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Text Indexing

Lecture 03: Suffix Trees and Suffix Arrays

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Recap: Compact Trie

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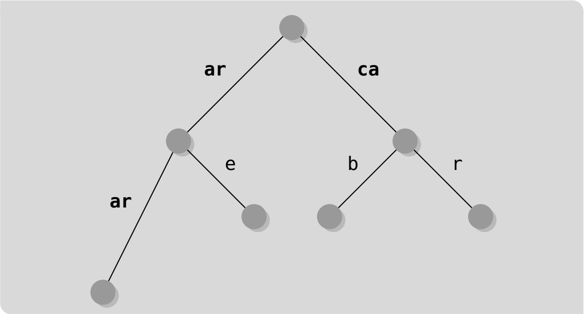
Definition: Compact Trie

- A compact trie is a trie where all branchless paths are replaced by a single edge.
- The label of the new edge is the concatenation of the replaced edges' labels.



Next

A full-text index for a text T is

- a data structure that
- allows to answer queries on T faster than naive
- we are interested in *pattern matching* queries
- how to use tries to create full-text index

Suffix Tree (1/4)

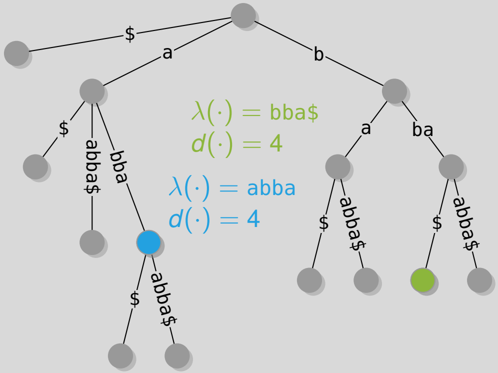
Definition: Suffix Tree [Wei73]

A suffix tree (ST) for a text T of length n is a

- compact trie
- over $S = \{T[1..n], T[2..n], \ldots, T[n..n]\}$
  - suffixes are prefix-free due to sentinel

Let $G = (V, E)$ be a compact trie with root r and a node $v \in V$, then

- $\lambda(v)$ is the concatenation of labels from r to v
- $d(v) = |\lambda(v)|$ is the string-depth of v
  - string depth $\neq$ depth



1	2	3	4	5	6	7	8	9
a	b	b	a	a	b	b	a	$

Suffix Tree (2/4)

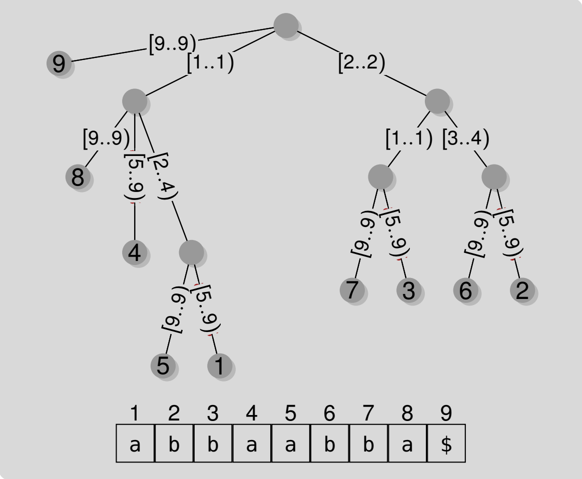
Representing Labels

- explicit edge labels require $O(n^2)$ words space
- references require only $O(n)$ words space

- for simplicity, we show text

Suffix Information

- label leaves with corresponding suffix
  - ℹ will be important later on

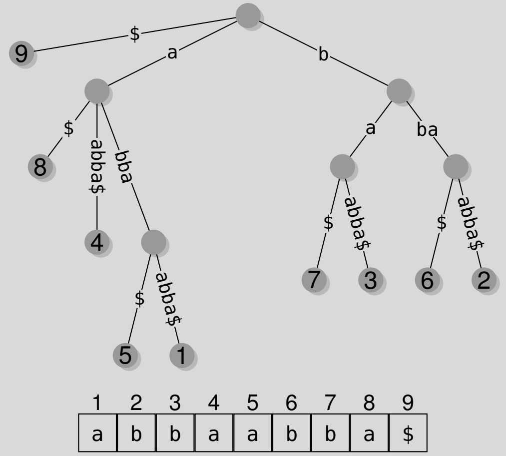


Suffix Tree (3/4)

