Karlsruhe Institute of Technology

Text Indexing

Lecture 06: Burrows-Wheeler Transform

Florian Kurpicz

PINGO

https://pingo.scc.kit.edu/886630

Recap: Text-Compression

Karlsruhe Institute of Technology

Definition: LZ77 Factorization [ZL77]

Given a text T of length n over an alphabet Σ, the **LZ77 factorization** is

- a set of z factors $f_1, f_2, \ldots, f_z \in \Sigma^+$, such that
- $T = f_1 f_2 \ldots f_z$ and for all $i \in [1, z]$ f_i is
- single character not occurring in $f_1 \ldots f_{i-1}$ or
- longest substring occurring ≥ 2 times in $f_1 \ldots f_i$

$T =$ abababbbbaba\$

- $f_1 =$ a
- $f_2 =$ b
- $f_3 =$ abab
- $f_4 =$ bbb
- $f_5 =$ aba
- $f_6 =$ \$

Definition: LZ78 Factorization [ZL78]

Given a text T of length n over an alphabet Σ, the **LZ78 factorization** is

- a set of z factors $f_1, f_2, \ldots, f_z \in \Sigma^+$, such that
- $T = f_1 f_2 \ldots f_z$, $f_0 = \epsilon$ and for all $i \in [1, z]$
- if $f_1 \ldots f_{i-1} = T[1..j-1]$, then f_i is the longest prefix of $T[j..n]$, such that

$$\exists k \in [0, i), \alpha \in \Sigma \cup \{\$\} : f_k = f_i \alpha$$

$T =$ abababbbbaba\$

- $f_1 =$ a
- $f_2 =$ b
- $f_3 =$ ab
- $f_4 =$ abb
- $f_5 =$ bb
- $f_6 =$ aba
- $f_7 =$ \$

Burrows-Wheeler Transform [BW94] (1/2)

Karlsruhe Institute of Technology

Definition: Burrows-Wheeler Transform

Given a text T of length n and its suffix array SA, for $i \in [1, n]$ the **Burrows-Wheeler transform** is

$$BWT[i] = \begin{cases} T[SA[i] - 1] & SA[i] > 1 \\ \$ & SA[i] = 1 \end{cases}$$

- character before the suffix in SA-order
- choose characters cyclic ℹ $ for first suffix

- can compute BWT in $O(n)$ time
- for binary alphabet $O(n/\sqrt{\lg n})$ time and $O(n/\lg n)$ words space is possible [KK19]

	1	2	3	4	5	6	7	8	9	10	11	12	13
T	a	b	a	b	c	a	b	c	a	b	b	a	\$
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
LCP	0	0	1	2	2	5	0	2	1	1	4	0	3
BWT	a	b	\$	c	c	b	b	a	a	a	a	b	b

- definition is not very descriptive
- easy way to compute BWT
- what can we do with the BWT

- **PINGO** can the BWT be reversed?

Burrows-Wheeler Transform (2/2)

Definition: Cyclic Rotation

Given a text T of length n, the i-th **cyclic rotation** is

$$T^{(i)} = T[i..n]T[1..i)$$

- i-th cyclic rotation is concatenation of i-th suffix and $(i-1)$-th prefix

Definition: Burrows-Wheeler Transform (alt.)

Given a text T and a matrix containing all its cyclic rotations in lexicographical order as columns, then the **Burrows-Wheeler transform** of the text is the last row of the matrix

$T = $ ababcabcabba$

$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$	$T^{(5)}$	$T^{(6)}$	$T^{(7)}$	$T^{(8)}$	$T^{(9)}$	$T^{(10)}$	$T^{(11)}$	$T^{(12)}$	$T^{(13)}$
a	b	a	b	c	a	b	c	a	b	b	a	$
b	a	b	c	a	b	c	a	b	b	a	$	a
a	b	c	a	b	c	a	b	b	a	$	a	b
b	c	a	b	c	a	b	b	a	$	a	b	a
c	a	b	c	a	b	b	a	$	a	b	a	b
a	b	c	a	b	b	a	$	a	b	a	b	c
b	c	a	b	b	a	$	a	b	a	b	c	a
c	a	b	b	a	$	a	b	a	b	c	a	b
a	b	b	a	$	a	b	a	b	c	a	b	c
b	b	a	$	a	b	a	b	c	a	b	c	a
b	a	$	a	b	a	b	c	a	b	c	a	b
a	$	a	b	a	b	c	a	b	c	a	b	b
$	a	b	a	b	c	a	b	c	a	b	b	a

$T = $ ababcabcabba$

First and Last Row

- two important rows in the matrix
- other rows are not needed at all
- there is a special relation between the two rows
  - later this lecture

