Karlsruhe Institute of Technology

Advanced Data Structures

Lecture 02: Succinct Trees

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PINGO

Karlsruhe Institute of Technology

https://pingo.scc.kit.edu/306589

Recap: Rank Queries on Bit Vectors (1/2)

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$\text{rank}_\alpha(i)$ # of αs before i

$\text{select}_\alpha(j)$ position of j-th α

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	1	0	0

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$\text{rank}_0(5)$

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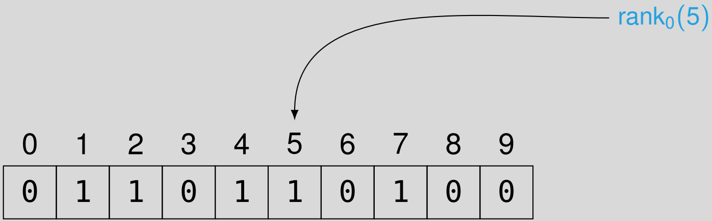

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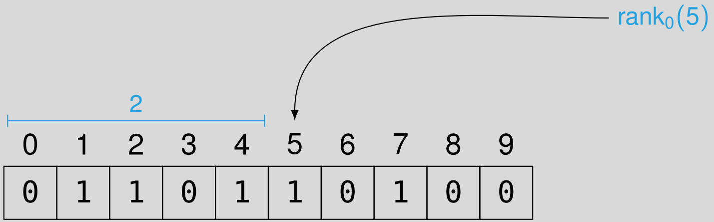

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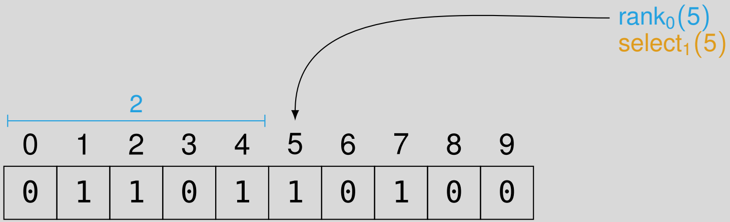

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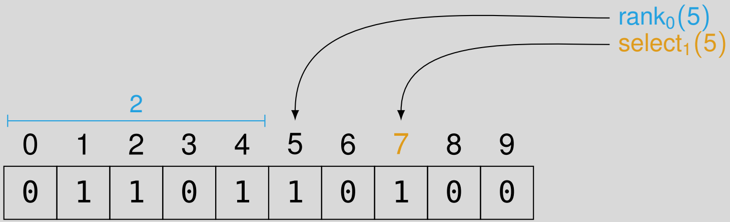

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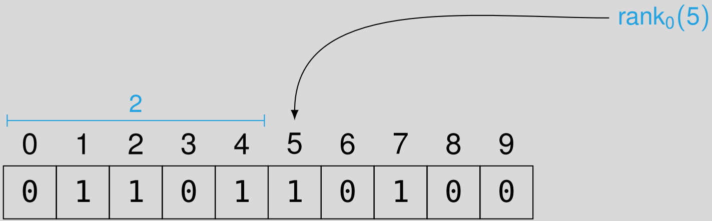

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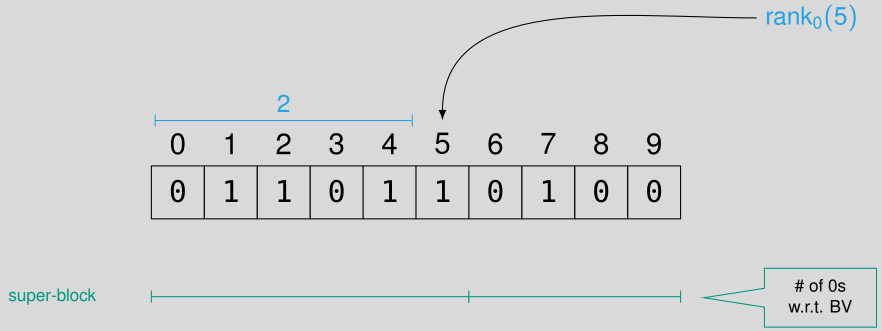

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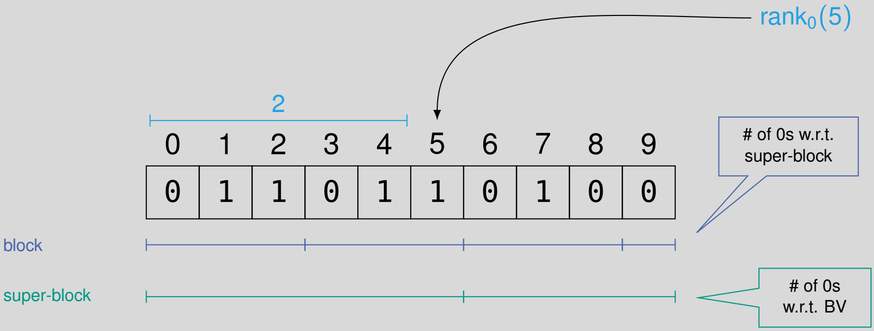

