_k(T)$

LZ-Compression

- references to previous occurrences
- each LZ factor can be encoded in $O(1)$ space
- good for repetitions
- index in this lecture

BWT-Compression

- used in powerful index
- theoretical insight in this lecture

LZ-Compressed Index

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Definition: LZ77 Factorization [ZL77]

Given a text T of length n over an alphabet Σ, the **LZ77 factorization** is

- a set of z factors $f_1, f_2, \ldots, f_z \in \Sigma^+$, such that
- $T = f_1 f_2 \ldots f_z$ and for all $i \in [1, z]$ f_i is
- single character not occurring in $f_1 \ldots f_{i-1}$ or
- longest substring occurring ≥ 2 times in $f_1 \ldots f_i$

Now

- LZ-compressed replacement for wavelet trees
- *rank* and *access* queries ℹ *select* also supported
- LZ-compression better than H_k-compression

$T =$ `abababbbbaba$`

- $f_1 =$ `a`
- $f_2 =$ `b`
- $f_3 =$ `abab`
- $f_4 =$ `bbb`
- $f_5 =$ `aba`
- $f_6 =$ `$`

Block Trees [Bel+21] (1/4)

Definition: Block Tree (1/4)

Given a text T of length n over an alphabet of size σ

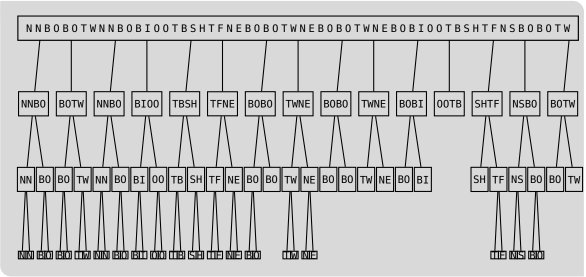
- $\tau, s \in \mathbb{N}$ greater 1
- assume that $n = s \cdot \tau^h$ for some $h \in \mathbb{N}$
  - ℹ append \$s until n has this form

A **block tree** is a

- perfectly balanced tree with height h
- that may have leaves at higher levels

such that

- the root has s children,
- each other inner node has τ children



Block Trees (2/4)

Definition: Block Tree (2/4)

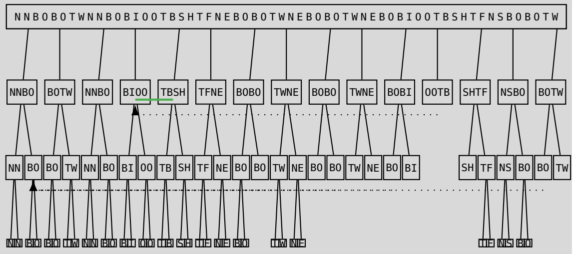
In a block tree, leaves at

- the last level store characters or substrings of T
- at higher levels store special leftward pointer

Each node u

- represents a block B^u
- which is a substring of T identified by a position

The root represents T and its children consecutive blocks of T of size n/s



Block Trees (3/4)

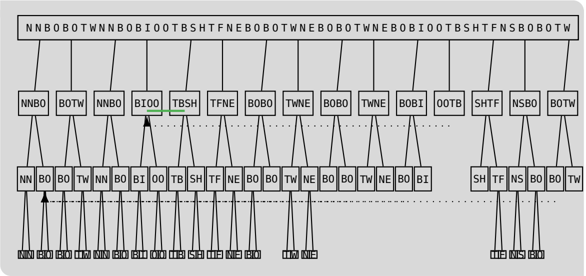
Definition: Block Tree (3/4)

Let ℓ_u be the level (depth) of node u

- the level of the root is 0

Let $B_1, B_2, \ldots$ be the blocks represented at level ℓ_u from left to right

- for any i, B_i and B_{i+1} are consecutive in T
- if $B_i B_{i+1}$ are the leftmost occurrence in T, the nodes representing the blocks are marked



Block Trees (4/4)

