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Advanced Data Structures

Lecture 04: Predecessor and Range Minimum Query Data Structures

Florian Kurpicz

PINGO

https://pingo.scc.kit.edu/267787

Recap

Succinct Planar Graphs

- using spanning tree of graph and
- special spanning tree of dual graph
- both represented succinctly
- represent planar graph succinctly
- store whether edge is in spanning tree or not

Predecessor and Successor

Setting

- assume universe $\mathcal{U} = [0, u)$
- let $u = 2^w$
- sorted array of n integers $A \subseteq \mathcal{U}$
- $\log n \leq w$ ⓘ since $n \leq u$

Definition: Predecessor & Successor

Given an array A of n integers from an universe $\mathcal{U}$ and an integer $x \in \mathcal{U}$, the predecessor and successor of x in A are

- $pred(A, x) = \max\{y \in A \colon y \leq x\}$
- $succ(A, x) = \min\{y \in A \colon y \geq x\}$

0	1	2	3	4	5	6	7	8	9
0	1	2	4	7	10	20	21	22	32

- $pred(3) = 2$
- $pred(10) = 10$
- $succ(23) = 32$

- in what time and space can we solve this using bit vectors? **PINGO**

Predecessor and Successor: Simple Solutions

- binary search
- $O(\log n)$ query time
- no space overhead

- using bit vector
- $O(1)$ query time
- $u + o(u)$ bits space

Predecessor of x in Bit Vector

- $z = rank_1(x + 2)$
- predecessor is $select_1(z)$

0	1	2	3	4	5	6	7	8	9
0	1	2	4	7	10	20	21	22	32

- $pred(3) = 2$

`111010010010000000001110000000001`

- $rank_1(21) = 6$
- $select_1(6) = 10$
- $pred(19) = 10$

Elias-Fano Coding [Eli74; Fan71] (1/3)

- n integers from universe $\mathcal{U} = [0, u)$
- split number in upper and lower halves
- upper half: $\lceil \log n \rceil$ most significant bits
- lower half: $\lceil \log u - \log n \rceil$ remaining bits

Upper Half

- monotonous sequence of $\lceil \log n \rceil$ bit integers
- not strictly monotonous
- let $p_0, \ldots, p_{n-1}$ be sequence
- use bit vector of length $2n + 1$ bits
- represent p_i with a 1 at position $i + p_i$
- rank and select support requires $o(n)$ bits

Lower Half

- store lower half plain using $\lceil \log \frac{u}{n} \rceil$ bits
- $n \log \lceil \frac{u}{n} \rceil$ bits for lower half

0	1	2	3	4	5	6	7	8	9
0	1	2	4	7	10	20	21	22	32

- 0: `000000`
- 1: `000001`
- 2: `000010`
- 4: `000100`
- 7: `000111`
- 10: `001010`
- 20: `010100`
- 21: `010101`
- 22: `010110`
- 30: `100000`

Elias-Fano Coding (2/3)

Access i-th Element

- upper: $select_1(i) - i$
- lower: corresponding bits from lower bit vector

Predecessor x

- let x' be $\lceil \log n \rceil$ MSB of x
- $p = select_0(x')$ ℹ $select_0(0)$ returns 0
- scan corresponding values in lower till predecessor is found
- how many elements do we have to scan?
  PINGO
- scanning $O(u/n)$ elements

0	1	2	3	4	5	6	7	8	9
0	1	2	4	7	10	20	21	22	32

- 0: 000000
- 1: 000001
- 2: 000010
- 4: 000100
- 7: 000111
- 10: 001010
- 20: 010100
- 21: 010101
- 22: 010110
- 30: 100000

upper: 1110110100011100010 0

lower: 00 01 10 00 11 10 00 01 10 00

Elias-Fano Coding (3/3)

Lemma: Elias-Fano Coding

Given an array containing n distinct integers from a universe $\mathcal{U} = [0, u)$, the array can be represented using

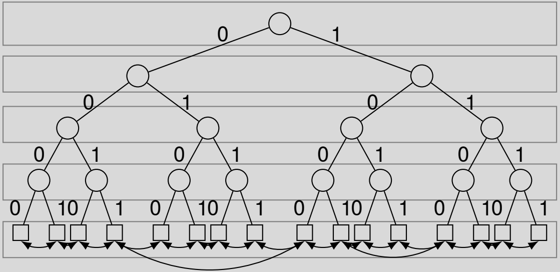
$$n(2 + \log\lceil \frac{u}{n} \rceil) \text{ bits}$$

while allowing $O(1)$ access time and $O(\log \frac{u}{n})$ predecessor/successor time

x-Fast Tries

- each number has *w* bits
- build binary tree where leaves represent numbers
- edges are labeled `0` or `1`
- labels on path from root to leaf are value represented in leaf

- store nodes in hash tables with bit prefix as key
- also store pointer to *min* and *max* in right and left subtree
- leaves are stored in doubly linked list
- using perfect hashing on each level requires $O(wn)$ space



- pointers to *min* and *max* are missing
- tree most likely not complete

x-Fast Tries: Queries

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- traversing tree requires $O(w)$ time
- using binary search on levels requires $O(\log w)$ time
- if value not found go to *min* or *max* depending on query
- if value is found use doubly linked list to find predecessor or successor

- example on the board

y-Fast Tries

- x-fast trie requires $O(wn)$ space
- group w consecutive objects into one block B_i
- for each block B_i choose maximum m_i as representative
- build x-fast trie for representatives
- store blocks in balanced binary trees