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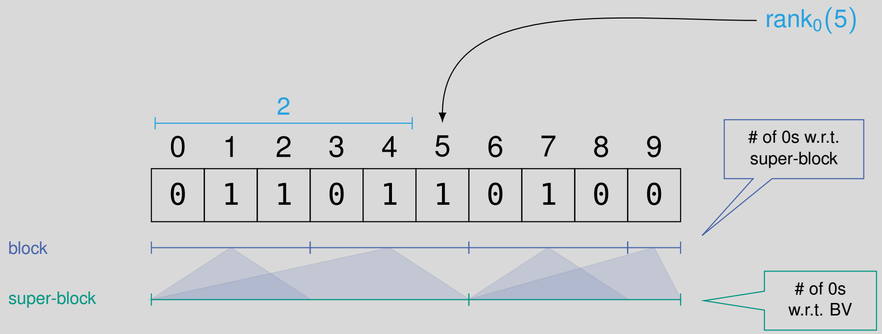

Recap: Rank Queries on Bit Vectors (2/2)

Lemma: Binary Rank- and Select-Queries

Given a bit vector of size n, there exist data structures that can be computed in time $O(n)$ of size $o(n)$ bits that can answer rank and select queries on the bit vector in $O(1)$ time

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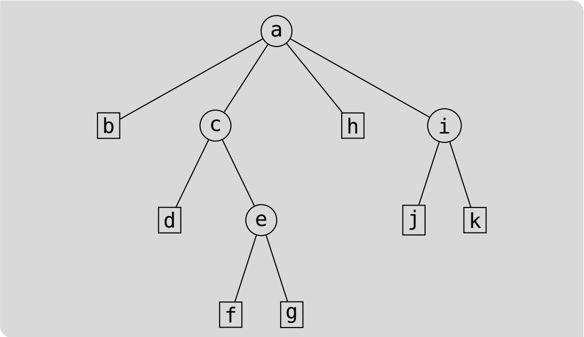
Given a bit vector of size n, there exist data structures that can be computed in time $O(n)$ of size $o(n)$ bits that can answer rank and select queries on the bit vector in $O(1)$ time

Word RAM

- unlimited memory
- words of size w ℹ $w = \Theta \log n$
- constant time load and store
- constant time bit operations on words

Plan for Today

- represent tree with n nodes using $2n$ bits
- make succinct tree fully-functional using additional $o(n)$ bits



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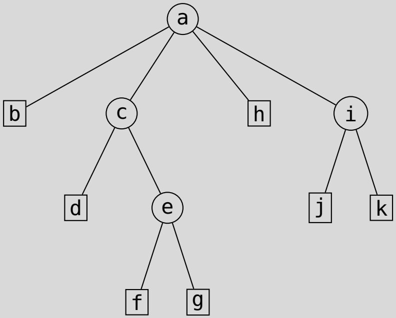
- trees are important
  - searching for keys
  - maintaining directories
  - representations of parsings
  - ...



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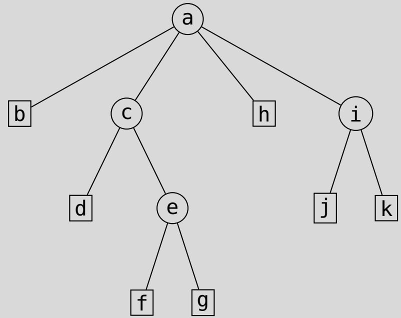
- supporting different operations



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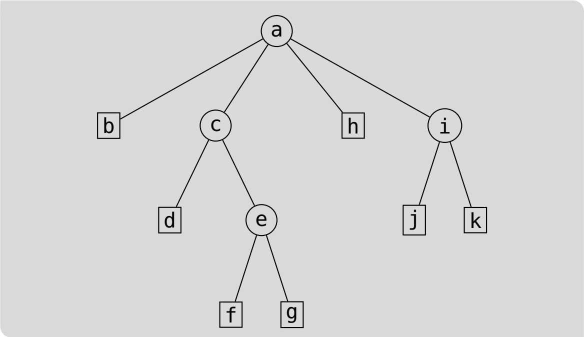
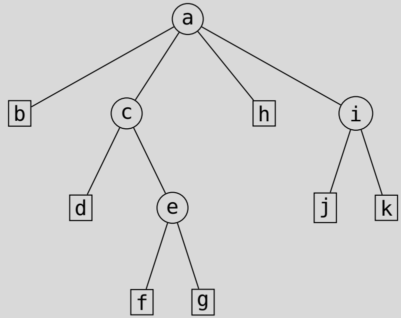


