Karlsruhe Institute of Technology

# Advanced Data Structures

## Lecture 09: String B-Trees and Temporal Data Structures 2

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# PINGO

https://pingo.scc.kit.edu/172581

# External Memory Model [AV88]

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## Definition: External Memory Model

- internal memory of $M$ words
- instances of size $N \gg M$
- unlimited external memory
- transfer blocks of size $B$ between memories

- measure number of blocks I/Os
- scanning $N$ elements: $\Theta(N/B)$
- sorting $N$ elements: $\Theta(\frac{N}{B} \log_{\frac{M}{B}} \frac{N}{B})$

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## Set of Strings

- alphabet $\Sigma$ of size $\sigma$
- $k$ strings $\{s_1, \ldots, s_k\}$ over the alphabet $\Sigma$
- total size of strings is $N = \sum_{i=1}^{k} |s_i|$
- queries ask for pattern $P$ of length $m$

# String Dictionary

Given a set $S \subseteq \Sigma^\star$ of prefix-free strings, we want to answer:

- is $x \in \Sigma^\star$ in $S$
- add $x \notin S$ to $S$
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## Definition: Trie

Given a set $S = \{S_1, \ldots, S_k\}$ of prefix-free strings, a trie is a labeled rooted tree $G = (V, E)$ with:

1. $k$ leaves
2. $\forall S_i \in S$ there is a path from the root to a leaf, such that the concatenation of the labels is $S_i$
3. $\forall v \in V$ the labels of the edges $(v, \cdot)$ are unique

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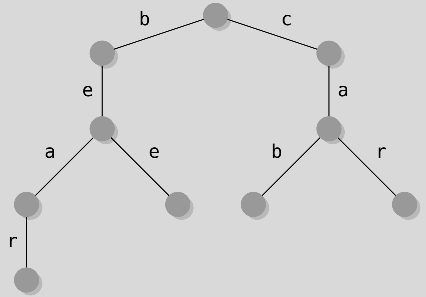


$S = \{\mathtt{bear}, \mathtt{bee}, \mathtt{cab}, \mathtt{car}\}$

# Theoretical Comparison

| Representation | Query Time (Contains) | Space in Words |
|---|---|---|
| arrays of variable size | $O(m \cdot \sigma)$ | $O(N)$ |
| arrays of fixed size | $O(m)$ | $O(N \cdot \sigma)$ |
| hash tables | $O(m)$ w.h.p. | $O(N)$ |
| balanced search trees | $O(m \cdot \lg \sigma)$ | $O(N)$ |
| weight-balanced search trees | $O(m + \lg k)$ | $O(N)$ |
| two-levels with weight-balanced search trees | $O(m + \lg \sigma)$ | $O(N)$ |

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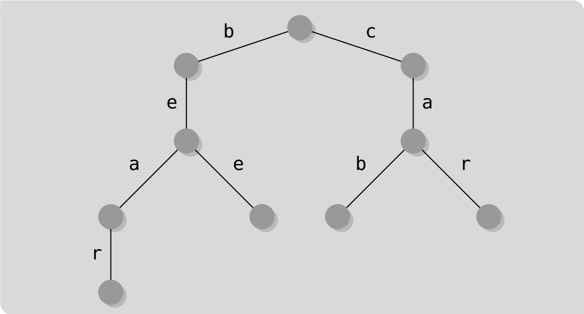
- more details in lecture Text Indexing

# Compact Trie

- tries have unnecessary nodes
- branchless paths can be removed
- edge labels can consist of multiple characters



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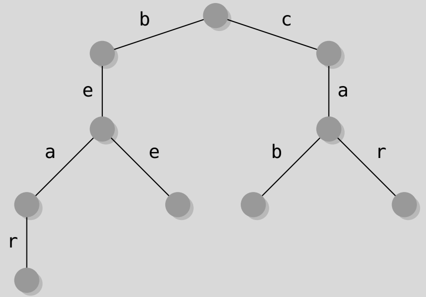