First Row **F**

- contains all characters or the text in sorted order

Last Row **L**

- is the *BWT* itself

$T = \mathtt{ababcabcabba\$}$

	$T^{(13)}$	$T^{(12)}$	$T^{(1)}$	$T^{(9)}$	$T^{(6)}$	$T^{(3)}$	$T^{(11)}$	$T^{(2)}$	$T^{(10)}$	$T^{(7)}$	$T^{(4)}$	$T^{(8)}$	$T^{(5)}$
F	$	a	a	a	a	a	b	b	b	b	b	c	c
	a	$	b	b	b	b	a	a	b	c	c	a	a
	b	a	a	b	c	c	$	b	a	a	a	b	b
	a	b	b	a	a	a	a	c	$	b	b	b	c
	b	a	c	$	b	b	b	a	a	b	c	a	a
	c	b	a	a	b	c	a	b	b	a	a	$	b
	a	c	b	b	a	a	b	c	a	$	b	a	b
	b	a	c	a	$	b	c	a	b	a	b	b	a
	c	b	a	b	a	b	a	b	c	b	a	a	$
	a	c	b	c	b	a	b	b	a	a	$	b	a
	b	a	b	a	a	$	c	a	b	b	a	c	b
	b	b	a	b	b	a	a	$	c	c	b	a	a
L	a	b	$	c	c	b	b	a	a	a	a	b	b

Properties of the BWT: Rank of Characters

Definition: Rank

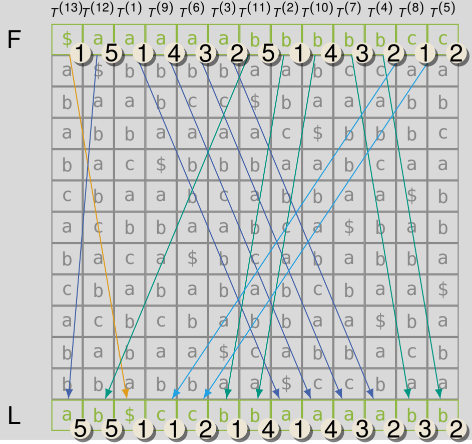
Given a text T over an alphabet *Sigma*, the **rank** of a character at position $i \in [1, n]$ is

$$rank(i) = |\{j \in [1, i] \colon T[i] = T[j]\}|$$

- rank is number of same characters that occur before in the text
- mark ranks of characters w.r.t. text not *BWT*
- order of ranks is the same in first and last row

T	a	b	a	b	c	a	b	c	a	b	b	a	\$
rank	1	1	2	2	1	3	3	2	4	4	5	5	1

T = ababcabcabba\$



LF-Mapping (1/2)

- want to map characters from last to first row
- why do we want this?
  - helps with pattern matching
  - transform *BWT* back to *T*

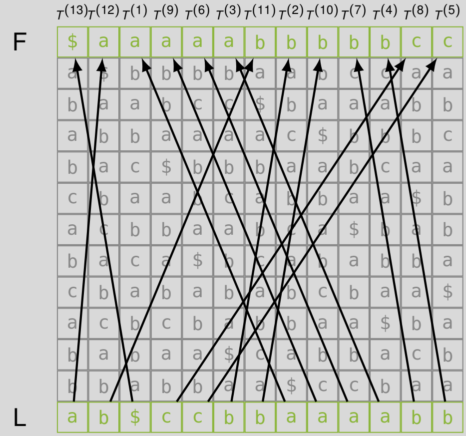
Definition: *LF*-mapping

Given a text T of length n and its suffix array SA, then the LF-mapping is a permutation of $[1, n]$, such that

$$LF(i) = j \iff SA[j] = SA[i] - 1$$

- similar to definition of *BWT*
- requires *SA* or explicitly saving *LF*-mapping

$T = \texttt{ababcabcabba\$}$



LF-Mapping (2/2)

Definition: *C*-Array and *Rank*-Function

Given a text T of length n over an alphabet Σ, $\alpha \in \Sigma$, and $i \in [1, n]$ then

$$C[\alpha] = |i \in [1, n]\colon\ T[i] < \alpha|$$

and

$$rank_\alpha(i) = |\{j \in [1, i]\colon\ T[j] = \alpha\}|$$

- C contains total number of smaller characters
- $rank_\alpha$ contains number of α's in prefix $T[1..i]$
- $rank_\alpha$ can be computed in $O(1)$ time [FM00]

T	a	b	a	b	c	a	b	c	a	b	b	a	\$
rank	1	1	2	2	1	3	3	2	4	4	5	5	1

- rank now on *BWT*
- C is exclusive prefix sum over histogram

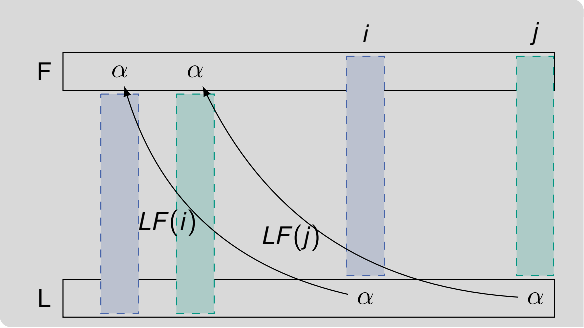
Definition: *LF*-Mapping (alt.)