Handout

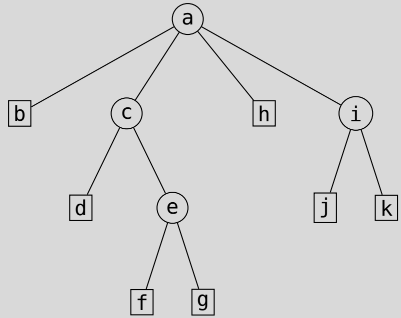
Preliminaries

- a tree is an acyclic connected graph $G = (V, E)$ with a root $r \in V$
- degree δ is the number of children
- leaves have degree 0
- depth of a node is the length of the path from the root to that node



Level Ordered Unary Degree Sequence (1/2) [Jac88]

- represent tree level-wise
- use $\leq$ 2 bits per node



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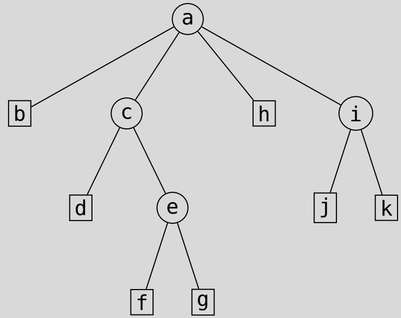
Definition: LOUDS

Starting at the root, all nodes on the same depth

- are visited from left to right and
- for node v, $\delta(v)$ 1's followed by a 0 are

appended to the bit vector that contains an initial `10`



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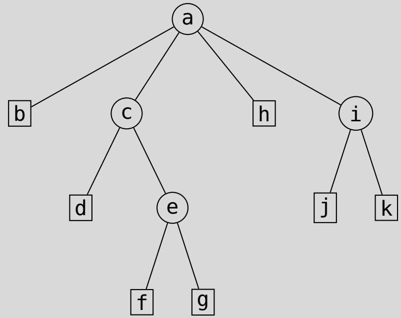
Lemma: Space Usage of LOUDS

Representing a tree with n nodes requires $2n + 1$ bits using LOUDS



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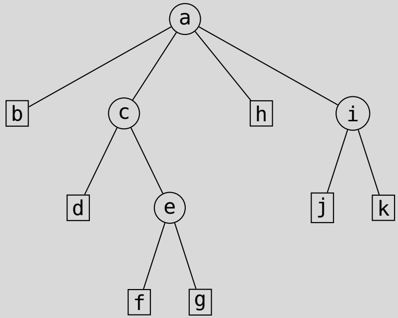
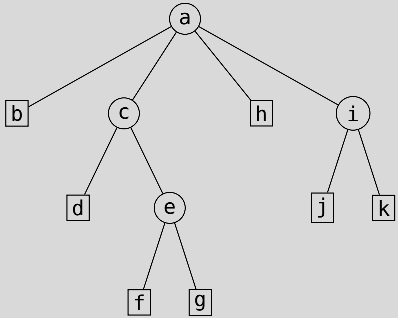


- write down the LOUDS representation of this example tree

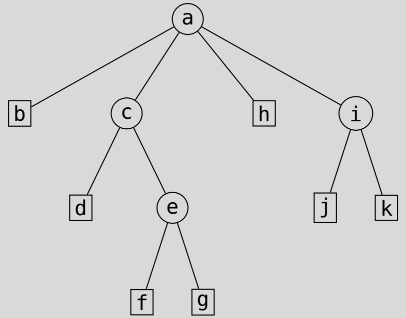
Level Ordered Unary Degree Sequence (2/2)

1011110



Level Ordered Unary Degree Sequence (2/2)

10111100110010



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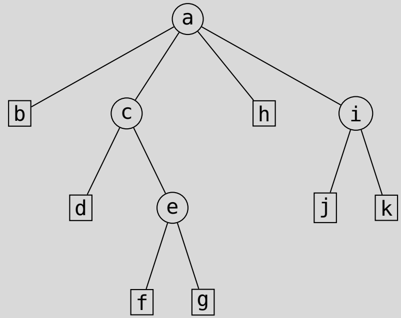

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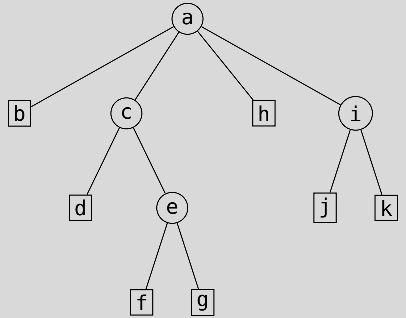
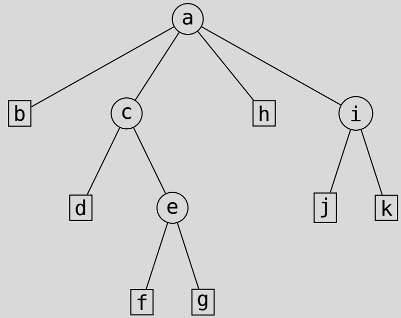


Level Ordered Unary Degree Sequence (2/2)

```
          ab  ch  id  ejkfg
10111100110011001100000
```

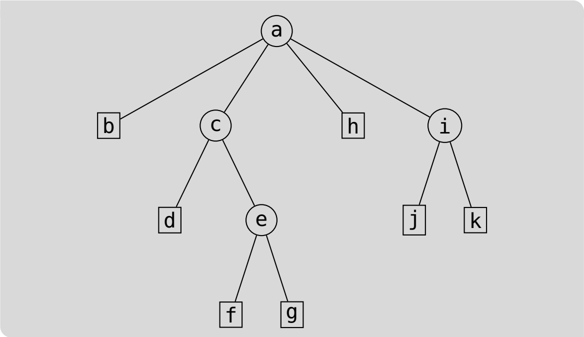
- node start at pertinent `0`



What is Fully-Functional?