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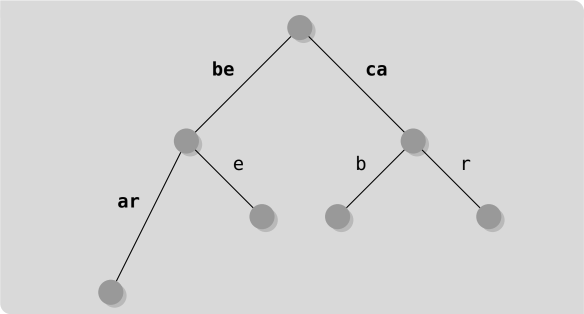


# (Recap) B-Trees

- search tree with out-degree in $[b, 2b)$
- works well in external memory
- uses separators to find subtree
- can be dynamic
- who knows B-trees **PINGO**

- example on the board

### From Atomic Values to Strings

- strings take more time to compare
- load as few strings from disk as possible

# String B-Tree [FG99]

- strings are stored in EM
- strings are identified by starting positions

- B-tree layout for sorted suffixes ℹ identified by position
- at least $b = \Theta(B)$ children
- tree height $O(\log_B N)$

- given node $v$
- $L(v)$ is lexicographically smallest string at $v$
- $R(v)$ is lexicographically largest string at $v$

- given node $v$ with children $v_0, \ldots, v_k$ with $k \in [b, 2b)$
- inner: store separators $L(v_0), R(v_0), \ldots, L(v_k), R(v_k)$
- leaf: store strings and link leaves

# Search in String B-Tree

- task: find all occurrences of pattern $P$
- two traversals of String B-Tree
- identify leftmost/rightmost occurrence
- output all strings in $O(occ/B)$

- at every node with children $v_0, \ldots, v_k$
- binary search for $P$ in $L(v_0), \ldots, R(v_k)$
  - if $R(v_i) < P < L(v_{i+1})$: not found
  - if $L(v_i) \leq P \leq R(v_i)$: continue in $v_i$

## Lemma: String B-Tree

Using a String B-tree, a pattern $P$ can be found in a set of strings with total length $N$ in $O(|P|/B \log N)$ I/Os

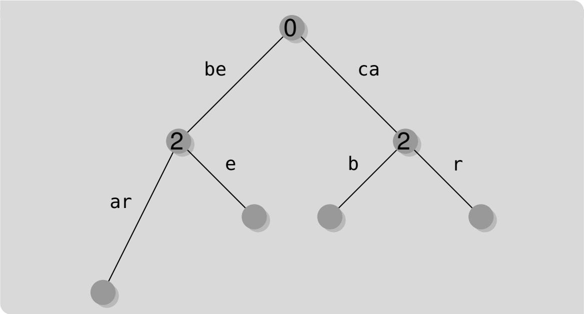
## Proof (Sketch)

- String B-Tree has height $\log_B N$
- load separators of node: $O(1)$ I/O
- load strings for binary search: $O(|P|/B)$ I/Os
- total: $O(\log_B N \cdot \log B \cdot |P|/B) = O(|P|/B \log N)$ I/Os

# Improving String B-Tree with Patricia Tries (1/2)

## Patricia Trie

- for strings $S = \{S_0, \ldots, S_{k-1}\}$
- a compact trie where only branching characters are stored
- additionally the string depth is stored
- size $O(k)$ for $k$ strings



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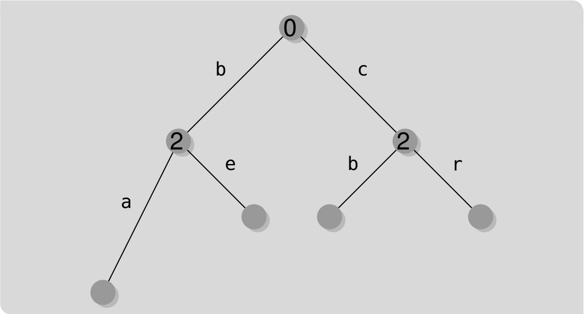

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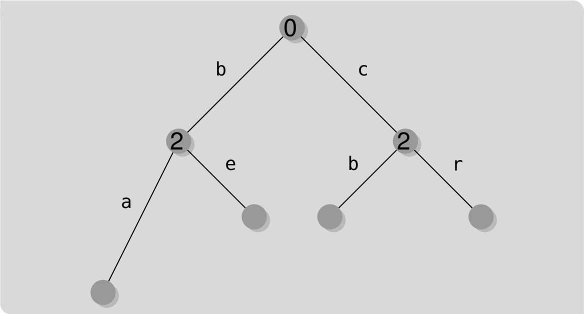


