Advanced Data Structures

Lecture 11: Learned Data Structures

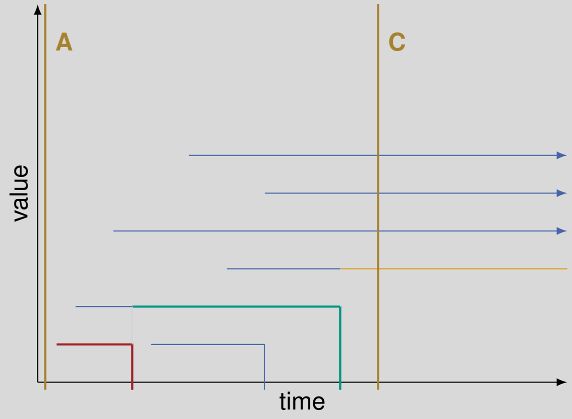
Florian Kurpicz

PINGO

https://pingo.scc.kit.edu/524651

Recap: Retroactive Data Structures

- BBST for Q_∞ ⓘ changed for each update
- BBST where leaves are inserts ordered by time augmented with
  - for each node x store $\max\{v' \notin Q_\infty : v' \text{ inserted in subtree of } x\}$
- BBST where leaves are all updates ordered by time augmented with
  - leaves store 0 for inserts with $v \in Q_\infty$, 1 for inserts with $v \notin Q_\infty$ and -1 for delete-mins
  - inner nodes store subtree sums
  - inner nodes store smallest prefix sum in subtree



Setting: Rank and Select Dictionary

Given ordered integers S from a universe $\mathcal{U} = [1, u]$ a rank and select index can answer

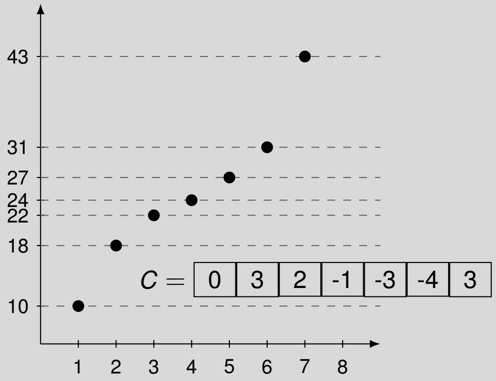
- $rank(x) = |\{y \in S \colon y < x\}|$
- $select(i) = S[\arg\min_j(rank(j) = i + 1)]$

- can be use to answer predecessor queries
- a bit vector with rank and select suffices

- bit vector requires $u + o(u)$ bits
- compressing bit vector to save space (if sparse)
- Elias-Fano requires how much space?
  PINGO
  - $|S|(2 + \log\lceil \frac{u}{|S|} \rceil)$ bits

Learned Rank and Select Index [BFV21]

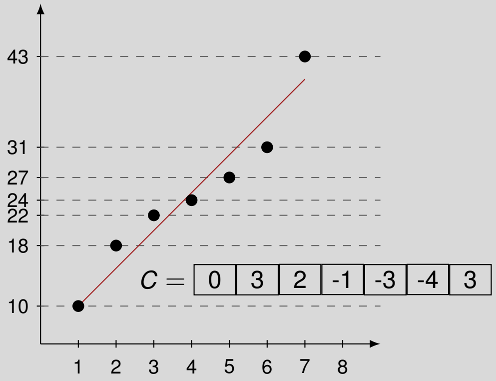
- how to represent S from a universe $\mathcal{U} = [1, u]$
- let $S = \langle x_1, x_2, \ldots, x_n \rangle$ be a sorted sequence
- map each element $x_i \in S$ to point (i, x_i)
- points are in Cartesian plane



- $S = \langle 10, 18, 22, 24, 27, 31, 43 \rangle$

Approximating S in Cartesian Plane

- find function f passing through all points
  - $x_i = f(i)$
- BUT f should be fast to compute and require little space
- use linear approximation with error ϵ
- store error correction in array C
- correction can be very big



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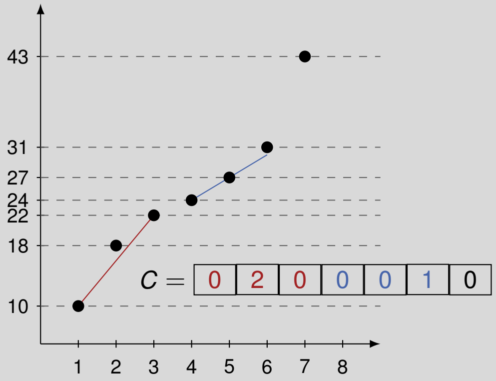
Piece-Wise Linear Approximation (1/2)