Operations

- degree ℹ is leaf
- *i*-th child
- parent
- subtree size



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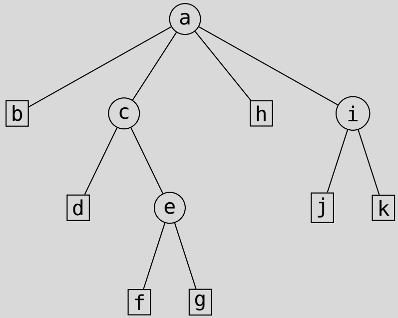
- depth
- lowest common ancestor
- rank (pre- or post-order)
- . . .



Making LOUDS Fully-Functional

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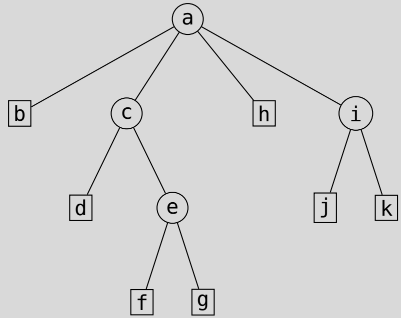
- degree of p: $p - select_0(rank_0(p)) - 1$

- explanation on the board



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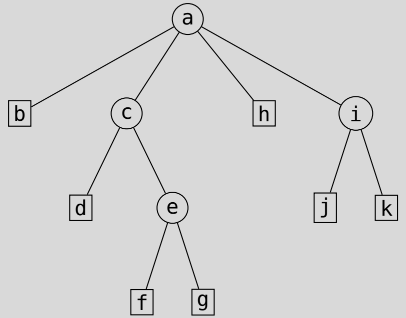

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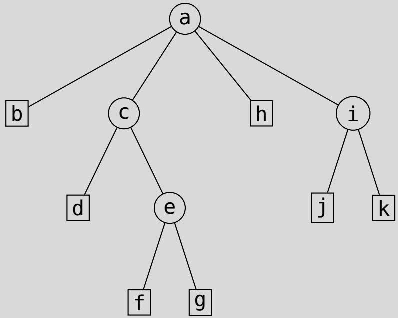

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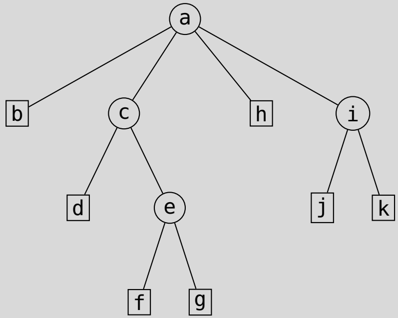
- subtree size **PINGO**



From Bit Vectors to Parentheses

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- instead of `0` and `1`
- use `(` and `)`

- requires the same space
- can add relation between parentheses

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A string of parentheses is balanced, if for each left parenthesis there exist unique right parenthesis to its right

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- how can we answer *excess* queries **PINGO**

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- here, a little bit simpler

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- $excess(i) = rank_{\text{“(”}}(i+1) - rank_{\text{“)”}}(i+1)$
- $fwd\_search(i, d) = \min\{j > i \colon excess(j) - excess(i-1) = d\}$
- $bwd\_search(i, d) = \max\{j < i \colon excess(i) - excess(j-1) = d\}$

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- $findclose(i) = fwd\_search(i, 0)$
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- can be answered with a min-max-tree

Range Min-Max Trees (1/2)

Definition: Range Min-Max Tree

Given a bit vector B of length n and a block size b, store for each consecutive block (from s to e) of BV

- total excess in block:
  $excess(e) - excess(s-1)$
- minimum left-to-right excess in block:
  $\min\{excess(p) - excess(s-1) \colon p \in [s, e)\}$

and build a binary tree over these blocks, where each node stores the same total information for blocks in all its leaves

- example on the board

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Lemma: Range Min-Max Tree Space

A range min-max tree with block size b for a bit vector of size n requires $n + O((n/b)\log n)$ bits of space

Range Min-Max Trees (2/2)

fwdsearch in a Range Min-Max Tree

- scan block
- if not found traverse tree
- identify block in tree
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- process c bits at a time
- first align with next c bits
- requires $O(c + b/c)$ time

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- scanning last block requires $O(c + b/c)$ time