Definition: Block Tree (4/4)

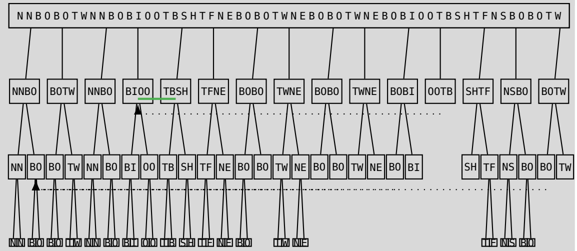
If node u is marked, then

- it is an internal node
- with τ children

otherwise, if node u is not marked, then

- u is a leaf storing
- pointers to nodes v_i, v_{i+1} at the same level
  - that represent blocks B_i and B_{i+1}
  - covering the leftmost occurrence of B^u
- offset to the occurrence of B^u in $B_i B_{i+1}$

leaves on last level store text explicitly



- $|B^u| = n/(s\tau^{\ell_u - 1})$
- if $|B_u|$ is small enough, store text explicitly
  ℹ $|B^u \in \Theta(\lg_\sigma n)|$
- **PINGO** how many blocks are there per level?

Block Trees are LZ Compressed (1/2)

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Lemma: Number of Blocks per Level

The number of blocks in any level > 0 in the block tree is at most $3\tau z$

- $O(\tau z)$ blocks per level
- unmarked block requires $O(\lg n)$ bits of space
- marked block requires $O(\tau \lg n)$ bits of space
  charged to child
- last level has $O(\tau z)$ blocks with plain text
  - $O(\lg_\sigma n)$ symbols of $\lceil \lg n \rceil$ bits
  - requiring $O(\lg \sigma)$ bits per block
- $h = \lg_\tau \frac{n \lg \sigma}{s \lg n}$ and $O(s)$ pointers to top level
- rounding up length adds $\leq O(\tau)$ blocks per level

Proof (Sketch)

Let $\ell > 0$ be a level in the block tree and

- $C = B_{i-1} B_i B_{i+1}$ a concatenation of three consecutive blocks at level $\ell - 1$
- not containing the end of an LZ factor
- thus a leftwards occurrence in T

B_{i-1} and B_{i+1} can only be marked if B_i is marked

- B_i is marked if it contains end of LZ factor
- there are only z LZ factors

Each marked block results in τ children

Block Trees are LZ Compressed (2/2)

Lemma: Space Requirements of Block Trees

Given a text T of length n over an alphabet of size σ and integers $s, \tau > 1$, a block tree of T has height $h = \lg_\tau \frac{n \lg \sigma}{s \lg n}$. The block tree requires

$$O((s + z\tau \lg_\tau \frac{n \lg \sigma}{s \lg n}) \lg n) \text{ bits of space,}$$

where z is the number of LZ77 factors of T

- $s = z$ results in a tree of height $O(\lg_\tau \frac{n \lg \sigma}{z \lg n})$
- space requirements $O(z\tau \lg_\tau \frac{n \lg \sigma}{z \lg n} \lg n)$ bits
- however z not known

Access Queries in Block Trees

- queries are easy to realize
- if not supported directly, additional information can be stored for blocks

Access Query

Given position i return $T[i]$

- follow nodes that represent block containing $T[i]$
- of not marked follow pointer and consider offset
- at leaf, if last level, return character
- else, follow pointer and continue

- time $O(\lg_\tau \frac{n \lg \sigma}{s \lg n})$

- example on the board

- **PINGO** can we answer rank queries the same way?

Rank Queries in Block Trees

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- for each block add histogram $Hist_{B_u}$ for prefix of T up to block (not containing)
- $O(\sigma(s + z\tau \lg_\tau \frac{n \lg n}{s \lg \sigma}) \lg n)$ bits of space

Rank Query

Given position i and character α return $rank_\alpha(T, i)$

- follow nodes that represent block containing $T[i]$
- remember $Hist_{B_u}[\alpha]$
- of not marked follow pointer and consider offset
- at leaf, if last level, compute local rank binary rank for each character
- else, follow pointer and continue

- time $O(\lg_\tau \frac{n \lg \sigma}{s \lg n})$

- example on the board

- **PINGO** what can be problematic with block tree construction?

