Karlsruhe Institute of Technology

# Advanced Data Structures

## Lecture 06: Orthogonal Range Searching and BSP Trees

Florian Kurpicz and *Stefan Walzer*

# PINGO

# Motivation: Query Set of Points

- given set of points $P = \{p_1, \ldots, p_n\}$ with $p_i = (x_i, y_i)$
- find all points in $[x, y] \times [x', y']$
- higher dimensions are possible

- think about database queries
- each dimension is a property
- searching for objects fulfilling all properties of range

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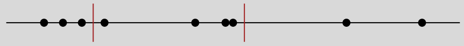

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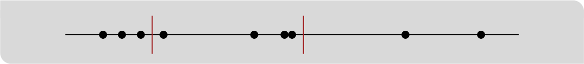

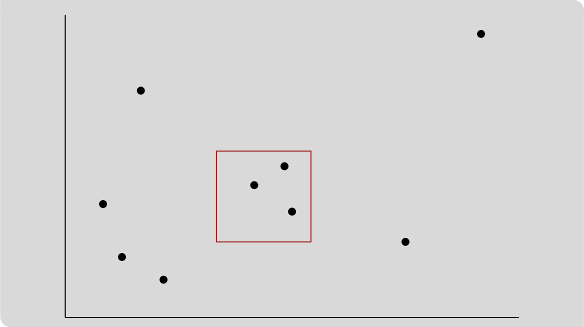

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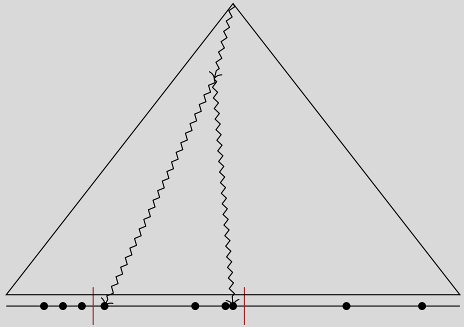

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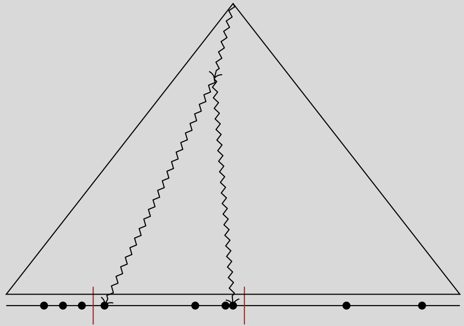
- query for both $x$ and $x'$
- find leaves $b$ and $e$ for $x$ and $x'$
- let node $v$ be node where paths to leaves split
- report all leaves between $b$ and $e$



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**Lemma: 1-Dimensional Range Searching**

Let $P$ be a set of $n$ 1-dimensional points. $P$ can be stored in a BBST that requires $O(n)$ words space, can be constructed in $O(n \log n)$ time, and can answer range searching queries in $O(\log n + occ)$ time

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**Proof (Sketch Time)**

- reporting all children in a subtree requires $O(occ)$ time
- BBST has depth $O(\log n)$
- search paths starting at $v$ have length $O(\log n)$
- report all subtrees to the right of the left path
- report all subtrees to the left of the right path

# 2-Dimensional Rectangular Range Searching

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**Important**

- assume no two points have the same $x$- or $y$-coordinate $\Rightarrow$ general position

- generalize 1-dimensional idea
- 1-dimensional
  - split number of points in half at each node
  - points consist of one value
- 2-dimensional
  - points consist of two values
  - split number of points in half w.r.t. one value
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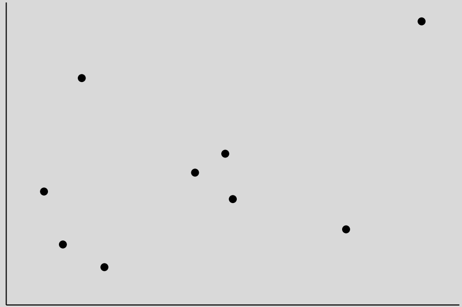

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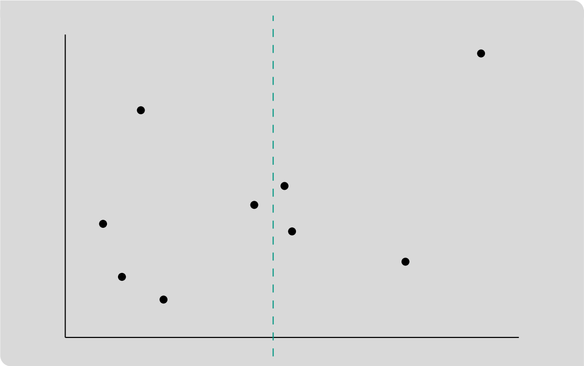