Pattern Matching using Suffix Trees

- Pattern $P[1..m]$
- start at the root and follow edges
- query time depends on representation of children

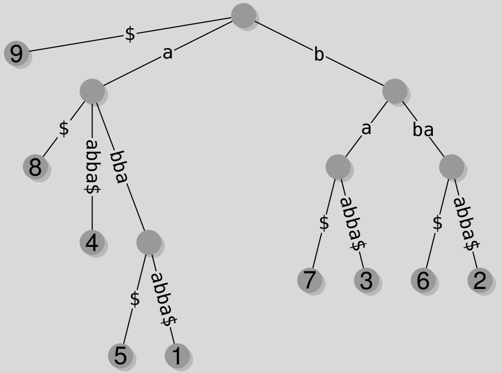
- $O(m)$ time using $O(n\sigma)$ words space
- $O(m \cdot \lg \sigma)$ time with $O(n)$ words space
- $O(m + \lg \sigma)$ time with $O(n)$ words space



Suffix Tree (4/4)

- very (most?) powerful text-index
- suffix trees require ≈ 8–20 bytes per character
- efficient direct construction in $O(n)$ time [Ukk95]
- also possible for integer alphabets [Far97]
- *SA* and *LCP*-array can replace suffix tree
- can answer all queries in the same time
- Mohamed Ibrahim Abouelhoda, Stefan Kurtz, and Enno Ohlebusch. "Replacing Suffix Trees with Enhanced Suffix Arrays". In: *J. Discrete Algorithms* 2.1 (2004), pages 53–86. DOI: 10.1016/S1570-8667(03)00065-0

next, suffix array construction



1	2	3	4	5	6	7	8	9
a	b	b	a	a	b	b	a	$

Suffix Array and LCP-Array

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Definition: Suffix Array [GBS92; MM93]

Given a text T of length n, the **suffix array** (SA) is a permutation of $[1..n]$, such that for $i \leq j \in [1..n]$

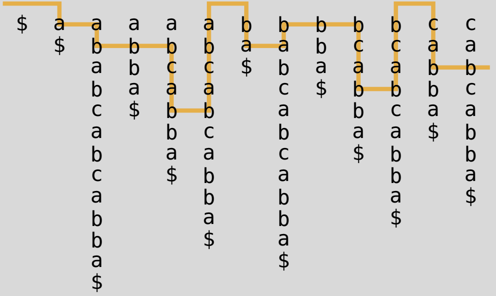
$$T[SA[i]..n] \leq T[SA[j]..n]$$

Definition: Longest Common Prefix Array

Given a text T of length n and its SA, the **LCP-array** is defined as

$$LCP[i] = \begin{cases} 0 & i = 1 \\ \max\{\ell \colon T[SA[i]..SA[i]+\ell) = \\ \quad T[SA[i-1]..SA[i-1]+\ell)\} & i \neq 1 \end{cases}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	\$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
LCP	0	0	1	2	2	5	0	2	1	1	4	0	3



Pattern Matching with the Suffix Array (1/2)

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Function SeachSA(T, $SA[1..n]$, $P[1..m]$)**:**

- $\ell = 1, r = n + 1$
- **while** $\ell < r$ **do** ℹ Find left border
  - $i = \lfloor (\ell + r)/2 \rfloor$
  - **if** $P > T[SA[i]..SA[i] + m)$ **then**
    - $\ell = i + 1$
  - **else** $r = i$
- $s = \ell, \ell = \ell - 1, r = n$
- **while** $\ell < r$ **do** ℹ Find right border
  - $i = \lceil \ell + r/2 \rceil$
  - **if** $P = T[SA[i]..SA[i] + m)$ **then** $\ell = i$
  - **else** $r = i - 1$
- **return** $[s, r]$

