Karlsruhe Institute of Technology

Advanced Data Structures

Lecture 06: Orthogonal Range Searching and BSP Trees

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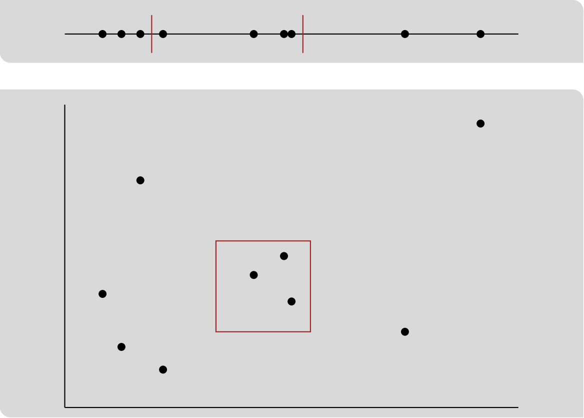
PINGO

Motivation: Query Set of Points

- given set of points $P = \{p_1, \ldots, p_n\}$ with $p_i = (x_i, y_i)$
- find all points in $[x, y] \times [x', y']$
- higher dimensions are possible

- think about database queries
- each dimension is a property
- searching for objects fulfilling all properties of range

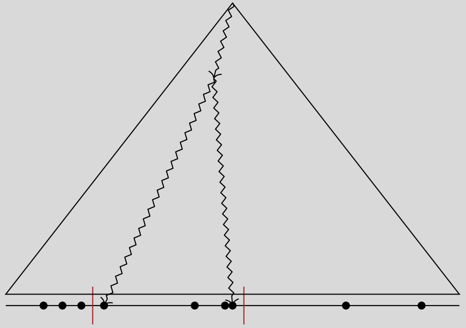


1-Dimensional Range Searching (1/2)

- consider 1-dimensional problem
- range is $[x..x']$
- points $P = \{x_1, \ldots, x_n\}$ are just numbers

- build BBST where each leaf contains a point
- inner node v store splitting value x_v

- query for both x and x'
- find leaves b and e for x and x'
- let node v be node where paths to leaves split
- report all leaves between b and e



1-Dimensional Range Searching (2/2)

- how long does it take to report all children of a subtree with k leaves in a BBST? **PINGO**

Lemma: 1-Dimensional Range Searching

Let P be a set of n 1-dimensional points. P can be stored in a BBST that requires $O(n)$ words space, can be constructed in $O(n \log n)$ time, and can answer range searching queries in $O(\log n + occ)$ time

Proof (Sketch Time)

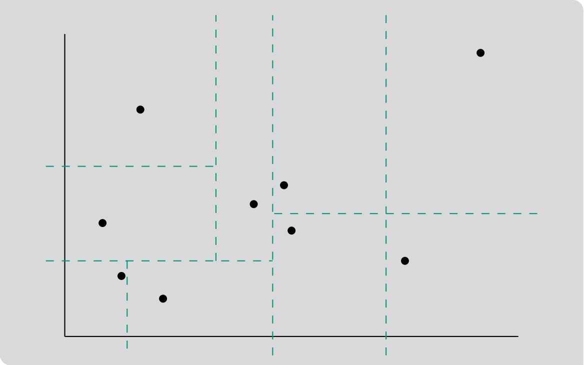
- reporting all children in a subtree requires $O(occ)$ time
- BBST has depth $O(\log n)$
- search paths starting at v have length $O(\log n)$
- report all subtrees to the right of the left path
- report all subtrees to the left of the right path

2-Dimensional Rectangular Range Searching

Important

- assume no two points have the same x- or y-coordinate ⇒ general position

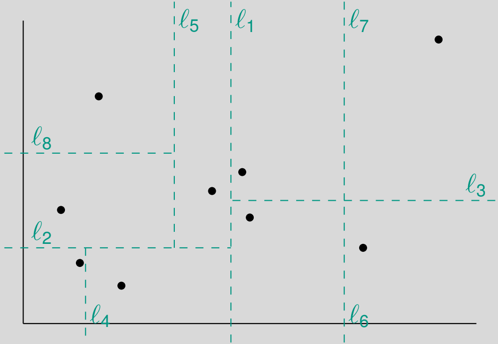
- generalize 1-dimensional idea
- 1-dimensional
  - split number of points in half at each node
  - points consist of one value
- 2-dimensional
  - points consist of two values
  - split number of points in half w.r.t. one value
  - switch between values depending on depth