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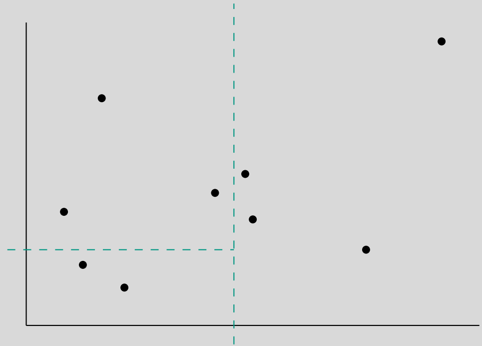

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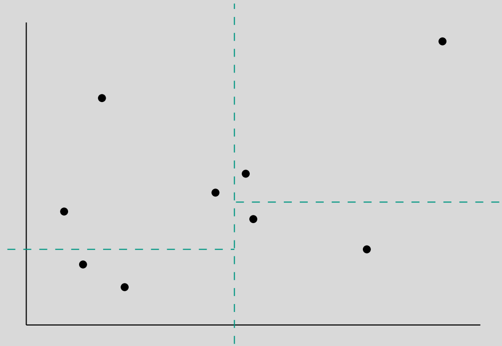

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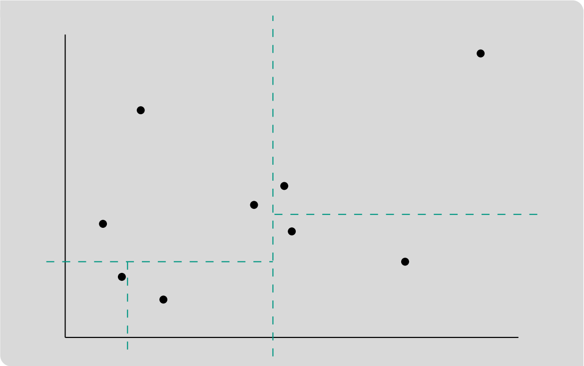

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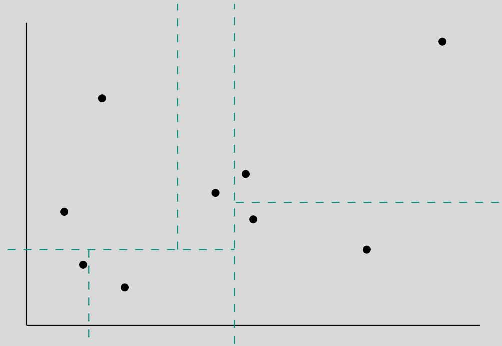

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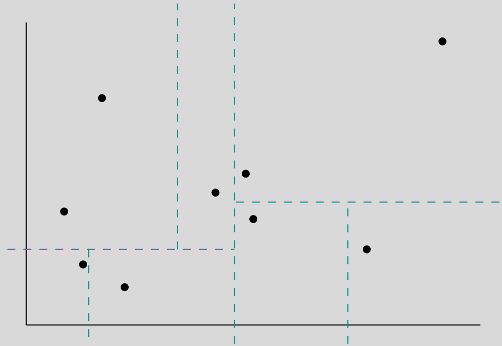

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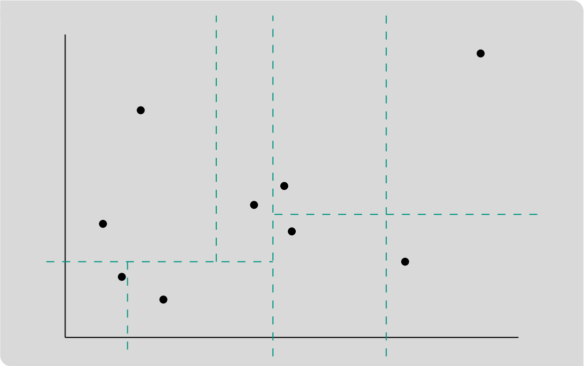