- pattern $P = \texttt{abc}$

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

Pattern Matching with the Suffix Array (2/2)

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Function SeachSA(T, $SA[1..n]$, $P[1..m]$)**:**

- $\ell = 1, r = n + 1$
- **while** $\ell < r$ **do**
  - $i = \lfloor (\ell + r)/2 \rfloor$
  - **if** $P > T[SA[i]..SA[i] + m)$ **then**
    - $\ell = i + 1$
  - **else** $r = i$
- $s = \ell, \ell = \ell - 1, r = n$
- **while** $\ell < r$ **do**
  - $i = \lceil \ell + r/2 \rceil$
  - **if** $P = T[SA[i]..SA[i] + m)$ **then** $\ell = i$
  - **else** $r = i - 1$
- **return** $[s, r]$

Lemma: Running Time SeachSA

The SeachSA answers counting queries in $O(m \lg n)$ time and reporting queries in $O(m \lg n + occ)$ time

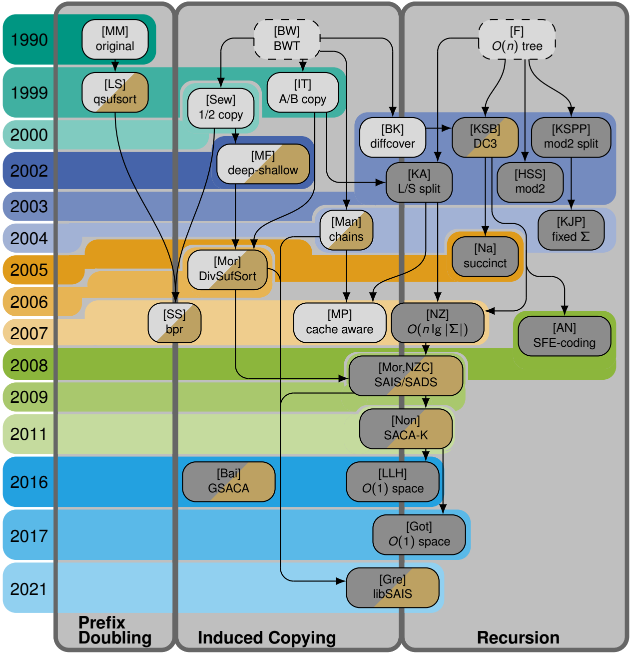
Proof (Sketch)

- two binary searches on the SA in $O(lgn)$ time
- each comparison requires $O(m)$ time
- counting in $O(1)$ additional time
- reporting in $O(occ)$ additional time

Preview: Improving Running Time with LCP-Array

- next lecture: $O(m + \lg n)$ and $O(m + \lg n + occ)$ time
- requires additional indices on LCP-array

- now: how to compute the suffix array directly ℹ without the suffix tree



Timeline Sequential Suffix Sorting

- based on [Bah+19; Bin18; Kur20; PST07]
- **darker grey**: linear running time
- **brown**: available implementation

Special Mentions

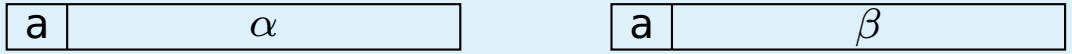
- DC3 first $O(n)$ algorithm
- $O(n)$ running time and $O(1)$ space for integer alphabets possible
- until 2021: DivSufSort fastest in practice with $O(n \lg n)$ running time
- since 2021: libSAIS fastest in practice with $O(n)$ running time

Suffix Array Induced Sorting: Overview

The Idea: Inducing

Given a text T of length n and two positions $i, j \in [1..n]$ with $T[i] = T[j]$, then

$$T[i..n] < T[j..n] \iff T[i+1..n] < T[j+1..n]$$



The Algorithm: SAIS

- using inducing for everything
- described in [NZC11]

Suffix Array Construction in 3 Phases

- classification
- sort special substrings/suffixes recursively
- induce all non-sorted suffixes

- classification helps identifying special suffixes
- everything in linear time

Roadmap

- classification
- inducing
- sorting special suffixes

Suffix Array Induced Sorting: Classification (1/2)

