Advanced Data Structures

Lecture 02: Succinct Trees

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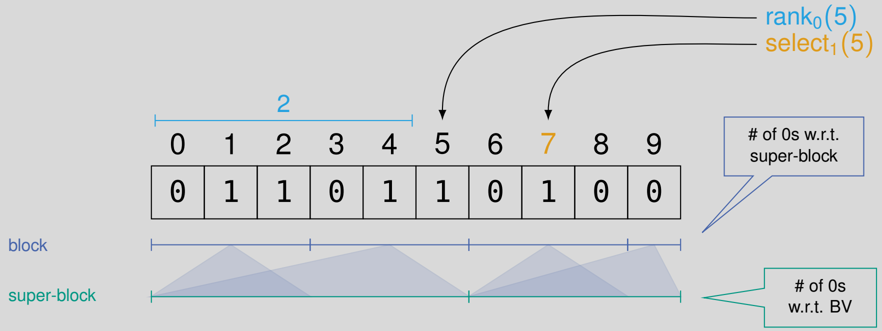
PINGO

https://pingo.scc.kit.edu/306589

Recap: Rank Queries on Bit Vectors (1/2)

$\text{rank}_\alpha(i)$ # of αs before i

$\text{select}_\alpha(j)$ position of j-th α



Recap: Rank Queries on Bit Vectors (2/2)

Lemma: Binary Rank- and Select-Queries

Given a bit vector of size n, there exist data structures that can be computed in time $O(n)$ of size $o(n)$ bits that can answer rank and select queries on the bit vector in $O(1)$ time

Word RAM

- unlimited memory
- words of size w ℹ $w = \Theta \log n$
- constant time load and store
- constant time bit operations on words

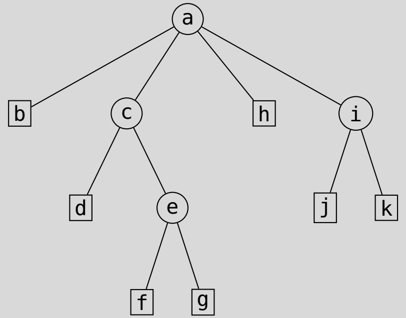
Plan for Today

Karlsruhe Institute of Technology

- represent tree with n nodes using $2n$ bits
- make succinct tree fully-functional using additional $o(n)$ bits

- trees are important
  - searching for keys
  - maintaining directories
  - representations of parsings
  - ...

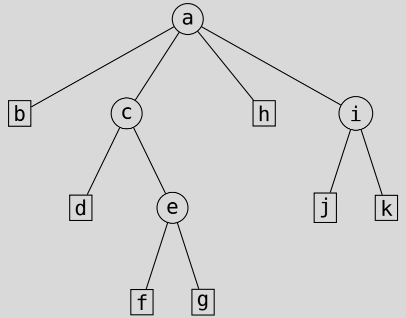
- different representations
- supporting different operations



Handout

Preliminaries

- a tree is an acyclic connected graph $G = (V, E)$ with a root $r \in V$
- degree δ is the number of children
- leaves have degree 0
- depth of a node is the length of the path from the root to that node



Level Ordered Unary Degree Sequence (1/2) [Jac88]

- represent tree level-wise
- use ≤ 2 bits per node

Definition: LOUDS

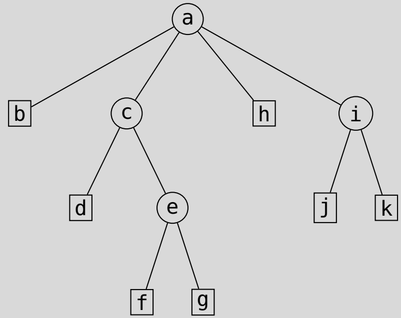
Starting at the root, all nodes on the same depth

- are visited from left to right and
- for node v, $\delta(v)$ 1's followed by a 0 are

appended to the bit vector that contains an initial `10`

Lemma: Space Usage of LOUDS

Representing a tree with n nodes requires $2n+1$ bits using LOUDS

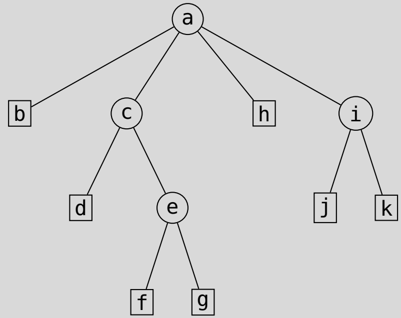


- write down the LOUDS representation of this example tree

Level Ordered Unary Degree Sequence (2/2)

```
       ab  ch  id  ejkfg
10111100110011001100000
```

- node start at pertinent `0`

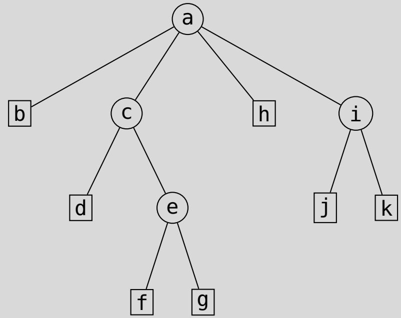


What is Fully-Functional?