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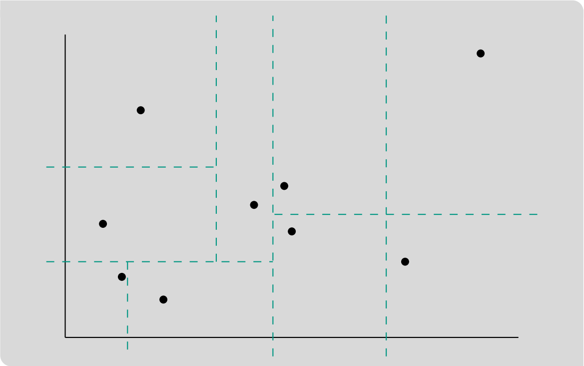

# Kd-Trees (1/4)

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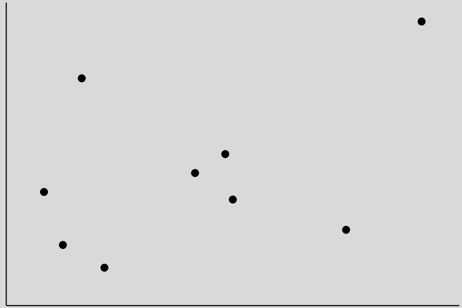

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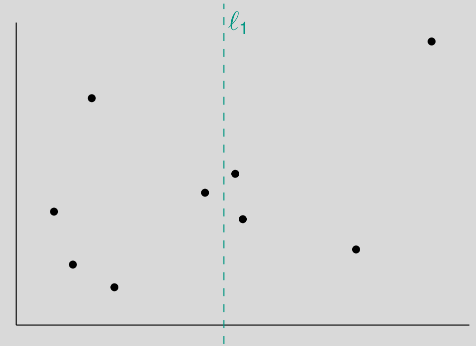


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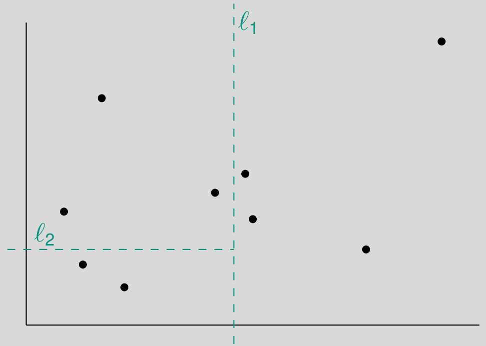


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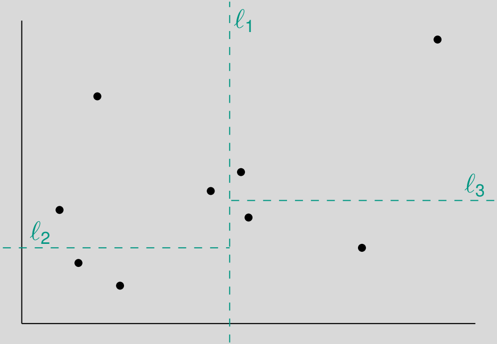


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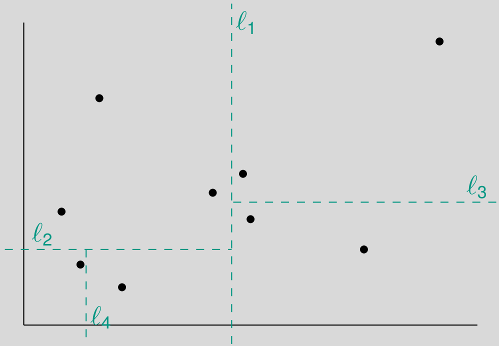

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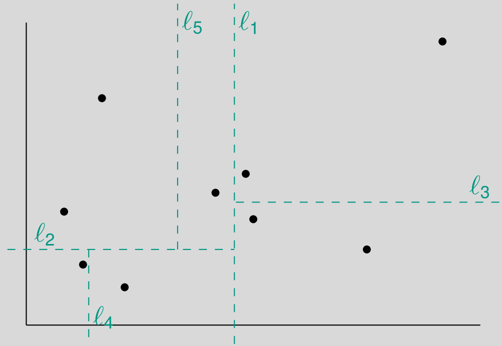


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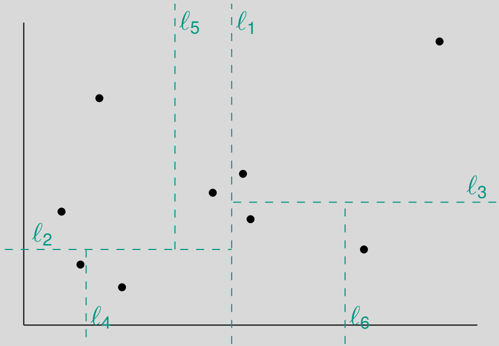