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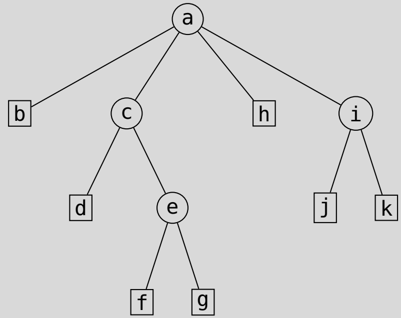
Improvements

- two level approach
- build range min-max trees for chunks of size $\Theta(\log^3 n)$
- $O(\log \log n)$ query time inside a chunk
- can result in total query time of $O(\log \log n)$

Balanced Parentheses (1/2) [MR01]

- represent tree as depth-first traversal
- using balanced parentheses



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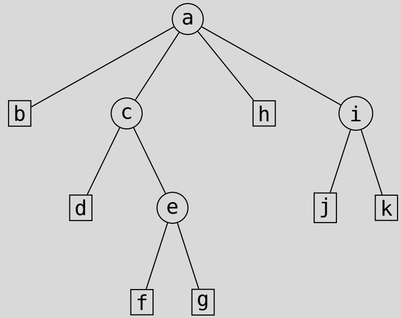
Definition: BP

Starting at the root, traverse the tree in depth-first order and append a

- left parenthesis if a node is visited the first time
- right parenthesis if a node is visited the last time

to the bit vector



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- represent tree as depth-first traversal
- using balanced parentheses

Definition: BP

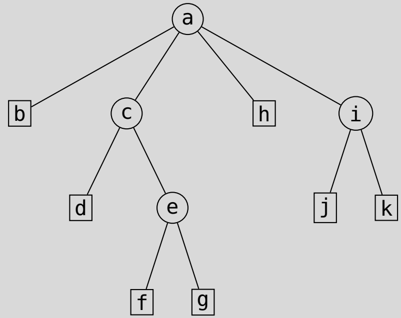
Starting at the root, traverse the tree in depth-first order and append a

- left parenthesis if a node is visited the first time
- right parenthesis if a node is visited the last time

to the bit vector

Lemma: Space Usage of BP

Representing a tree with n nodes requires $2n$ bits using *BP*



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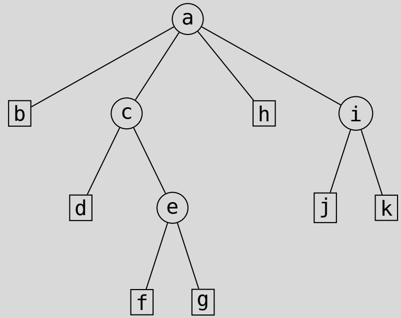
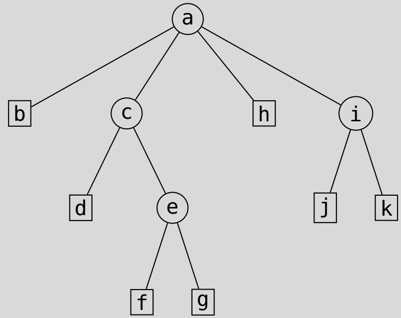


- write down the BP representation of this example tree

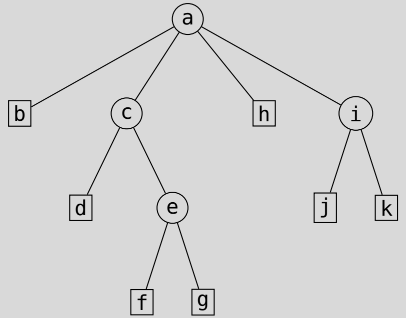
Balanced Parentheses (2/2)

```
a
(
```



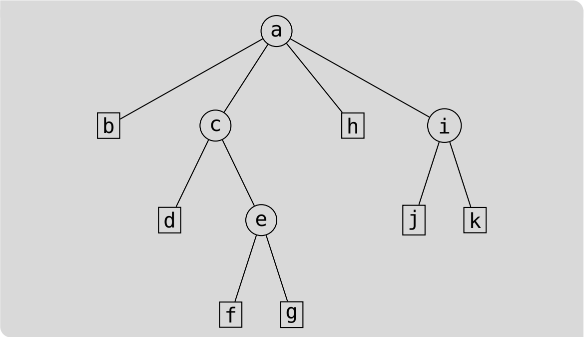
Balanced Parentheses (2/2)

```
ab
(()
```



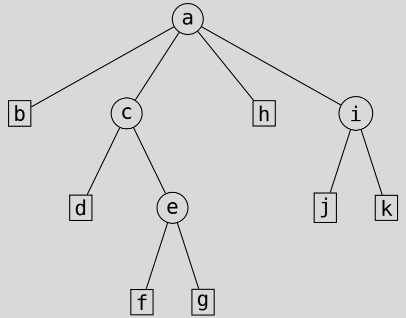
Balanced Parentheses (2/2)

```
ab cd
(()(()
```