Construction of Block Trees

$O(n)$ Working Space

- build Aho-Corasick automaton for containing all pairs of consecutive unmarked blocks
- identify unmarked blocks on next level
- $O(n(1 + \lg_\tau \frac{z}{s}))$ time and $O(n)$ space

Pruning

- size of block tree can be reduced further
- some blocks not necessary
- those blocks can easily be identified

$O(s + z\tau)$ Working Space

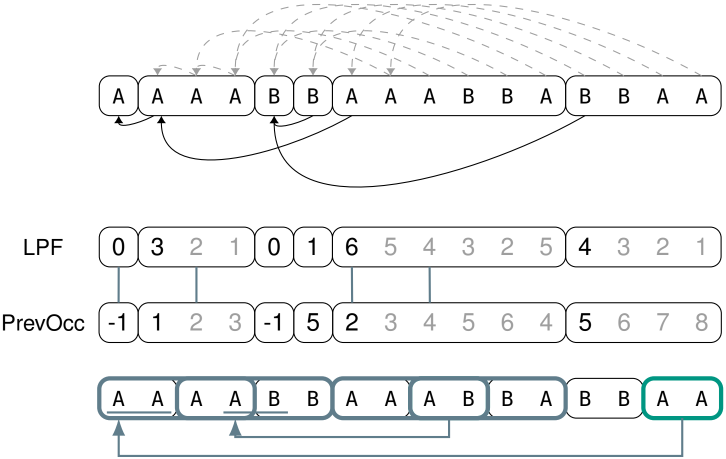
- replace Aho-Corasick automaton with Karp-Rabin fingerprints
- validate if matching fingerprints due to matching strings ℹ Monte Carlo algorithm
- $O(n(1 + \lg_\tau \frac{z}{s}))$ expected time and $O(n)$ space
- only expected construction time!

- queries very fast in practice
- construction very slow in practice
- space-efficient construction of block trees

State-of-Block-Tree-Construction

Method	Reference	Working Space	Time	I
Aho-Corasic	[Bel+21]	$O(n)$	$O(n(1 + \log_\tau(z\tau/s)))$	n
Fingerprints	[Bel+21]	$O(s + z\tau \log_\tau(\frac{n \log \sigma}{s \log n}))$	$O(n(1 + \log_\tau(z\tau/s)))$ expected	y
LPF Array	[**KopplKM2023LPFBlockTrees**]	$O(n)$	$O(n(1 + \log_\tau(z\tau/s)))$	y

Our Algorithm (Marking of Nodes)



Experimental Evaluation

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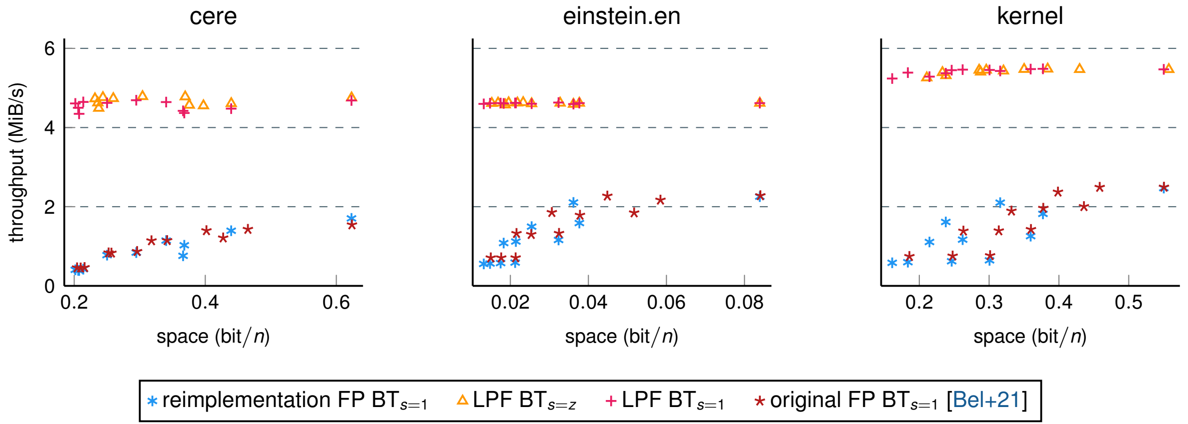
- highly tuned implementation
- tree consists only of bit and compact vectors
- tuning parameter
  - degree root $s = \{1, z\}$ (only we have $s = z$)
  - degree other nodes $\tau = \{2, 4, 8, 16\}$
  - number characters in leaves $b = \{2, 4, 8, 16\}$