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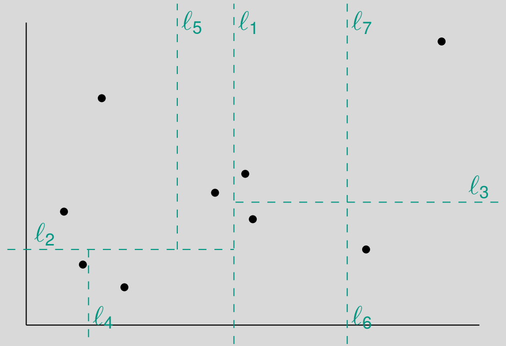


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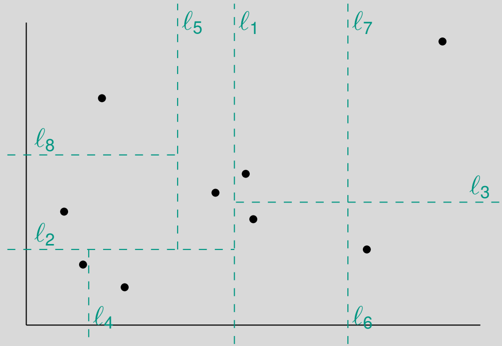


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## Proof (Sketch: Time)

- finding the splitter is easy due to presorted points
- splitting requires $T(n)$ time with

$$T(n) = \begin{cases} O(1) & n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & n > 1 \end{cases}$$

- results in $O(n \log n)$ running time
- presorting in same time bound

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## Proof (Sketch)

- $O(occ)$ time necessary to report points
- look at number of regions intersected by any vertical line
- upper bound for the regions intersected by query (for left and right edge of rectangle)
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## Proof (Sketch, cnt.)

- for vertical lines consider every inner node at odd depth
- starting at root's children
- can intersect two regions
- number of nodes is $\lceil n/4 \rceil$ ℹ halved at each level
- number of intersected regions is $Q(n)$ with

$$Q(n) = \begin{cases} O(1) & n = 1 \\ 2 + 2Q(\lceil n/4 \rceil) & n > 1 \end{cases}$$

- results in $Q(n) = O(\sqrt{n})$
- $O(\sqrt{n} + k)$ total running time

# Teaser: Other Space-Partitioning Search Trees

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  - recursively group nearby objects into minimal bounding boxes (bottom-up)
  - works also for complex shapes, not only points
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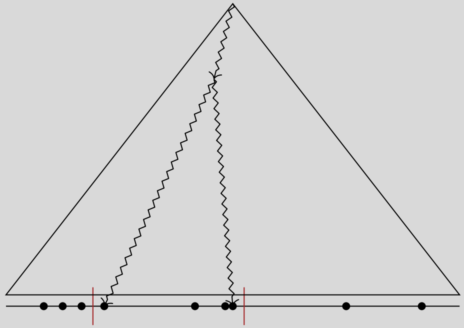
- one BBST build on the *x*-coordinates
  - same as for 1-dimensional queries
- each inner node is associated with a set of points
- build a BBST for the *y*-coordinates of associated points for each inner node
  - store points in leaves not just *y*-coordinates
  - this BBST is used for reporting

- space-query-time trade-off
- faster queries but larger



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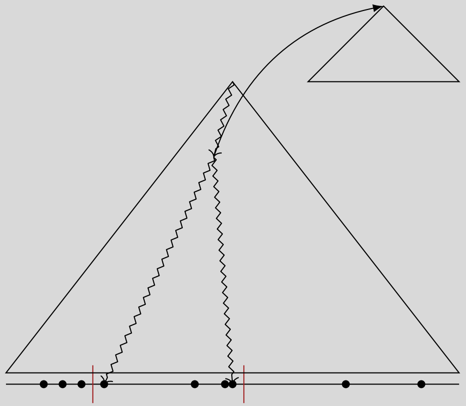