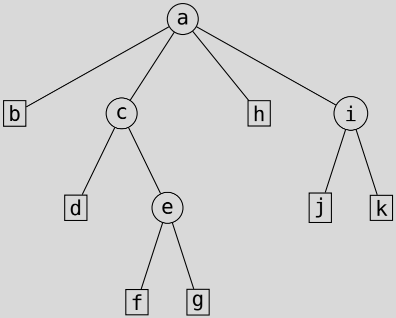
Balanced Parentheses (2/2)

```
ab cd ef g
(()(()(()())))
```



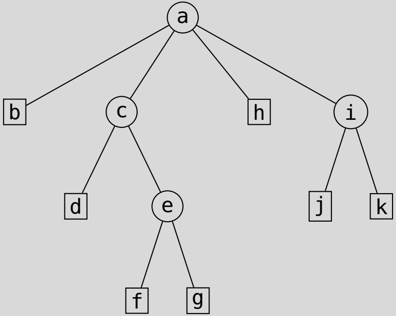
Balanced Parentheses (2/2)

```
ab cd ef g   h
(()((()(())))()
```



Balanced Parentheses (2/2)

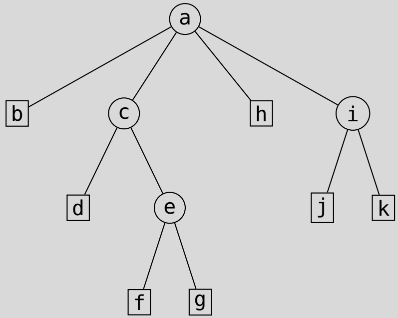
```
ab cd ef g   h ij k
(()(()(()()))()(()()))
```



Balanced Parentheses (2/2)

```
ab cd ef g   h ij k
(()(()(()()))()(()()))
```

- node starts at first parenthesis
- subtree structure is encoded in parentheses

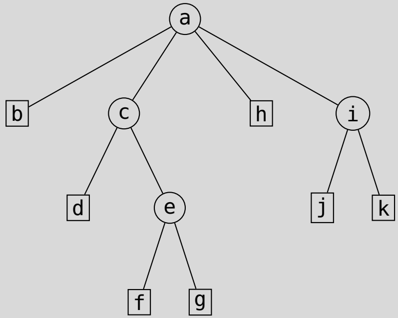


Making BP Fully-Functional

```
ab cd ef g   h ij k
(()(()(()()))()(()()))
```

- subtree size of p: $(\mathit{findclose}(p) - p + 1)/2$

- explanation on the board



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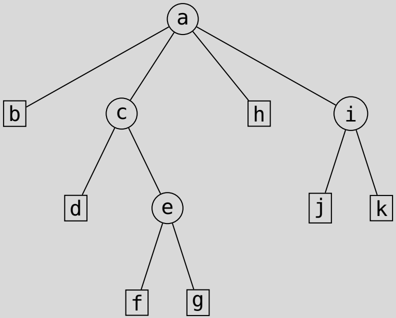

Making BP Fully-Functional

```
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```

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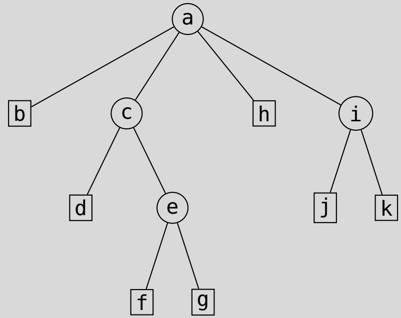
Complicated Constant Time [NS14]

- degree
- i-th child



Advantages and Disadvantages of Both Approaches

- LOUDS cannot answer subtree size
- BP cannot easily answer i-th child and degree

- all other operations can be done easily

Depth First Unary Degree Sequence (1/2) [Ben+05]

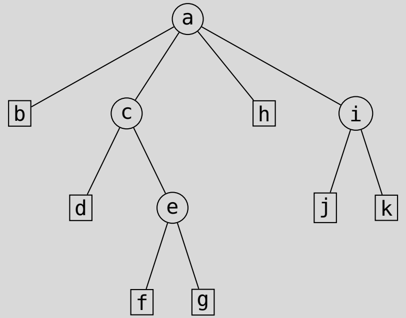
Karlsruhe Institute of Technology

Definition: DFUDS

Starting at the root, traverse tree in depth-first order and append

- for node v, $\delta(v)$ left parentheses and
- a right parenthesis if v is visited the first time

to the bit vector that initially contains a left parenthesis ℹ to make them balanced



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Karlsruhe Institute of Technology

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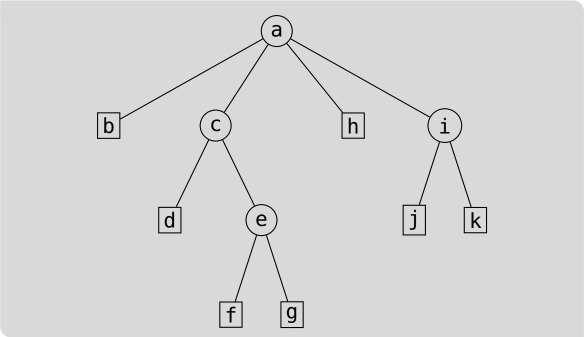
Lemma: Space Usage of DFUDS

Representing a tree with n nodes requires $2n$ bits using DFUDS



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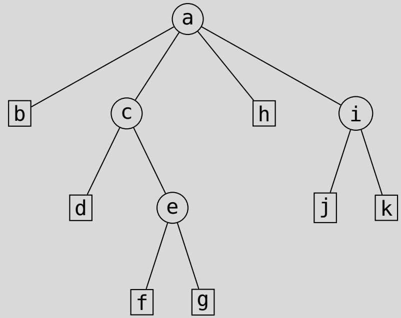
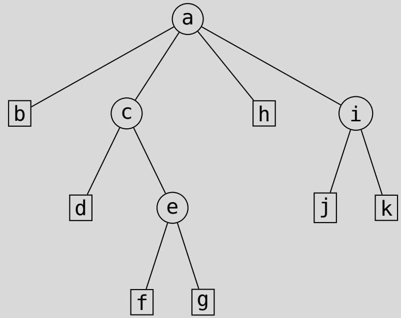


- write down the DFUDS representation of this example tree

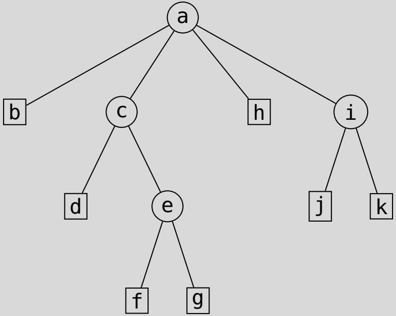
Depth First Unary Degree Sequence (2/2)