Kd-Trees (1/4)

- considering the 2-dimensional case
- each inner node at an even depth
  - splits the leaves in its subtree in half
  - using the x-coordinate
- each inner node at an odd depth
  - splits the leaves in its subtree in half
  - using the y-coordinate
- until each region contains a single point
- each leaf represents a point

- splitting in linear time is complicated
- better presort based on x- and y-coordinate
- inner nodes store splitter (line)



Kd-Trees (2/4)

Lemma: Kd-Tree Construction

A kd-tree for a set of n points requires $O(n)$ words space and can be constructed in $O(n \log n)$ time

Proof (Sketch: Space)

- there are $O(n)$ leaves
- there are $O(n)$ inner nodes
- a binary tree requires $O(1)$ words per node
- O(n) words total space

Proof (Sketch: Time)

- finding the splitter is easy due to presorted points
- splitting requires $T(n)$ time with

$$T(n) = \begin{cases} O(1) & n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & n > 1 \end{cases}$$

- results in $O(n \log n)$ running time
- presorting in same time bound

Kd-Trees (3/4)

- use splitter depending on depth to identify paths through tree
- if a region is fully contained in query: report region
- if a region is intersected by query: check if point has to be reported

- precomputation of query not necessary
- current region can be computed during query
- using splitters

- example on the board

Kd-Trees (4/4)

Lemma: Kd-Tree Query

A query with an axis-parallel rectangle in a Kd-tree storing n points in the plane can be performed in $O(\sqrt{n} + occ)$ time

Proof (Sketch)

- $O(occ)$ time necessary to report points
- look at number of regions intersected by any vertical line
- upper bound for the regions intersected by query (for left and right edge of rectangle)
- upper bound for top and bottom edges are the same

Proof (Sketch, cnt.)

- for vertical lines consider every inner node at odd depth
- starting at root's children
- can intersect two regions
- number of nodes is $\lceil n/4 \rceil$ ℹ halved at each level
- number of intersected regions is $Q(n)$ with

$$Q(n) = \begin{cases} O(1) & n = 1 \\ 2 + 2Q(\lceil n/4 \rceil) & n > 1 \end{cases}$$

- results in $Q(n) = O(\sqrt{n})$
- $O(\sqrt{n} + k)$ total running time

Teaser: Other Space-Partitioning Search Trees

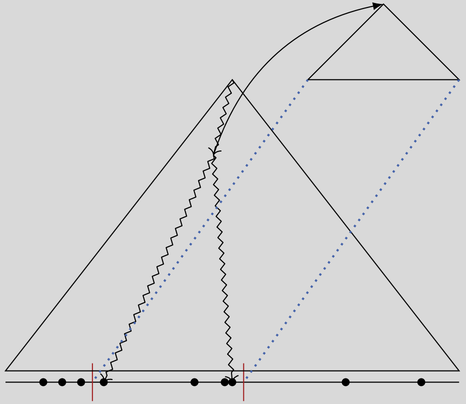
- Quadtrees
  - recursive partition of input space into four children (top-down)
  - generalizes to higher dimensions (Octtree)
  - often used in computer graphics
- R-Trees
  - recursively group nearby objects into minimal bounding boxes (bottom-up)
  - works also for complex shapes, not only points
  - many variants exist (R^*-Trees, R+Trees)
  - often used in spatial databases

Example on the board

Range Trees (1/4)

- one BBST build on the *x*-coordinates
  - same as for 1-dimensional queries
- each inner node is associated with a set of points
- build a BBST for the *y*-coordinates of associated points for each inner node
  - store points in leaves not just *y*-coordinates
  - this BBST is used for reporting

- space-query-time trade-off
- faster queries but larger



Range Trees (2/4)

- the BBST for the x-coordinates requires $O(n)$ words of space
- how much space do the associated BBSTs require in total? **PINGO**

Lemma: Space Range Tree

A range tree on a set of n points in the plane requires $O(n \log n)$ words space

Proof (Sketch)

- BBST for x-coordinates has depth $O(\log n)$
- all points are represented on each depth exactly once

Proof (Sketch, cnt.)