- search requires two steps
- first blind search using only trie
- blind search can result in false matches
- second a comparison with resulting string
- use any leaf after matching pattern

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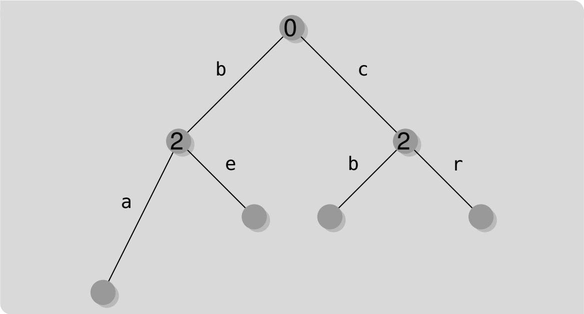


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- How do Patricia tries help? **PINGO**

# Improving String B-Tree with Patricia Tries (2/2)

- in each inner node build Patricia trie for separators
- if blind search finds leaf $w$
- compute $L = lcp(P, w)$
- let $u$ be first node on root-to-$w$ path with $d \geq L$

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**$d = L$**

- find matching children $v_i$ and $v_{i+1}$ of $w$ with
- branching characters $c_i < P[L+1] < c_{i+1}$
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## $d > L$

- consider next branching character $c$ on path
- if $P[L+1] < c$ continue in leftmost leaf
- if $P[L+1] > c$ continue in rightmost leaf

# Searching in Improved String B-Tree

- at every node with children $v_0, \ldots, v_k$
- load Patricia trie for $L(v_0), \ldots, R(v_k)$
- search Patricia trie for $w$ ℹ result of blind search
- load one string and compare with $P$
- identify child and continue

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**Lemma: String B-Tree with PTs**

Using a string B-tree with Patricia tries, a pattern $P$ can be found in a set of strings with total length $N$ with $O(|P|/B \log_B N)$ I/Os

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## Proof (Sketch)

- loading PT: $O(1)$ I/Os
- blind search: no I/Os
- loading one string: $O(|P|/B)$ I/Os
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- total $O(|P|/B \log_B N)$ I/Os

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- How can this be improved even further?
  **PINGO**

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# Improving Search with LCP-Values

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- search for pattern in nodes
- path in String B-tree $p_0, p_1, p_2, \ldots$
- in Patricia tries $PT_{p_i}$ compute $L = lcp(P, w)$
- all strings in $p_i$ have prefix $P[0..L)$
- do not compare previously matched characters
- load only $|P| - L$ characters at next node
- pass $L$ down the String B-tree

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**Lemma: String B-Tree with PTs and LCP**

Using a String B-tree with Patricia tries and passing down the LCP-value, a pattern $P$ can be found in a set of strings with total length $N$ in $O(|P|/B + \log_B N)$ I/Os

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## Proof (Sketch)

- passing down LCP-value: no I/Os
- telescoping sum $\sum_{i \leq h} \frac{L_i - L_{i-1}}{B}$
- $h = \log_B N$ height of String B-tree
- $L_i$ is LCP-value on Level $i$
- $L_0 = 0$ and $L_h \leq |P|$
- total: $O(|P|/B + \log_B N)$ I/Os

# Recap: Persistent Data Structures

- lecture based on: `http://courses.csail.mit.edu/6.851/spring12/lectures/L01`

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**Definition: Partial Persistence**

Only the latest version can be updated

**Definition: Full Persistence**

Any version can be updated

**Definition: Confluent Persistence**

Like full persistence, but two versions can be combined to a new version

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Nodes cannot be modified, only new nodes can be created

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**Persistence**

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**Retroactivity**

- change in the past affects future
- make change in earlier version changes all later versions

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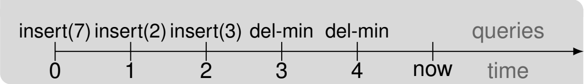
# Retroactive Data Structures