- use piece-wise linear approximation (PLA)
- sequence of segments with error bound by ϵ
- smallest number of segments can be computed in $O(n)$ time [O'R81]
- let there be ℓ segments

Definition: Representation of a Segment

The i-th segment starting with (j, x_j) is represented as triple $s_i = (r_i, \alpha_i, \beta_i)$, where

- $r_i = j$,
- α_i is the slope, and
- β_i is the intercept



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Piece-Wise Linear Approximation (2/2)

- use function to approximate point in i-th segment

$$f_i(j) = (j - r_i) \cdot \alpha_i + \beta_i$$

- use correction to obtain correct value

$$\lfloor f_i(j) \rfloor + C[j] = x_j$$

- $C[j] = x_j - \lfloor f_i(j) \rfloor$
- $C[j] \in \{-\epsilon, -\epsilon + 1, \ldots, -1, 0, 1, \ldots, \epsilon - 1, \epsilon\}$

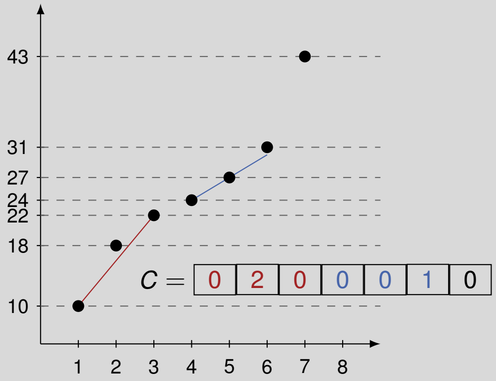
- let $c \geq 2$ be the number of bits used per correction
- $\epsilon = 2^c - 1$
- $c = 0$ results in $\epsilon = 0$
- $c = 1$ possible? **PINGO**

- what time does it take to recover x_j? **PINGO**
- $O(\log \ell)$ time to find the segment
- constant time within segment

What is Missing?

- use linear functions to approximate values
- corrections allow recovering values

- compression of data structure
- rank and select support
- (space-)optimal segmentation



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Compressing the Representation (1/2)

- larger ϵ results in smaller "expected" number of segments ℓ
- smaller c results in smaller correction and in larger ℓ ⓘ $\epsilon = \max\{0, 2^c - 1\}$

- ℓ depends on the distribution of points
- $\ell \leq \min\{u/(2\epsilon), n/2\}$ [FV20]

Definition: Number of Segments

Let $\{x_1, \ldots, x_n\}$ be a sorted sequence of distinct integers from in $[1, u]$. Given $0 \leq c \leq \lfloor \log u \rfloor$, there are ℓ segments in the optimal PLA of maximum error $\epsilon = \max\{0, 2^c - 1\}$ for $\{(i, x_i) \colon i = 1, \ldots, n\}$

Lemma: Space-Requirements (uncompressed)

$\{x_1, \ldots, x_n\}$ as defined before can be represented using $nc + 2\ell(\log n + \log u)$ bits of space.

Proof (Sketch)

Each segment $s_i = (r_i, \alpha_i, \beta_i)$ requires

- r_i: $\log n$ bits of space,
- α_i: $\log u + \log n$ bits of space ⓘ rational number
- β_i: $\log u$ bits of space

- can we compress r_i's, α_i's, or β_i's? **PINGO**

Compressing the Representation (2/2)

Lemma: Space-Requirements (Elias-Fano)

$\{x_1, \dots, x_n\}$ as defined before can be represented using $nc + \ell(2\log\frac{un}{\ell} + 4 + o(1))$ bits of space.

Proof (Sketch)

- r_i's are increasing sequence of ℓ integers in $[1, n]$
- β_i's are increasing sequence of ℓ integers in $[1, u]$
- use Elias-Fano coding

- C can also be compressed
- using entropy compressed indices

Lemma: Space-Requirements (Compressed)

$\{x_1, \dots, x_n\}$ as defined before can be represented using $nH_0(C) + o(nc) + \ell(2\log\frac{un}{\ell} + 4 + o(1))$ bits of space. Access time is $O(c)$.