Operations

- degree ℹ is leaf
- *i*-th child
- parent
- subtree size

- depth
- lowest common ancestor
- rank (pre- or post-order)
- ...



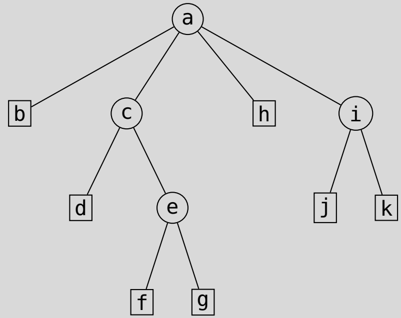
Making LOUDS Fully-Functional

```
             ab  ch  id  ejkfg
      1011110011001100110000 0
```

- degree of p: $p - select_0(rank_0(p)) - 1$
- i-th child of p:
  $select_0(rank_1(select_0(rank_0(p))) + i + 1)$
- parent of p:
  $select_0(rank_0(select_1(rank_0(p))) + 1)$

- explanation on the board

- subtree size **PINGO**



From Bit Vectors to Parentheses

- instead of `0` and `1`
- use `(` and `)`

- requires the same space
- can add relation between parentheses

Definition: Balanced String of Parentheses

A string of parentheses is balanced, if for each left parenthesis there exist unique right parenthesis to its right

- *findclose*(i): find the right parenthesis matching the left parenthesis at position i
- *findopen*(i): find the left parenthesis matching the right parenthesis at position i
- *excess*(i): find the difference between the number of left and right parentheses before position i
- *enclose*(i): given a parentheses pair with the left parenthesis at position i, return the position of the closest left parenthesis belonging to the parentheses pair enclosing it

- how can we answer *excess* queries **PINGO**

From Bit Vectors to Parentheses

- all parentheses operations can be answered in $O(1)$ time using $o(n)$ bits space
- here, a little bit simpler

- $excess(i) = rank_{\text{“(”}}(i+1) - rank_{\text{“)”}}(i+1)$
- $fwd\_search(i, d) = \min\{j > i \colon excess(j) - excess(i-1) = d\}$
- $bwd\_search(i, d) = \max\{j < i \colon excess(i) - excess(j-1) = d\}$

- $findclose(i) = fwd\_search(i, 0)$
- $findopen(i) = bwd\_search(i, 0)$
- $enclose(i) = bwd\_search(i, 2)$

- can be answered with a min-max-tree

Range Min-Max Trees (1/2)

Definition: Range Min-Max Tree

Given a bit vector B of length n and a block size b, store for each consecutive block (from s to e) of BV

- total excess in block: $\mathit{excess}(e) - \mathit{excess}(s-1)$
- minimum left-to-right excess in block: $\min\{\mathit{excess}(p) - \mathit{excess}(s-1) \colon p \in [s, e)\}$

and build a binary tree over these blocks, where each node stores the same total information for blocks in all its leaves

- example on the board

Lemma: Range Min-Max Tree Space

A range min-max tree with block size b for a bit vector of size n requires $n + O((n/b)\log n)$ bits of space

Range Min-Max Trees (2/2)

fwdsearch in a Range Min-Max Tree

- scan block
- if not found traverse tree
- identify block in tree
- scan block

- process c bits at a time
- first align with next c bits
- requires $O(c + b/c)$ time

- going up and down tree in $O(\log(n/b))$ time
- scanning last block requires $O(c + b/c)$ time

- by choosing $b = c \log n$ this requires
- $O(\log n)$ time and
  $n + O(n/(c \log n)) = n + o(n)$ bits space

Improvements

- two level approach
- build range min-max trees for chunks of size $\Theta(\log^3 n)$
- $O(\log \log n)$ query time inside a chunk
- can result in total query time of $O(\log \log n)$

Balanced Parentheses (1/2) [MR01]

- represent tree as depth-first traversal
- using balanced parentheses

Definition: BP

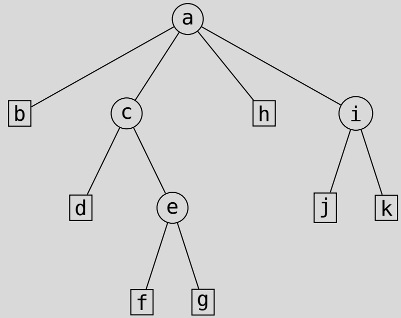
Starting at the root, traverse the tree in depth-first order and append a