## Operations

- $\texttt{INSERT}(t, \mathit{operation})$: insert operation at time $t$
- $\texttt{DELETE}(t)$: delete operation at time $t$
- $\texttt{QUERY}(t, \mathit{query})$: ask *query* at time $t$

- for a priority queue updates are
  - insert
  - delete-min
- time is integer ⓘ for simplicity otherwise use order-maintenance data structure



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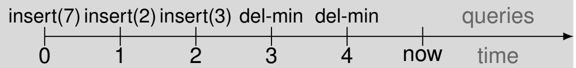


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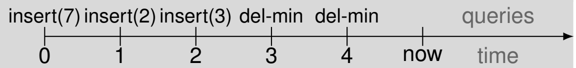


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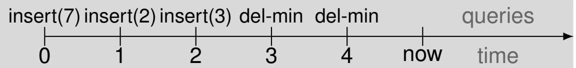


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## Definition: Nonoblivious Retroactivity

`INSERT`, `DELETE`, and `QUERY` at any time $t$ but also identify changed `QUERY` results

# Easy Cases: Partial Retroactivity

- commutative updates
- invertible updates
  - operation $op^{-1}$ such that $op^{-}1(op(\cdot)) = \emptyset$
  - `DELETE` becomes `INSERT` inverse operation
- makes partial retroactivity easy
- $\texttt{INSERT}(t, \mathit{operation}) = \texttt{INSERT}(\infty, \mathit{operation})$
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**Partial Retroactivity**

- hashing
- dynamic dictionaries
- array with updates only $A[i] += value$

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- range minimum queries

- nearest neighbor
- point location
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- these types of problems are also “easy”

# Decomposable Search Problems: Full Retroactivity

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## Proof (Sketch)

- use balances search tree segment tree
- each leaf corresponds to an update
- node $n$ corresponds to interval of time $[s_n, e_n]$
- if an object exists in the time interval $[s, e]$, then it appears in node $n$ if $[s_n, e_n] \subseteq [s, e]$ if none of $n$'s ancestors' are $\subseteq [s, e]$
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## Proof (Sketch, cnt.)

- to query find leaf corresponding to $t$
- look at ancestors to find all objects
- $O(\log m)$ results which can be combined in $O(\log m)$ time

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## Lemma: Full Retroactivity for DSP

Every decomposable search problems can be made fully retroactive with a $O(\log m)$ overhead in space and time, where $m$ is the number of operations

## Proof (Sketch)

- use balances search tree segment tree
- each leaf corresponds to an update
- node $n$ corresponds to interval of time $[s_n, e_n]$
- if an object exists in the time interval $[s, e]$, then it appears in node $n$ if $[s_n, e_n] \subseteq [s, e]$ if none of $n$'s ancestors' are $\subseteq [s, e]$
- each object occurs in $O(\log n)$ nodes

## Proof (Sketch, cnt.)

- to query find leaf corresponding to $t$
- look at ancestors to find all objects
- $O(\log m)$ results which can be combined in $O(\log m)$ time

- data structure is stored for each operation!
- $O(\log m)$ space overhead!

# General Full Retroactivity

**Lemma: Lower Bound**

Rewinding $m$ operations has a lower bound of $\Omega(m)$ overhead

- general case

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**Proof (Sketch)**

- two values $X$ and $Y$
- initially $X = \emptyset$ and $Y = \emptyset$
- supported operations
  - $X = x$
  - $Y+ = value$
  - $Y = X \cdot Y$
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**Proof (Sketch, cnt.)**

- perform operations
  - $Y+= a_n$
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  - $Y+= a_{n=1}$
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  - $Y+= a_0$
- what are we computing here? **PINGO**

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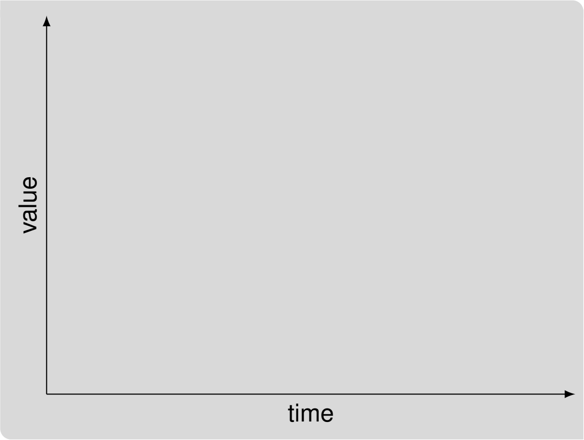
- evaluate polynomial at $X = x$ using `t=0,X=x`
- this requires $\Omega(n)$ time [FHM01]