Definition: Type L/S Suffixes

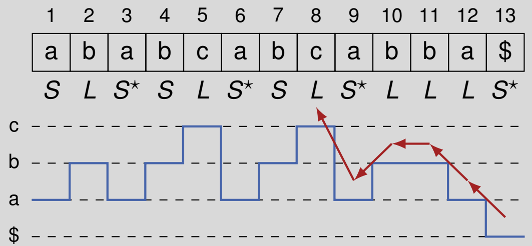
Given a text T of length n and $i \in [1..n]$, then

- $T[i] < T[i+1]$ or $i = n \Rightarrow T[i..n]$ has **type S**
- $T[i] > T[i+1] \Rightarrow T[i..n]$ has **type L**
- $T[i] = T[i+1] \Rightarrow T[i..n]$ has $T[i+1..n]$'s type.

Definition: Leftmost S Suffixes

Given a text T of length n, $i \in [2..n]$ such that $T[i..n]$ has type S and $T[i-1..n]$ has type L, then $T[i..n]$ is called **leftmost S suffix (LMS)**.

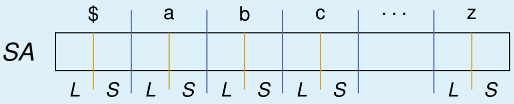
- denoted by $S^\star$



- scan text from right to left
- do not store types explicitly ℹ initially, we are only interested in LMS-suffixes

Suffix Array Induced Sorting: Classification (2/2)

- partition suffix array based text's histogram
- use types of suffixes to partition suffix array



Lemma: Order of L/S Suffixes

Given a text T of length n, a type L suffixes $T[i..n]$ and a type S $T[j..n]$ with $\alpha = T[i] = T[i]$, then

$$T[i..n] < T[j..n]$$

Proof (Sketch)

- $T[i..n]$ has type L
  - $T[i..n] = \alpha \underbrace{\alpha \ldots \alpha}_{\ell \geq 0 \text{ times}} \beta \ldots \$$
  - with $\beta < \alpha$
- $T[j..n]$ has type S
  - $T[j..n] = \alpha \underbrace{\alpha \ldots \alpha}_{\ell' \geq 0 \text{ times}} \gamma \ldots \$$
  - with $\alpha < \gamma$
- if $\ell < \ell'$ then $\alpha < \gamma$ and $T[i..n] < T[j..n]$
- if $\ell = \ell'$ then $\beta < \gamma$ and $T[i..n] < T[j..n]$
- if $\ell > \ell'$ then $\beta < \alpha$ and $T[i..n] < T[j..n]$

Suffix Array Induced Sorting: Inducing (1/2)

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Lemma: Inducing

If $T[i+1..n] < T[j+1..n]$ and $T[i] = T[j]$ then

$$T[i..n] < T[j..n]$$

Proof (Sketch)

- similar to order of L/S suffixes
- there is a leftmost character where $T[i+1..n]$ and $T[j+1..n]$ differ
- $T[i..n]$ and $T[j..n]$ differ at the same character

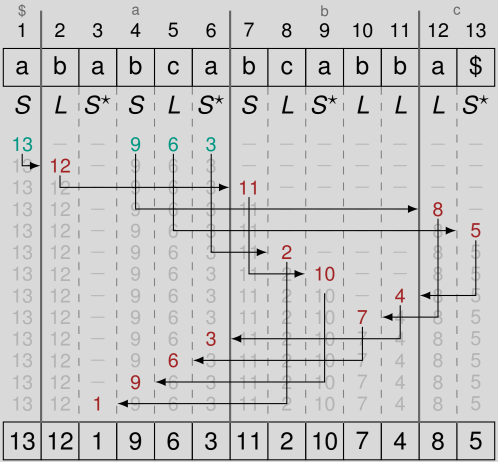
	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

Suffix Array Induced Sorting: Inducing (2/2)