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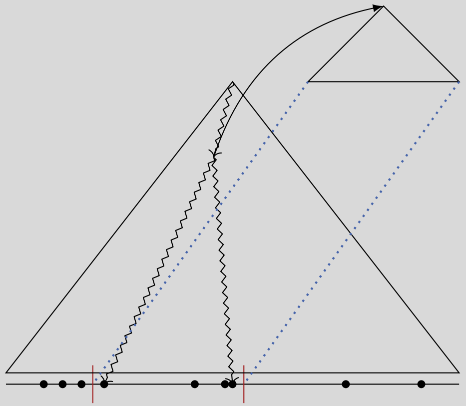

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**Lemma: Space Range Tree**

A range tree on a set of $n$ points in the plane requires $O(n \log n)$ words space

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- how much space do the associated BBSTs require in total? **PINGO**

**Lemma: Space Range Tree**

A range tree on a set of $n$ points in the plane requires $O(n \log n)$ words space

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- how much faster is the range tree?

# Range Trees (3/4)

- 2-dimensional rectangular range search reduced to two 1-dimensional range searches
- look in BBST for *x*-coordinates ℹ same as 1-dimensional case
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### Proof (Sketch)

- each search in an associated BBST $t$ requires $O(\log n + occ_t)$ time
- $O(\log n)$ associated BSSTs $T$ are searched ℹ as seen in 1-dimensional case
- total query time $\sum_{t \in T} O(\log n + occ_t)$
- $\sum_{t \in T} O(occ_t) = O(occ)$
- $\sum_{t \in T} O(\log n) = O(\log^2 n)$
- total time: $O(\log^2 n + occ)$

# Range Trees (4/4)

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**Proof (Sketch Construction Space)**

- recursive space $S_d(n)$ with $S_2(n) = O(n \log n)$ words
- $T_d(n) = O(n \log n) + O(\log n) \cdot T_{d-1}(n)$
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# Fractional Cascading (1/2)

- sorted sets $S_1, \ldots, S_m$
- $|S_1| = n$ and $S_{i+1} \subseteq S_i$
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- in addition to $S_i$ store pointers to $S_{i+1}$
- for each element in $S_i$ store pointer to successor in $S_{i+1}$
- possible because $S_{i+1} \subseteq S_i$

# Fractional Cascading (2/2)

**Lemma: Fractional Cascading**

Given sets $S_1, \ldots, S_m$ with $|S_1| = n$ and $S_{i+1} \subseteq S_i$, find a range in all $S_i$'s using fractional cascading requires $O(m + \log n + occ)$ time

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**Proof (Sketch)**

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- how to apply to range trees?
- instead of associated BBSTs store leaf data in arrays for all nodes but root
- each node has associated data
- store two successor pointers to the associated data in the left and right child
- two pointers to cover all possible paths
- this is a layered range tree

# Query Layered Range Trees

- search in BBST for $x$-coordinates remains the same
- to search $y$-coordinates first search associated BBST of root
- same as initial binary search for fractional cascading
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**Proof (Sketch)**

- the initial search requires $O(\log n)$ time
- the search in the associated BBST of the root requires $O(\log n)$ time
- remaining searches in associated data $a$ requires $O(1 + occ_a)$ time
- each point is reported once
- total time: $O(\log n + occ)$

# General Sets of Points (1/2)

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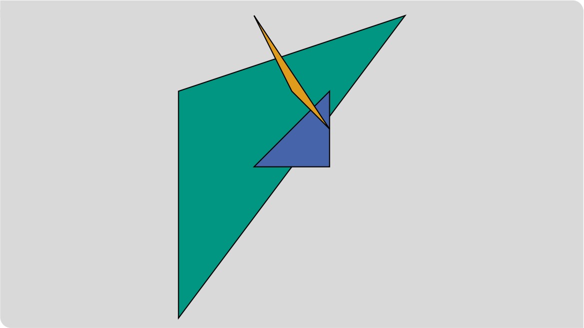
$$[(x-\epsilon)..(x'+\epsilon)] \times (y-\epsilon)..[(y'+\epsilon)]$$

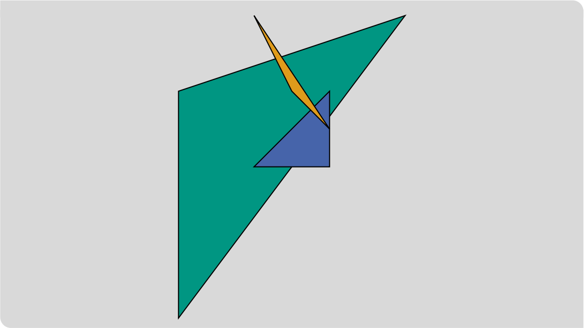
# Now: Render Object

- hidden surface removal
- which pixel is visible
- important for rendering



# $z$-Buffer Algorithm

- transform scene such that viewing direction is positive $z$-direction
- consider objects in scene in arbitrary order
- maintain two buffers
  - frame buffer ℹ currently shown pixel
  - $z$-buffer ℹ $z$-coordinate of object shown
- compare $z$-coordinate of $z$-buffer and object