# Priority Queues: Partial Retroactivity (1/6)

- priority queue with
  - insert
  - delete-min
- delete-min makes PQ non-commutative

**Lemma: Partial Retroactive PQ**

A priority queue can be partial retroactive with only $O(\log m)$ overhead per partially retroactive operation



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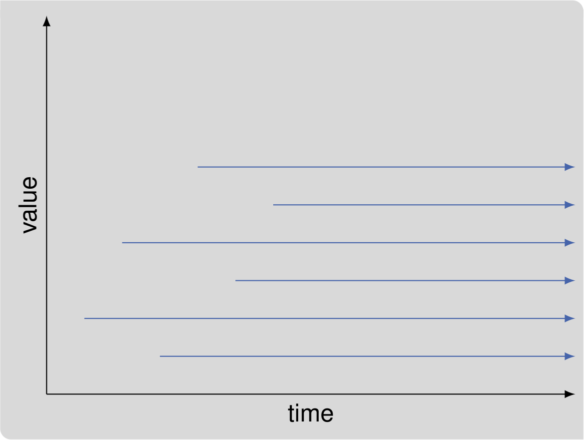

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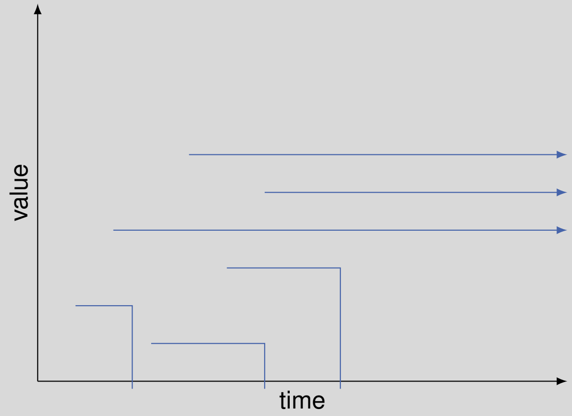

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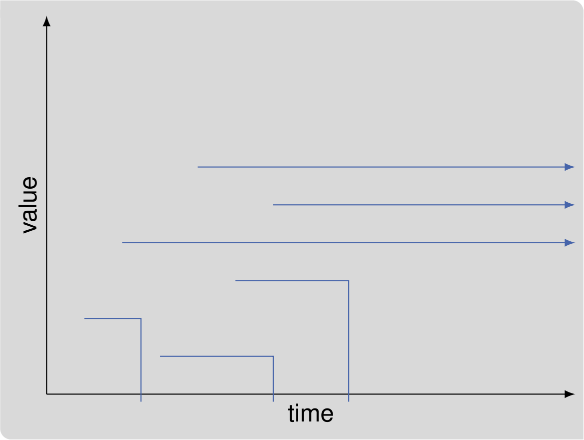

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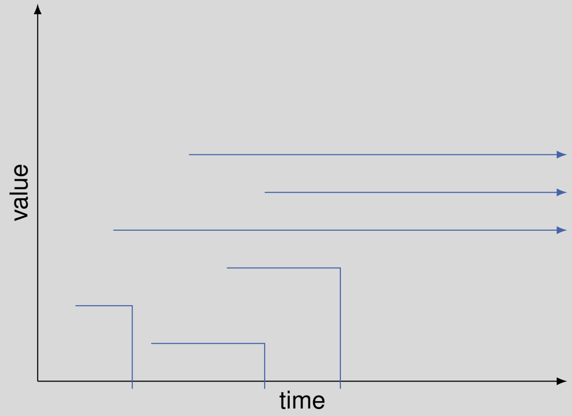