- x-fast trie requires $O(n)$ space
- search in x-fast trie requires $O(\log w) = O(\log \log n)$ time For large n
- search in balanced binary tree requires $O(\log w) = O(\log \log n)$ time

- example on the board

Dynamic y-Fast Trie

- use cuckoo hashing
- representative does not have to be maximum
- any element separating groups suffices
- merge and split blocks that are too small/too big
- query time only expected

Range Minimum Queries

Setting

- array of n integers
- not necessarily sorted

Definition: Range Minimum Queries

Given an array of A of n integers

$$rmq(A, s, e) = \arg\min_{s \le i \le e} A[i]$$

returns the position of minimum in $A[s, e]$

0	1	2	3	4	5	6	7	8	9
8	2	5	1	9	11	10	20	22	4

- $rmq(0, 9) = 3$
- $rmq(0, 2) = 1$
- $rmq(4, 8) = 4$

- naive in $O(1)$ time
- how much space does a naive $O(1)$-time solution need **PINGO**
- using $O(n^2)$ space $rmq(s, e) = M[s][e]$

Range Minimum Queries in $O(1)$ Time and $O(n \log n)$ Space

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- instead of storing all solutions
- store solutions for intervals of length 2^k for every k
- $M[0..n)[0..\lfloor \log n \rfloor)$

Queries

- query $rmq(A, s, e)$ is answered using two subqueries
- let $\ell = \lfloor log(e - s - 1) \rfloor$
- $m_1 = rmq(A, s, s + 2^\ell - 1)$ and $m_2 = rmq(A, e - 2^\ell + 1, e)$
- $rmq(A, s, e) = \arg\min_{m \in \{m_1, m_2\}} A[m]$

Construction

$$\begin{aligned} M[x][\ell] &= rmq(A, x, x + 2^\ell - 1) \\ &= \arg\min\{A[i] : i \in [x, x + 2^\ell)\} \\ &= \arg\min\{A[i] : i \in \{rmq(A, x, x + 2^{\ell-1} - 1), \\ &= \qquad rmq(A, x + 2^{\ell-1}, x + 2^\ell - 1)\}\} \\ &= \arg\min\{A[i] : i \in \{M[x][\ell - 1], \\ &= \qquad M[x + 2^{\ell-1}][\ell - 1]\}\} \end{aligned}$$

- how much time do we need to fill the table? **PINGO**
- dynamic programming in $O(n \log n)$ time

Range Minimum Queries in $O(1)$ Time and $O(n)$ Space (1/2)

- divide A into blocks of size $s = \frac{\log n}{4}$
- blocks $B_1, \ldots, B_m$ with $m = \lceil n/s \rceil$
- query $rmq(A, s, e)$ is answered using at most three subqueries
- one query spanning multiple block
- at most two queries within a block each

- example on the board

Query Spanning Blocks

- use array B containing minimum within each block
- B has m entries
- use $O(n \log n$ data structure for B
- $O(m \log m) = O(\frac{n}{s} \log \frac{n}{s}) = O(\frac{n}{\log n} \log \frac{n}{\log n}) = O(n)$
- use additional array B' storing position of minimum in each block

- for queries within block use Cartesian trees

Cartesian Trees (1/2)

Definition: Cartesian Tree

Given an array A of length n, a Cartesian tree $C(A)$ of a is a labeled binary tree with

- root r is labeled with $x = \arg\min\{A[i] : i \in [0, n)\}$
- left and right children of r are Cartesian trees $C(A[0, x))$ and $C(A[x+1, n))$ if interval exists

Lemma: Cartesian Tree Construction

A Cartesian tree for an array of size n can be computed in $O(n)$ time

Proof (Sketch)

- scan array from left to right
- insert each element by
  - following rightmost path from leaf to root till element can be inserted
  - everything below becomes left child of new node
- each node is removed at most once from the rightmost path
- moving subtree to left child in constant time gives $O(n)$ construction time

- example on the board

Cartesian Trees (2/2)

Lemma: Equality of Cartesian Trees

Given two arrays A and B of length n with equal Cartesian trees, then

$$rmq(A, s, e) = rmq(B, s, e)$$

for all $0 \leq s < e < n$

Proof (Sketch)

- proof by induction over the size of the array
- if the array has size one, this is true
- assuming this is correct for arrays of size n, showing this for arrays of size $n + 1$ uses recursive definition of Cartesian trees

Range Minimum Queries in $O(1)$ Time and $O(n)$ Space (2/2)

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Query Within a Block

- consider every possible Cartesian tree for arrays of size $s = \frac{\log n}{4}$
- tree can be represented using $2s + 1$ bits
- store bit representation of Cartesian tree for every block
- for every possible Cartesian tree and every start and end position store position of minimum
- $O(2^{2s+1} \cdot s \cdot s \cdot \log s) =$ $O(\sqrt{n} \log^2 n \cdot \log \log n) = O(n)$ space

Conclusion and Outlook

This Lecture

- successor and predecessor data structures
- range minimum query data structures

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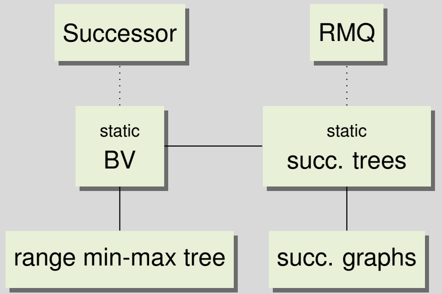

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