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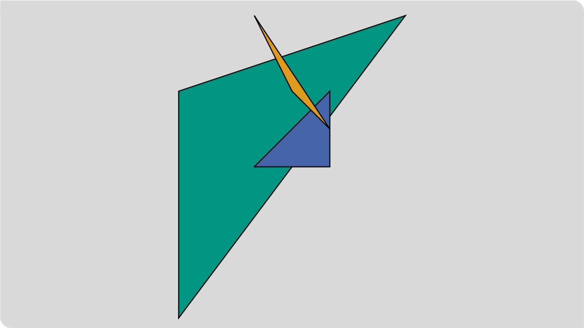
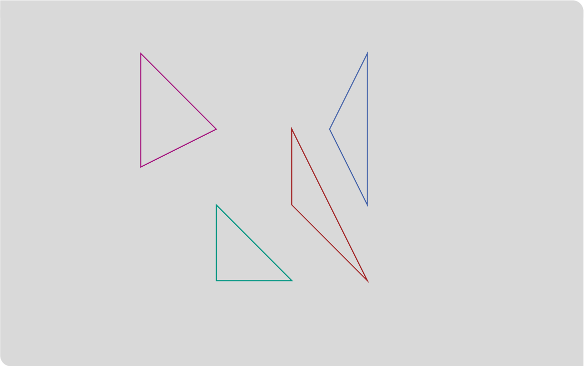
- first sort object in depth-order
- depth-order may not always exist
- how to efficiently sort objects?



# BSP Trees (1/2)

- partition space using hyperplanes
- binary partition ℹ similar to kd-tree
- hyperplanes create half-spaces and cut objects into fragments



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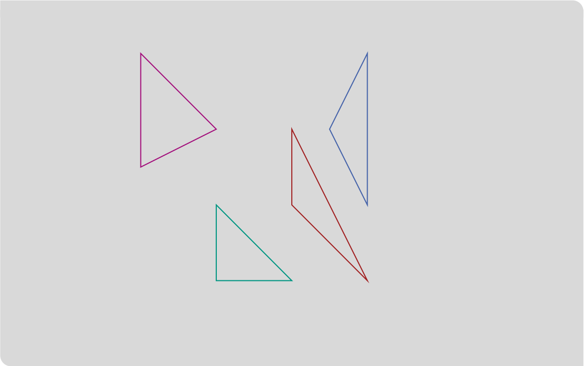

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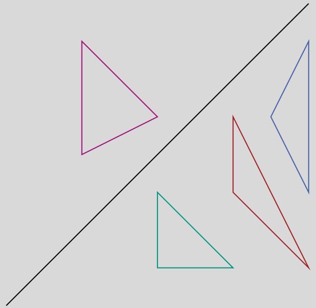

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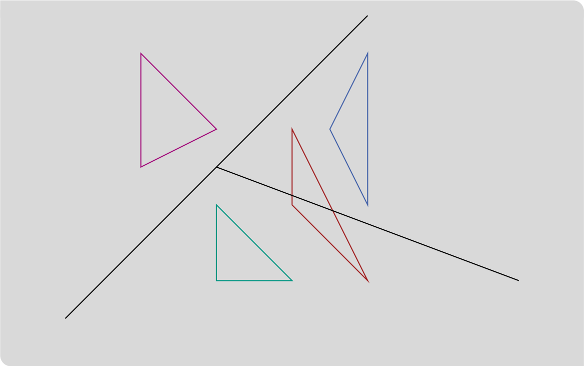

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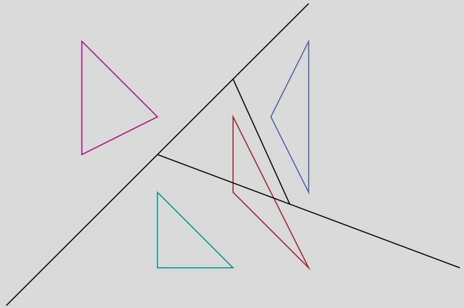