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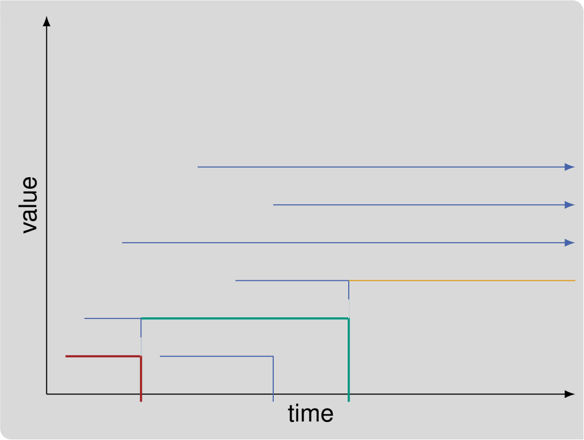

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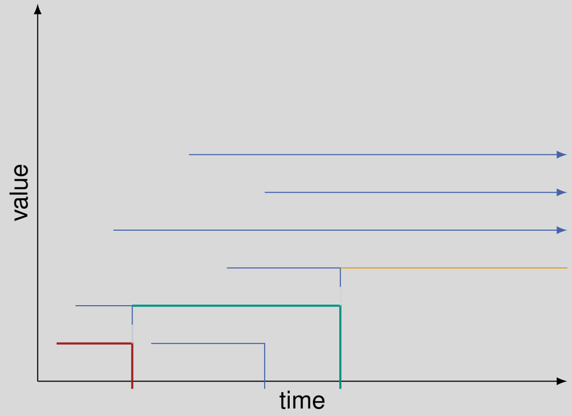

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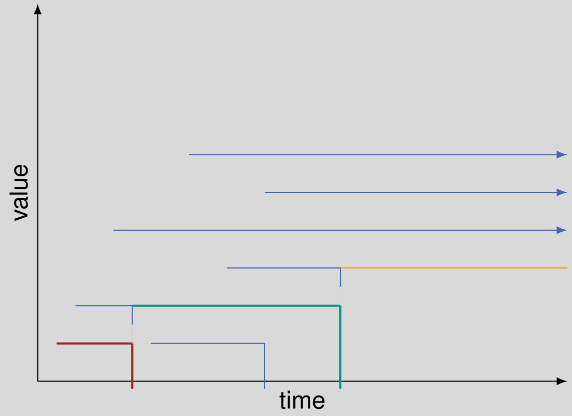

# Priority Queues: Partial Retroactivity (3/6)

- let $Q_t$ be elements in PQ at time $t$

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- what value inserts $\texttt{INSERT}(t, \text{insert}(v))$ in $Q_\infty$
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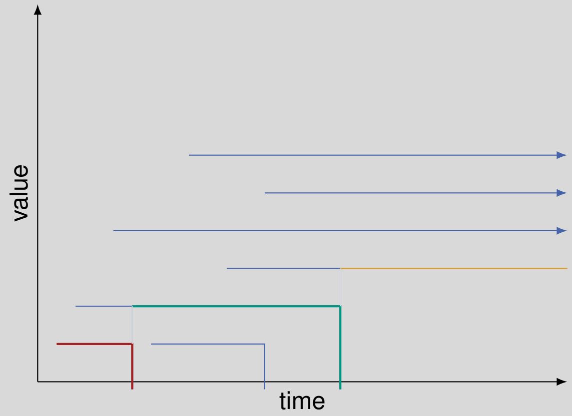

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A time $t'$ is a bridge if $Q_{t'} \subseteq Q_\infty$

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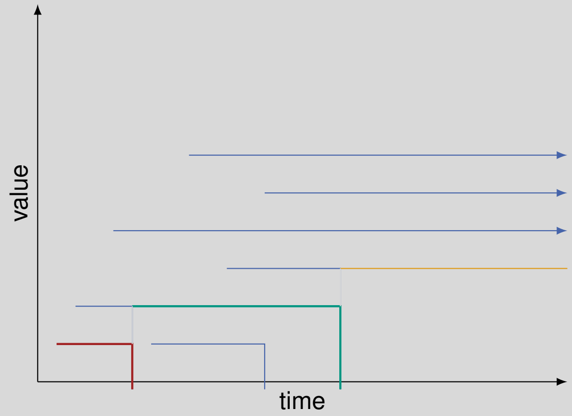