```
  a
((((()
```



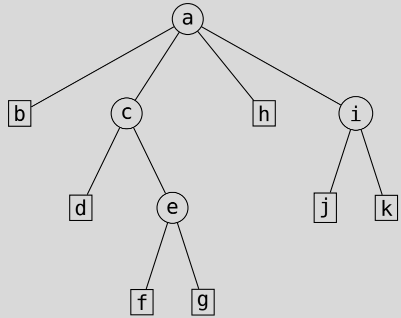
Depth First Unary Degree Sequence (2/2)

```
    a    b
(((((()) 
```



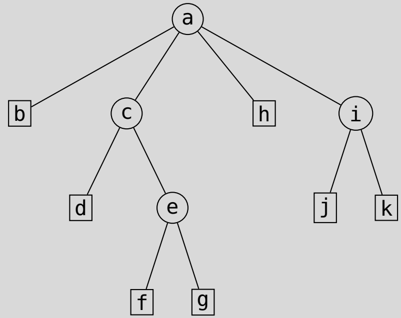
Depth First Unary Degree Sequence (2/2)

```
 a    bc
((((())(()
```



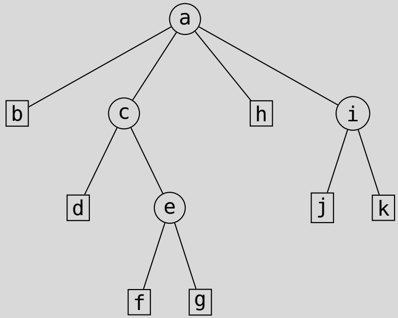
Depth First Unary Degree Sequence (2/2)

```
 a    bc  d
((((())(()))
```



Depth First Unary Degree Sequence (2/2)

```
  a    bc  de  fg
((((())(())(())))
```



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```
  a    bc  de  fgh
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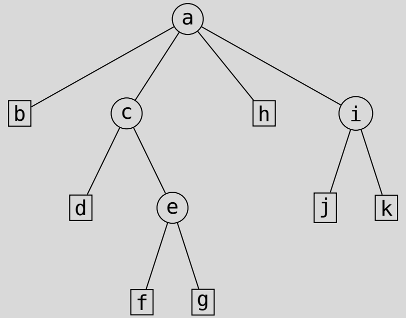

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```
 a   bc  de  fghi  jk
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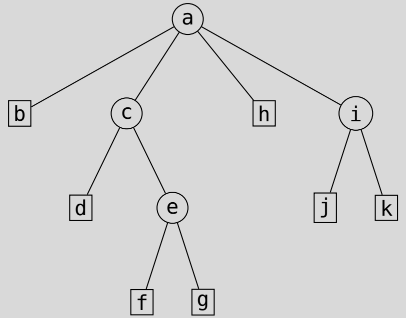
```



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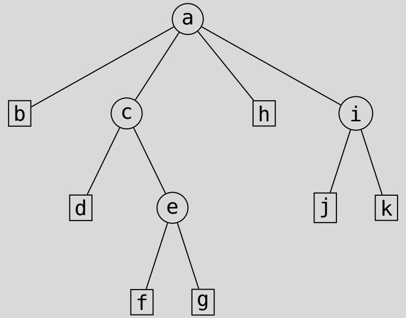
```
 a    bc  de  fghi  jk
((((())(())(())))(()))
```