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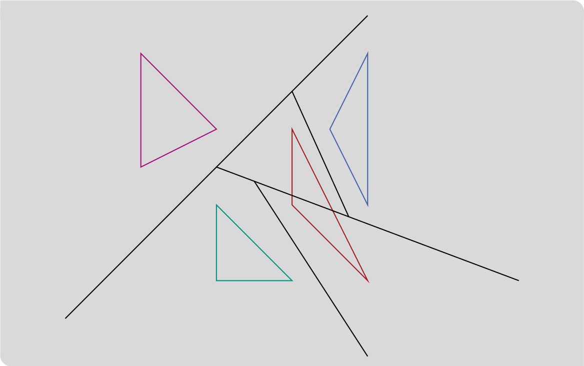

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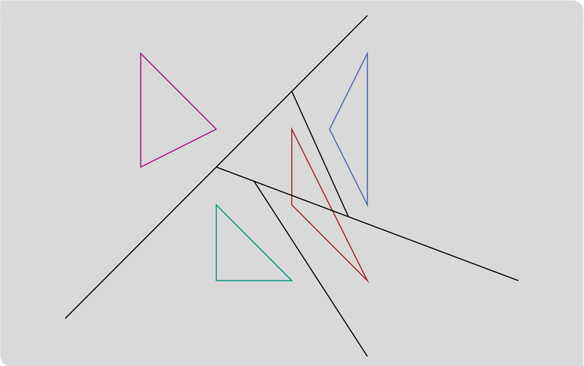
- each split creates two nodes in a tree
- if number of objects in space is one: leaf
- otherwise: inner node



# BSP Trees (2/2)

- for leaf: store object/fragment
- for inner node $v$: store hyperplane $h_v$ and the objects contained in $h_v$
- left child represents objects in upper half-space $h^+$
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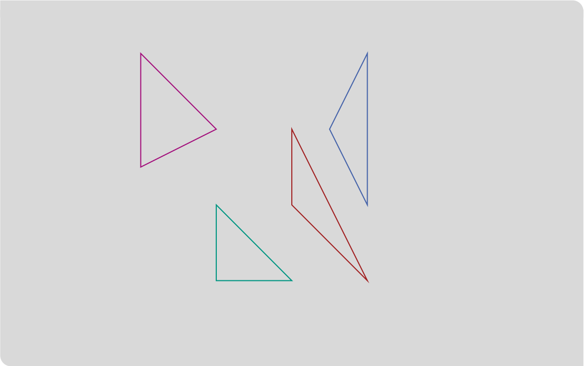

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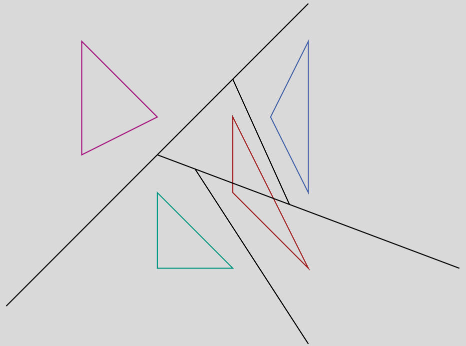


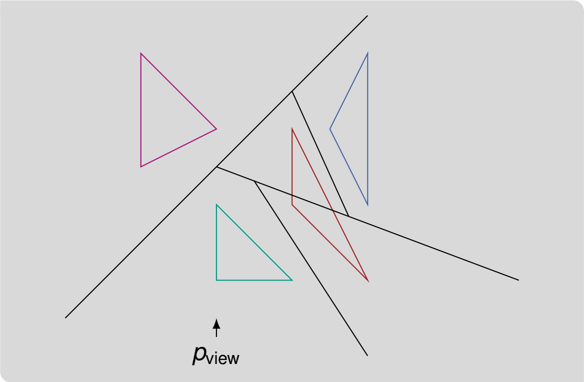
- space of BSP tree is number of objects stored at all nodes
- what about fragments?
- too many fragments can make the tree big

# Auto-Partitioning

- sorting points for kd-trees worked well
- BSP-tree is used to sort objects in depth-order
- auto-partitioning uses splitters through objects
  - 2-dimensional: line through line segments
  - 3-dimensional: half-plane through polygons