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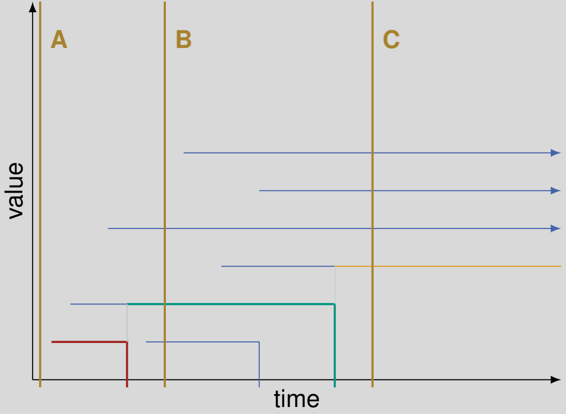


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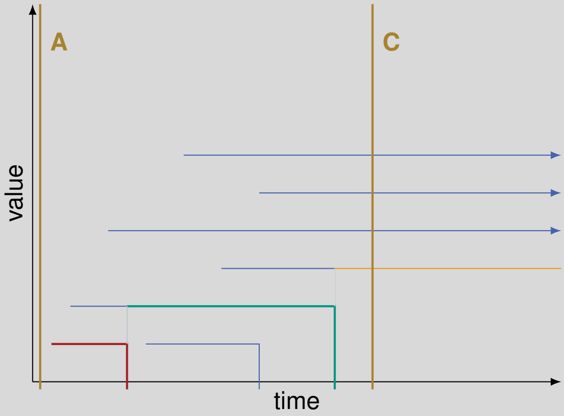


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# Priority Queues: Partial Retroactivity (4/6)

**Lemma: Deletions after Bridges**

If time $t'$ is closest bridge preceding time $t$, then

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- keep track of inserted values
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- update all BBSTs in $O(\log n)$ time

# Priority Queues: Partial Retroactivity (6/6)

**Lemma: Partial Retroactive PQ**

A priority queue can be partial retroactive with only $O(\log m)$ overhead per partially retroactive operation

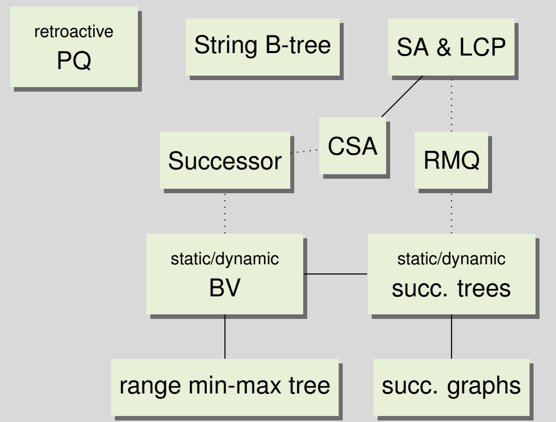
- requires three BBSTs
- updates need to update all BBSTs

# Conclusion and Outlook

**This Lecture**

- string B-tree
- retroactive data structures

**Advanced Data Structures**



# Conclusion and Outlook

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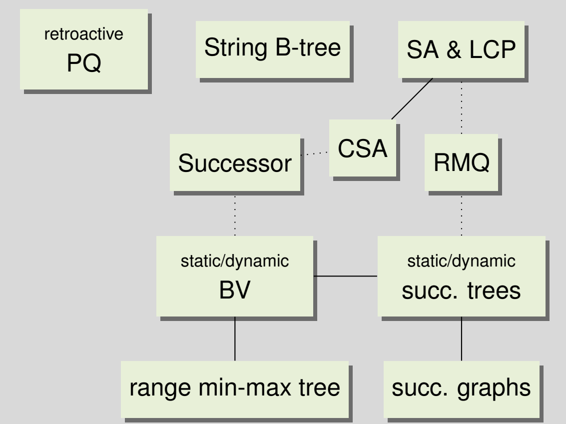
## This Lecture

- string B-tree
- retroactive data structures

## Next Lecture

- learned data structures

## Advanced Data Structures



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