Rank and Select Support

- rank and select use predecessor data structure on r_i's
- select is "easier" than rank

Lemma: Learned Select

Select on $\{x_1, \ldots, x_n\}$ as defined before is supported in $O(1)$ time requiring $n(c + 1 + o(1)) + \ell(2 \log u + \log n)$ bits of space.

Proof (Sketch)

- use bit vector marking r_i's
- $n + o(n)$ bits of space
- about one bit per element in S

- naive $rank(x)$ needs binary search
- find maximum i with $select(i) \leq x$
- requires $O(\log n)$ time

- better: binary search on segments
- within segment: get "position" of x
- use maximum error to find interval for binary search

Lemma: Learned Rank

Rank on $\{x_1, \ldots, x_n\}$ as defined before is supported in $O(\log \ell + c)$ time requiring no additional space.

Details on Rank Support

- error is bounded: $|f_j(i) - x_i| \leq \epsilon$
- search for $x_i \leq x < x_{i+1}$
- rank is one i with $f_j(i) - \epsilon \leq x \leq f_j(i) + \epsilon$

$f_j(i) = (i - r_i) \cdot \alpha_j + \beta_j$

- $(i - r_j) \cdot \alpha_j + \beta_j - \epsilon \leq x < (i + 1 - r_j) \cdot \alpha_j + \beta_j + \epsilon$
- solve for i
- $\frac{x - \beta_j}{\alpha_j} + r_j - (\frac{\epsilon}{\alpha_j} + 1) < i \leq \frac{x - \beta_j}{\alpha_j} + r_j + (\frac{\epsilon}{\alpha_j})$

Finding Optimal Data Partitioning

- fixed number c for corrections
- now: choose different error $\epsilon = \max\{0, 2^c - 1\}$ for each segment
- how to find optimal partitioning?

- let G be directed acyclic graph
- one node for each $\{x_1, \ldots, x_n\}$ plus sink node at end of sequence
- edge (i, j) with weight $w(i, j, c)$ indicates that there exists a segment compressing $x_i, \ldots, x_j$ using $w(i, j, c) = (j - i)c + \kappa$ bits of space

Lemma: Optimal Partitioning

The shortest path in G from node 1 to $n + 1$ corresponds to the PLA with minimal cost for $\{x_1, \ldots, x_n\}$

- finding shortest using brute-force not feasible
- requires $O(n^2 \log u)$ time [O'R81]
- can be done in $O(n \log u)$ time
  - solution is at most $\kappa\ell$ bits larger than optimal solution

From Encoding to Indexing

- data has been encoded (and compressed)
- now: indexing data
- in external memory model
- learned index is alternative to B-tree

Data Structure	Space	I/Os
B=tree	$\Theta(n)$	$O(\log_B(n))$
PGM-Index	$\Theta(m_{\text{opt}})$	$O(\log_B(m_{\text{opt}}))$

- $m_{\text{opt}} \leq n$ is optimal number of segments

Definition: External Memory Model (Recap)

- internal memory of M words
- instances of size $N \gg M$
- unlimited external memory
- transfer blocks of size B between memories

- measure number of blocks I/Os
- scanning N elements: $\Theta(N/B)$
- sorting N elements: $\Theta(\frac{N}{B} \log_{\frac{M}{B}} \frac{N}{B})$

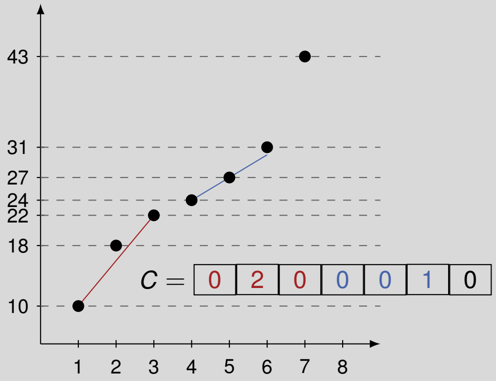
The PGM-Index [FV20]

- what do we not need when indexing instead of encoding? **PINGO**
- now S has to be stored
- how do we access elements in S
  - e.g., predecessor
- trick used before requires too much space

- store *key* instead position
- recurs on first *keys* of each segment

For Queries

- $\epsilon = \Theta(B)$
- load $2\epsilon + 1$ blocks per level



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Evaluation

https://onlineumfrage.kit.edu/evasys/online/online.php?p=CZSUW

Conclusion and Outlook

This Lecture

- learned data structures

Next Lecture

- one more interesting data structure
- results of the project/competition
- Q&A

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