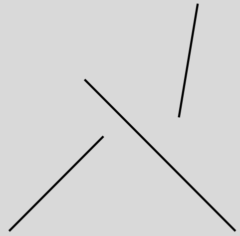
# Painter's Algorithm

- consider view point $p_{\text{view}}$
- traverse through tree and always recurse on half-space that does not contain $p_{\text{view}}$ first
- then scan-convert object contained in node
- then recurse on half-space that contains $p_{\text{view}}$



# Constructing Planar BSP Trees (1/3)

- use auto-partitioning
- construction similar to construction of kd-tree
- store all necessary information
  - hyperplane
  - objects in hyperplane
- how to determine next hyperplane?
- creating fragments increases size of BSP tree



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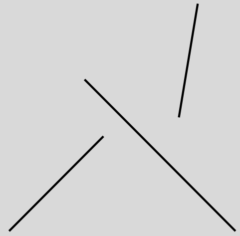

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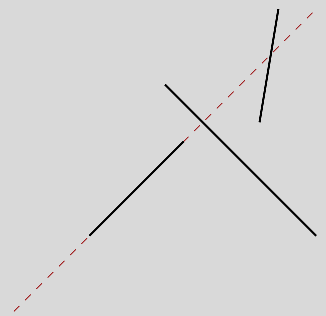

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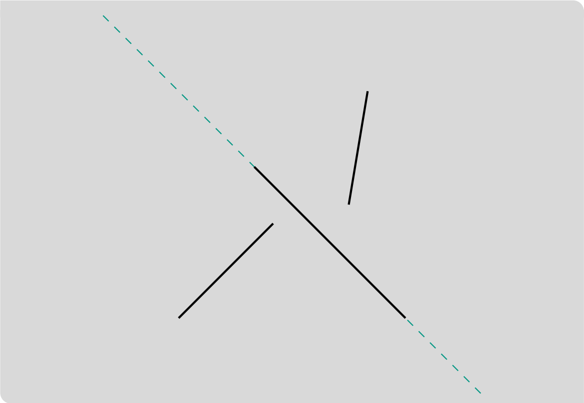

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## Proof (Sketch)

- distance of lines $dist_{s_i}(s_j) =$
$$\begin{cases} \#\text{ segments inters. } \ell(s_i) & \\ \text{ between } s_i \text{ and } s_j & \ell(s_i) \text{ inters. } s_j \\ \infty & \text{otherwise} \end{cases}$$
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## Proof (Sketch, cnt.)

- let $dist_{s_i}(s_j) = k$ and $s_{j_1}, \ldots, s_{j_k}$ be segments between $s_i$ and $s_j$
- what is the probability that $\ell(s_i)$ cuts $s_j$?
- this happens if no $s_{j_x}$ is processed before $s_i$
- since order is random

$$\mathbb{P}[\ell(s_i) \text{ cuts } s_j] \leq \frac{1}{dist_{s_i}(s_j) + 2}$$

# Constructing Planar BSP Trees (3/3)

## Proof (Sketch, cnt.)

- expected number of cuts

$$\mathbb{E}[\text{\# cuts generated by } s_i] \leq \sum_{j \neq i} \frac{1}{dist_{s_i}(s_j) + 2} \leq 2 \sum_{k=0}^{n-2} \frac{1}{k+2} \leq 2 \ln n$$

- all lines generate at most $2n \ln n$ fragments

# Constructing Planar BSP Trees (3/3)

## Proof (Sketch, cnt.)

- expected number of cuts

$$\mathbb{E}[\#\text{ cuts generated by } s_i] \leq \sum_{j \neq i} \frac{1}{dist_{s_i}(s_j) + 2} \leq 2 \sum_{k=0}^{n-2} \frac{1}{k+2} \leq 2 \ln n$$

- all lines generate at most $2n \ln n$ fragments

## Lemma: BSP Construction

A BSP tree of size $O(n \log n)$ can be computed in expected time $O(n^2 \log n)$

# Constructing Planar BSP Trees (3/3)

## Proof (Sketch, cnt.)

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- all lines generate at most $2n \ln n$ fragments

## Lemma: BSP Construction

A BSP tree of size $O(n \log n)$ can be computed in expected time $O(n^2 \log n)$

## Proof (Sketch)

- computing permutation in linear time
- construction is linear in number of fragments to be considered
- number of fragments in subtree is bounded by $n$
- number of recursions is $n \log n$

# Conclusion and Outlook

## This Lecture

- orthogonal range searching
- BSP trees

## Advanced Data Structures