- all associated BBSTs on each depth contain every point exactly once
- total size of all BBSTs on each depth is $O(n)$
- total space $O(n \log n)$ words

- how much faster is the range tree?

Range Trees (3/4)

- 2-dimensional rectangular range search reduced to two 1-dimensional range searches
- look in BBST for x-coordinates ℹ same as 1-dimensional case
- instead of reporting subtrees to the right/left of paths search associated BBSTs
- report results in leaves of associated BBSTs

Lemma: Range Tree Query Time

A query with an axis-parallel rectangle in a range tree storing n points requires $O(\log^2 n + occ)$ time

Proof (Sketch)

- each search in an associated BBST t requires $O(\log n + occ_t)$ time
- $O(\log n)$ associated BSSTs T are searched ℹ as seen in 1-dimensional case
- total query time $\sum_{t \in T} O(\log n + occ_t)$
- $\sum_{t \in T} O(occ_t) = O(occ)$
- $\sum_{t \in T} O(\log n) = O(\log^2 n)$
- total time: $O(\log^2 n + occ)$

Range Trees (4/4)

- range trees can be generalized to higher dimensions
- for each dimension add an additional associated BBST
- reporting in final BBST
- d-dimensional queries are d 1-dimensional queries

Lemma: Higher Dimensions Range Tree

A d-dimensional range tree (for $d \geq 2$) storing n points in the plane requires $O(n \log^{d-1} n)$ words space and can answer queries in $O(\log^d n + occ)$ time

Proof (Sketch Query Time)

- recursive query time $Q_d(n)$ with $Q_2(n) = O(\log^2 n)$
- $Q_d(n) = O(\log n) + O(\log n) \cdot Q_{d-1}(n)$
- solves to $Q_d(n) = O(\log^d n)$
- $O(occ)$ time for reporting

Proof (Sketch Construction Space)

- recursive space $S_d(n)$ with $S_2(n) = O(n \log n)$ words
- $T_d(n) = O(n \log n) + O(\log n) \cdot T_{d-1}(n)$
- solves to $S_d(n) = O(n \log^{d-1} n)$

Fractional Cascading (1/2)

- sorted sets $S_1, \dots, S_m$
- $|S_1| = n$ and $S_{i+1} \subseteq S_i$
- report elements in range $[x..x']$ in $S_1, \dots, S_m$

- how much time does a naive algorithm with binary search require? **PINGO**
- $O(m \log n + occ)$ time
- $O(m + \log n + occ)$ time possible with fractional cascading

- in addition to S_i store pointers to S_{i+1}
- for each element in S_i store pointer to successor in S_{i+1}
- possible because $S_{i+1} \subseteq S_i$

Fractional Cascading (2/2)

Lemma: Fractional Cascading

Given sets $S_1, \dots, S_m$ with $|S_1| = n$ and $S_{i+1} \subseteq S_i$, find a range in all S_i's using fractional cascading requires $O(m + \log n + occ)$ time

Proof (Sketch)

- binary search on S_1 requires $O(\log n)$ time
- following pointer to S_2 requires $O(1)$ time
- scanning S_2 requires $O(occ)$ time
- following pointer to S_3 requires $O(1)$ time
- repeat m times
- total: $O(m + \log n + occ)$ time

- how to apply to range trees?
- instead of associated BBSTs store leaf data in arrays for all nodes but root
- each node has associated data
- store two successor pointers to the associated data in the left and right child
- two pointers to cover all possible paths
- this is a layered range tree

Query Layered Range Trees

- search in BBST for x-coordinates remains the same
- to search y-coordinates first search associated BBST of root
- same as initial binary search for fractional cascading
- continue to follow pointers in associated data and scan to report queries

Lemma: Query time Layered Range Tree

A query with an axis-parallel rectangle in a layered range tree storing n points in the plane can be performed in $O(\log n + occ)$ time

Proof (Sketch)

- the initial search requires $O(\log n)$ time
- the search in the associated BBST of the root requires $O(\log n)$ time
- remaining searches in associated data a requires $O(1 + occ_a)$ time
- each point is reported once
- total time: $O(\log n + occ)$

General Sets of Points (1/2)

- all solutions requires unique x and y-coordinates
- big limitation for applications
- remember database motivation