- node starts at first parenthesis
- subtree structure is encoded

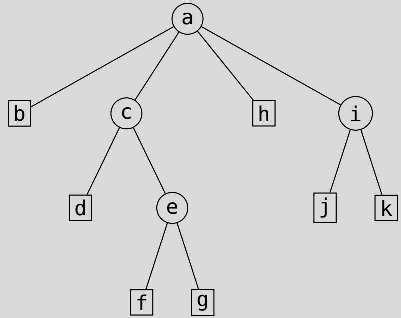


Making DFUDS Fully-Functional

```
a    bc  de  fghi  jk
((((()))(())(()))))(())))
```

- degree of p: $select_{``)''}(rank_{``)''}(p) + 1) - p$

- explanation on the board

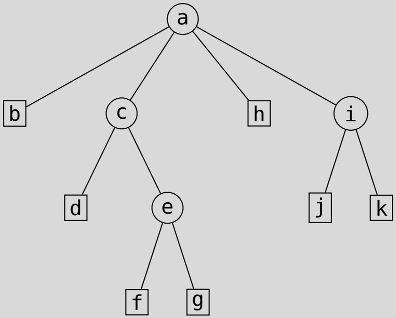


Making DFUDS Fully-Functional

Karlsruhe Institute of Technology

```
 a    bc  de  fghi  jk
((((()(())(())))(()))
```

- degree of p: $select_{\text{``})\text{''}}(rank_{\text{``})\text{''}}(p) + 1) - p$
- i-th child of p:
  $findclose(select_{\text{``})\text{''}}(rank_{\text{``})\text{''}}(p) + 1) - i) + 1$



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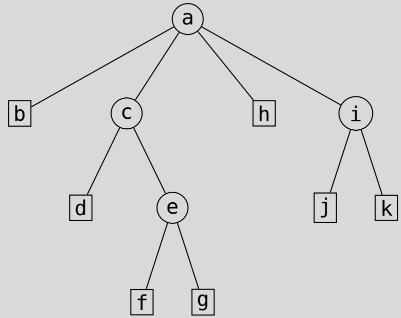
Making DFUDS Fully-Functional

```
   a    bc  de  fghi  jk
((((())(())(())))(()))
```

- degree of p: $\mathit{select}_{``)"}(\mathit{rank}_{``)"}(p)+1)-p$
- i-th child of p: $\mathit{findclose}(\mathit{select}_{``)"}(\mathit{rank}_{``)"}(p)+1)-i)+1$
- parent of p: $\mathit{select}_{``)"}(\mathit{rank}_{``)"}(\mathit{findopen}(p-1)))+1$



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Making DFUDS Fully-Functional

```
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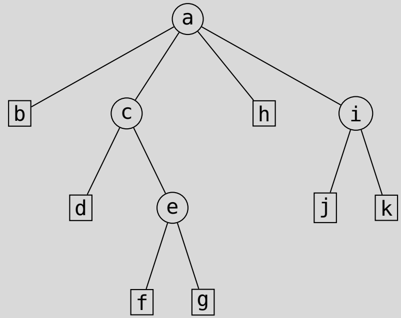
((((()(())(())))(()))
```

- degree of p: $\mathit{select}_{\text{``)''}}(\mathit{rank}_{\text{``)''}}(p) + 1) - p$
- i-th child of p: $\mathit{findclose}(\mathit{select}_{\text{``)''}}(\mathit{rank}_{\text{``)''}}(p) + 1) - i) + 1$
- parent of p: $\mathit{select}_{\text{``)''}}(\mathit{rank}_{\text{``)''}}(\mathit{findopen}(p - 1))) + 1$
- subtree size of p: $(\mathit{findclose}(\mathit{enclose}(p)) - p)/2 + 1$

- explanation on the board

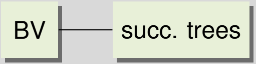


Conclusion and Outlook

This Lecture

- three succinct tree representations
- different advantages and disadvantages

Advanced Data Structures



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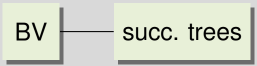
- min-max-trees

Advanced Data Structures



Conclusion and Outlook

Karlsruhe Institute of Technology

This Lecture

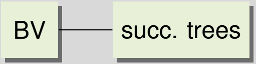
- three succinct tree representations
- different advantages and disadvantages

- min-max-trees

Next Lecture

- succinct graphs

Advanced Data Structures



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[Jac88] Guy Joseph Jacobson. “Succinct Static Data Structures”. PhD thesis. Carnegie Mellon University, 1988.

[MR01] J. Ian Munro and Venkatesh Raman. “Succinct Representation of Balanced Parentheses and Static Trees”. In: *SIAM J. Comput.* 31.3 (2001), pages 762–776. DOI: 10.1137/S0097539799364092.

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