- left parenthesis if a node is visited the first time
- right parenthesis if a node is visited the last time

to the bit vector

Lemma: Space Usage of BP

Representing a tree with n nodes requires $2n$ bits using *BP*

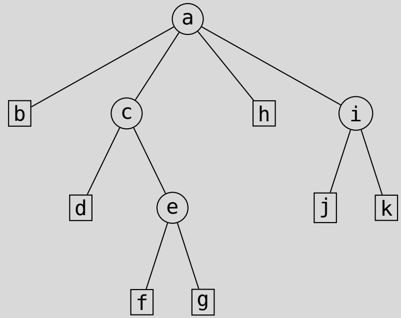


- write down the BP representation of this example tree

Balanced Parentheses (2/2)

```
ab cd ef g   h ij k
(()(()(()()))()(()()))
```

- node starts at first parenthesis
- subtree structure is encoded in parentheses



Making BP Fully-Functional

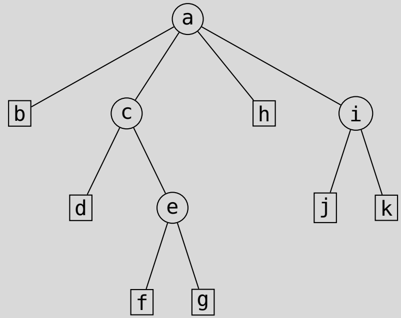
```
ab cd ef g   h ij k
(()((()(()()))()(()())))
```

- subtree size of p: $(\mathit{findclose}(p) - p + 1)/2$
- parent of p: $\mathit{enclose}(p)$

- explanation on the board

Complicated Constant Time [NS14]

- degree
- i-th child



Advantages and Disadvantages of Both Approaches

- LOUDS cannot answer subtree size
- BP cannot easily answer *i*-th child and degree

- all other operations can be done easily

Depth First Unary Degree Sequence (1/2) [Ben+05]

Definition: DFUDS

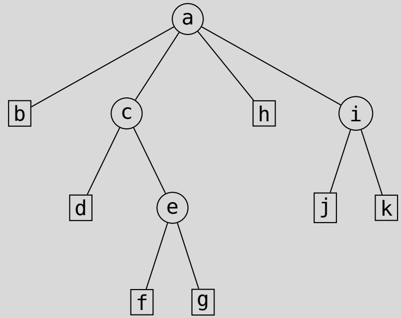
Starting at the root, traverse tree in depth-first order and append

- for node v, $\delta(v)$ left parentheses and
- a right parenthesis if v is visited the first time

to the bit vector that initially contains a left parenthesis ℹ to make them balanced

Lemma: Space Usage of DFUDS

Representing a tree with n nodes requires $2n$ bits using DFUDS

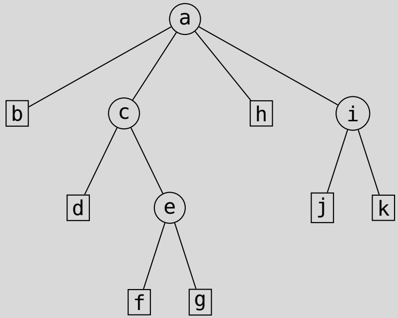


- write down the DFUDS representation of this example tree

Depth First Unary Degree Sequence (2/2)

```
  a    bc  de  fghi  jk
((((())(())(()))))(()))
```

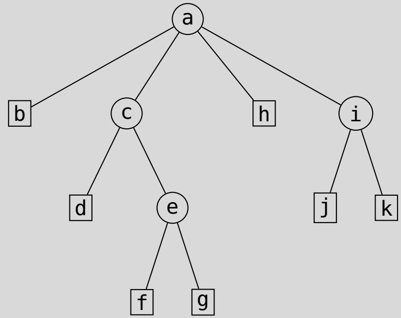
- node starts at first parenthesis
- subtree structure is encoded



Making DFUDS Fully-Functional

```
    a    bc  de   fghi   jk
  (((((())(())(())))(())))
```

- degree of p: $select_{``)''}(rank_{``)''}(p)+1)-p$
- i-th child of p: $findclose(select_{``)''}(rank_{``)''}(p)+1)-i)+1$
- parent of p: $select_{``)''}(rank_{``)''}(findopen(p-1)))+1$
- subtree size of p: $(findclose(enclose(p))-p)/2+1$



- explanation on the board

Conclusion and Outlook

This Lecture

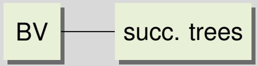
- three succinct tree representations
- different advantages and disadvantages

- min-max-trees

Next Lecture

- succinct graphs

Advanced Data Structures



Bibliography I

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[MR01] J. Ian Munro and Venkatesh Raman. “Succinct Representation of Balanced Parentheses and Static Trees”. In: *SIAM J. Comput.* 31.3 (2001), pages 762–776. DOI: 10.1137/S0097539799364092.

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