Inducing in SAIS

- Initialization
  - initialize each entry in SA with "−"
  - put sorted *LMS*-suffixes at the end of buckets
- Scan Left to Right ($i = 1, 2, \ldots, n$)
  - if $SA[i] \neq -$ and $T[SA[i] - 1..n]$ is L-type
  - then put $SA[i] - 1$ at beginning of bucket
- Scan Right to Left ($i = n, n-1, \ldots, 1$)
  - if $SA[i] \neq -$ and $T[SA[i] - 1..n]$ is S-type
  - then put $SA[i] - 1$ at end of bucket

- are all suffixes induced?
- now we only need to sort $S^\star$ suffixes



Suffix Array Induced Sorting: LMS-Substrings (1/2)

- how to sort $S^\star$ suffixes?
- slightly adopt algorithm

Definition: LMS-Prefix

Let $i < j$ or $i = j = n$ be text positions, such that $T[j..n]$ is LMS and $\nexists k \in (i, j)$ with $T[k..n]$ is LMS, then we call $T[i..j]$ **LMS-prefix**

Definition: LMS-Substring

Let $T[i..j]$ be an LMS-prefix and $T[i..n]$ be LMS, then $T[i..j]$ is an **LMS-substring**

Inducing LMS-Prefixes

- Initialization
  - initialize each entry in SA with "–"
  - put *LMS*-suffixes in text order at the end of buckets
- Scan Left to Right ($i = 1, 2, \ldots, n$)
  - if $SA[i] \neq -$ and $T[SA[i] - 1..n]$ is L-type
  - then put $SA[i] - 1$ at beginning of bucket
- Scan Right to Left ($i = n, n-1, \ldots, 1$)
  - if $SA[i] \neq -$ and $T[SA[i] - 1..n]$ is S-type
  - then put $SA[i] - 1$ at end of bucket

Suffix Array Induced Sorting: LMS-Substrings (2/2)

Lemma: Inducing LMS-Prefixes

The algorithm sorts all LMS-Prefixes correctly

Proof (Sketch)

- initially: only $T[n..n]$ sorted correctly
- L2R: *L*-type LMS-prefixes sorted correctly
  - only care for first character of next LMS
  - LMS in correct bucket
  - sorted correctly for first character
- R2L: *S*-type LMS-prefixes sorted correctly
  - only care for first character of next LMS
  - LMS in correct bucket
  - sorted correct for first character

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

Suffix Array Induced Sorting: Recursion

Lemma: Running Time Computation T'

Computing T' requires $O(n)$ time

Proof (Sketch)

- find LMS-substrings in $O(1)$ time ⓘ save S-buckets
- scan each LMS-substring twice
- each character is in at most two LMS-substrings

- construct text T' using ranks of LMS-substrings
- compare LMS-substrings character-wise

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
	$	a$	ababcabcabba$	abba$	abcabba$	abcabcabba$	ba$	babcabcabba$	bba$	bcabba$	bcabcabba$	cabba$	cabcabba$

- $T' = 0122\$$

Suffix Array Induced Sorting: Running Time

Lemma: SAIS Time Complexity

Given a text of length n, SAIS computes the suffix array in $O(n)$ time using

Proof (Sketch)

- classification, sorting of special suffixes, and inducing in $O(n)$ time
- the number of $S^\star$ suffixes is at most $\lfloor n/2 \rfloor$
- $\mathcal{T}(n) = \mathcal{T}(\lfloor n/2 \rfloor) + O(n) = O(n)$

Space Requirements

- naive: $O(n \lg n)$ bits
- better: $n\lceil \lg n \rceil + 2\sigma\lceil \lg n \rceil$ bits 

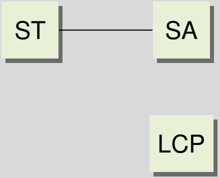
Conclusion and Outlook

This Lecture

- suffix trees and suffix arrays
- linear time suffix array construction

- suffix trees require ≈ 8–20 bytes per character
- suffix arrays require 5 bytes per character ⓘ for up to ≈ 1 TB text
- currently fastest implementation available at `https://github.com/IlyaGrebnov/libsais`

Linear Time Construction



Next Lecture

- linear time LCP-array construction
- interesting properties of LCP-array
- computing suffix trees using suffix array and LCP-array

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