- store $(x|k)$ as coordinate with x being the x-coordinate and k a unique key
- same for y-coordinates
- compare points using $(x|k) < (x'|k') \iff x < x'$ or $(x = x'$ and $k < k'))$

- range queries $[x..x'] \times [y..y']$ become

$$[(x|-\infty)..(x'|\infty)] \times (y|-\infty)..[(y'|\infty)]$$

General Sets of Points (2/2)

- all solutions requires unique x and y-coordinates
- big limitation for applications
- remember database motivation
- if exact positions are not important to application

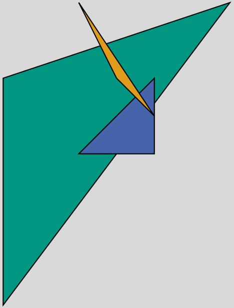
- random perturbation: $x + \delta \sim U(-\epsilon, \epsilon)$
- same for y-coordinates

- range queries $[x..x'] \times [y..y']$ become

$$[(x - \epsilon)..(x' + \epsilon)] \times (y - \epsilon)..[(y' + \epsilon)]$$

Now: Render Object

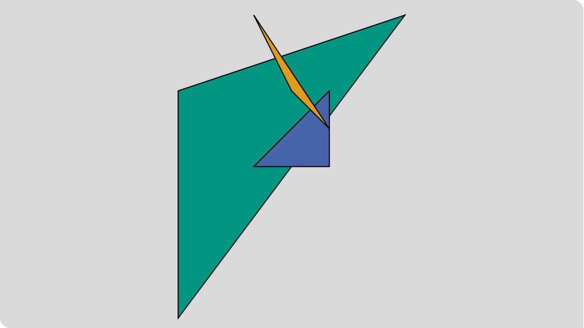
- hidden surface removal
- which pixel is visible
- important for rendering



z-Buffer Algorithm

- transform scene such that viewing direction is positive z-direction
- consider objects in scene in arbitrary order
- maintain two buffers
  - frame buffer ℹ currently shown pixel
  - z-buffer ℹ z-coordinate of object shown
- compare z-coordinate of z-buffer and object

- first sort object in depth-order
- depth-order may not always exist
- how to efficiently sort objects?

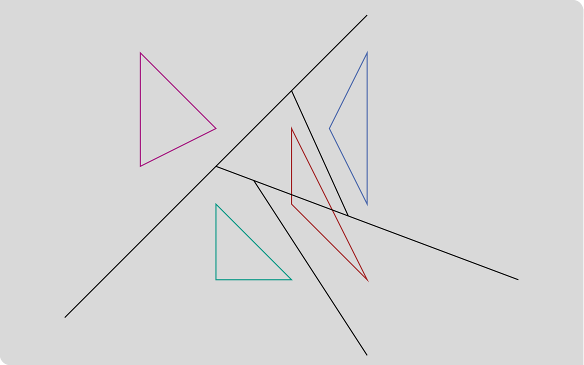


BSP Trees (1/2)

- partition space using hyperplanes
- binary partition ⓘ similar to kd-tree
- hyperplanes create half-spaces and cut objects into fragments

- $h^+ = \{(x_1, \dots, x_d) : a_1x_1 + \cdots + a_dx_d > c\}$
- $h^- = \{(x_1, \dots, x_d) : a_1x_1 + \cdots + a_dx_d < c\}$

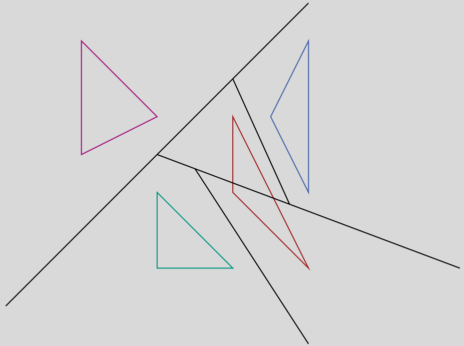
- each split creates two nodes in a tree
- if number of objects in space is one: leaf
- otherwise: inner node



BSP Trees (2/2)

- for leaf: store object/fragment
- for inner node v: store hyperplane h_v and the objects contained in h_v
- left child represents objects in upper half-space h^+
- right child represents objects in lower half-space h^-