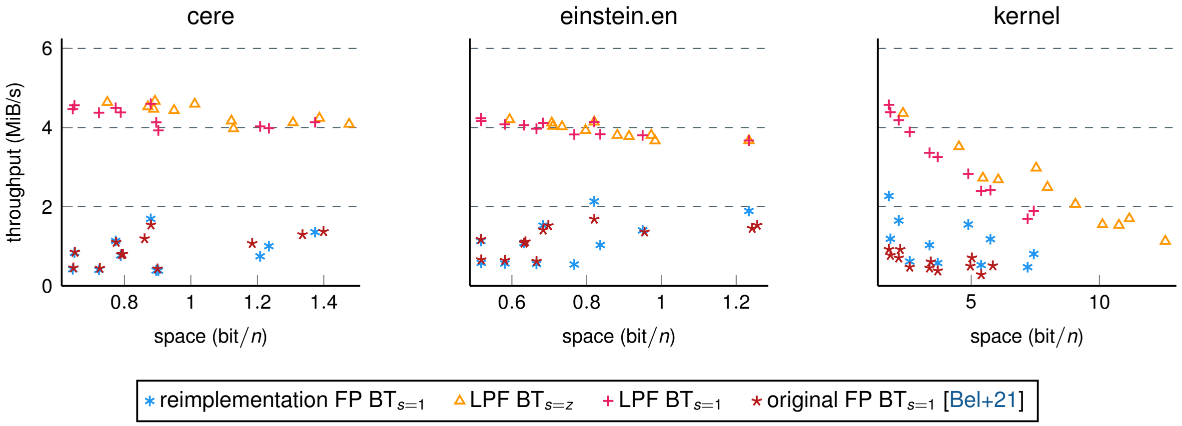
- original FP BT [Bel+21]
- our reimplementation of the original FP BT
- our LPF BT construction with $s = 1$ and $s = z$
- dynamic programming variants
- parallelization
- no comparison with wavelet trees (faster)

- repetitive instances from P&C corpus
- non-repetitive instances from P&C corpus

Highly Repetitive Inputs (Access Only)



Highly Repetitive Inputs (with Rank and Select Support)



Conclusion and Outlook

This Lecture

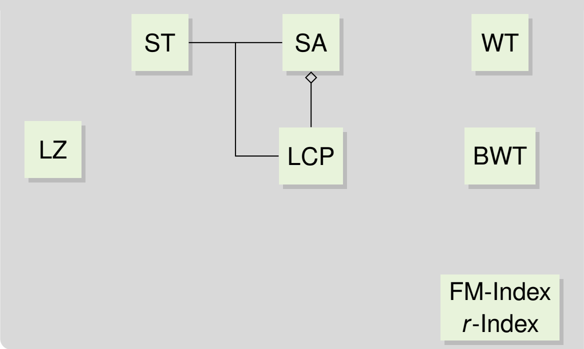
- block trees

- efficient block tree construction
- linear time block tree construction

Next Lecture

- move data structure and relation of BWT runs and LZ factors

Linear Time Construction



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