- space of BSP tree is number of objects stored at all nodes
- what about fragments?
- too many fragments can make the tree big

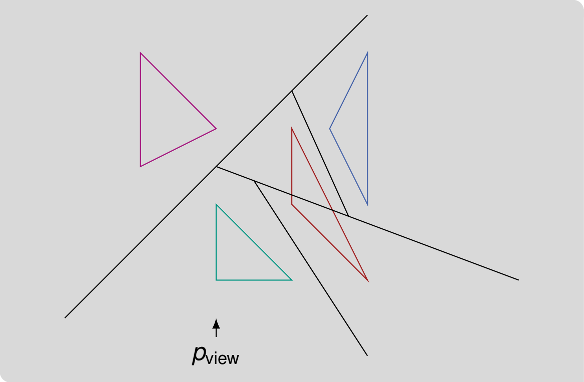


Auto-Partitioning

- sorting points for kd-trees worked well
- BSP-tree is used to sort objects in depth-order
- auto-partitioning uses splitters through objects
    - 2-dimensional: line through line segments
    - 3-dimensional: half-plane through polygons

Painter's Algorithm

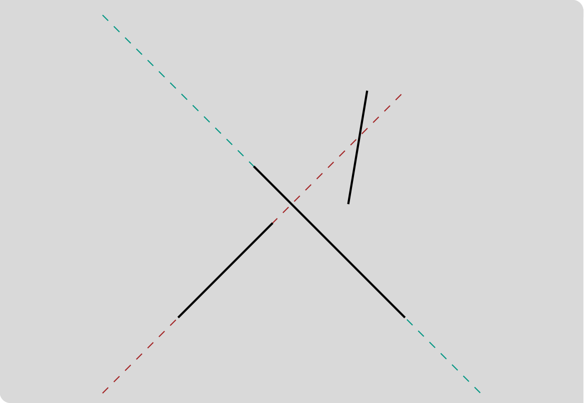
- consider view point p_{view}
- traverse through tree and always recurse on half-space that does not contain p_{view} first
- then scan-convert object contained in node
- then recurse on half-space that contains p_{view}



Constructing Planar BSP Trees (1/3)

- use auto-partitioning
- construction similar to construction of kd-tree
- store all necessary information
  - hyperplane
  - objects in hyperplane
- how to determine next hyperplane?
- creating fragments increases size of BSP tree

- let *s* be object and $\ell(s)$ line through object
- order matters



Constructing Planar BSP Trees (2/3)

Lemma: Number Line Fragments

The expected number of fragments generated when iterating through the line segments using a random permutation is $O(n \log n)$

Proof (Sketch)

- distance of lines $dist_{s_i}(s_j) =$

$$\begin{cases} \text{\# segments inters. } \ell(s_i) & \\ \text{ between } s_i \text{ and } s_j & \ell(s_i) \text{ inters. } s_j \\ \infty & \text{otherwise} \end{cases}$$

- example on the board

Proof (Sketch, cnt.)

- let $dist_{s_i}(s_j) = k$ and $s_{j_1}, \ldots, s_{j_k}$ be segments between s_i and s_j
- what is the probability that $\ell(s_i)$ cuts s_j?
- this happens if no s_{j_x} is processed before s_i
- since order is random

$$\mathbb{P}[\ell(s_i) \text{ cuts } s_j] \leq \frac{1}{dist_{s_i}(s_j) + 2}$$

Constructing Planar BSP Trees (3/3)

Proof (Sketch, cnt.)

- expected number of cuts

$$\mathbb{E}[\#\text{ cuts generated by } s_i] \leq \sum_{j \neq i} \frac{1}{dist_{s_i}(s_j) + 2} \leq 2 \sum_{k=0}^{n-2} \frac{1}{k+2} \leq 2 \ln n$$

- all lines generate at most $2n \ln n$ fragments

Lemma: BSP Construction

A BSP tree of size $O(n \log n)$ can be computed in expected time $O(n^2 \log n)$

Proof (Sketch)

- computing permutation in linear time
- construction is linear in number of fragments to be considered
- number of fragments in subtree is bounded by n
- number of recursions is $n \log n$

Conclusion and Outlook

This Lecture

- orthogonal range searching
- BSP trees

Advanced Data Structures