Given a *BWT*, its *C*-array, and its *rank*-Function, then

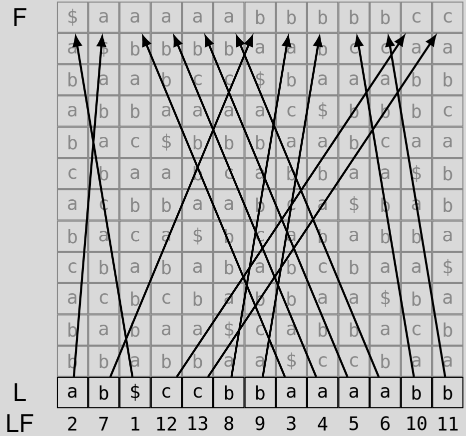
$$LF(i) = C[BWT[i]] + rank_{BWT[i]}(i)$$

Reversing the BWT (1/2)

- characters (w.r.t. text) preserve order in L and F
- LF-mapping returns previous character in text



$T = \texttt{ababcabcabba\$}$



Reversing the BWT (2/2)

- characters (w.r.t. text) preserve order in L and F
- LF-mapping returns previous character in text

- $T[n] = \$$ and $T^{(n)}$ in first row
- apply LF-mapping on result to obtain any character

$$T[n-i] = L[\underbrace{LF(LF(\ldots(}_{i-1 \text{ times}}LF(1))\ldots))]$$

	1	2	3	4	5	6	7	8	9	0	11	12	13
L	a	b	\$	c	c	b	b	a	a	a	a	b	b
LF	2	7	1	12	13	8	9	3	4	5	6	10	11

- $T[13] = \$$ and $k = 1$
- $T[12] = L[1] = \mathtt{a}$ and $k = LF(1) = 2$
- $T[11] = L[2] = \mathtt{b}$ and $k = LF(2) = 7$
- $T[10] = L[7] = \mathtt{b}$ and $k = LF(7) = 9$
- $T[9] = L[9] = \mathtt{a}$ and $k = LF(9) = 4$
- $T[9] = L[4] = \mathtt{c}$ and $k = LF(4) = 12$
- ...

Properties of the BWT: Runs

- *BWT* is reversible
- can be used for lossless compression

	1	2	3	4	5	6	7	8	9	0	11	12	13
L	a	b	$	c	c	b	b	a	a	a	a	b	b

- *BWT* contains lots of runs
- same context is often grouped together

Definition: Run (simplified)

Given a text T of length n, we call its substring $T[i..j]$ a **run**, if

- $T[k] = T[\ell]$ for all $k, \ell \in [i, j]$ and
- $T[i-1] \neq T[i]$ and $T[j+1] \neq T[j]$

(To be more precise, this is a definition for a run of a periodic substring with smallest period 1, but this is not important for this lecture )

Compressing the BWT: Run-Length Compression

Definition: Run-Length Encoding

Given a text T, represent each run $T[i..i+\ell)$ as tuple

$$(T[i], \ell)$$

$T = \texttt{ababcabcabba\$}$

	1	2	3	4	5	6	7	8	9	0	11	12	13
BWT	a	b	$	c	c	b	b	a	a	a	a	b	b

- $(\texttt{a}, 1)$
- $(\texttt{b}, 1)$
- $(\texttt{\$}, 1)$
- $(\texttt{c}, 2)$
- $(\texttt{b}, 2)$
- $(\texttt{a}, 4)$
- $(\texttt{b}, 2)$

Compressing the BWT: Move-to-Front

Definition: Move-To-Front Encoding

Given a text T over an alphabet $\Sigma = [1, \sigma]$, the **MTF encoding** $MTF(T)$ of the text is computed as follows

- start with a list $X = \Sigma[1], \Sigma[2], \ldots, \Sigma[\sigma]$
- scan text from left to right, for character $T[i]$
  - append position of $T[i]$ in X to $MTF(T)$ and
  - move $T[i]$ to front of X

- MTF encoding can easily be reverted
- consists of many small numbers
- runs are preserved
- use Huffman on encoding no theoretical improvement but good in practice

$T = \texttt{ababcabcabba\$}$

	1	2	3	4	5	6	7	8	9	0	11	12	13
BWT	a	b	$	c	c	b	b	a	a	a	a	b	b

- $X = \texttt{\$, a, b, c}$
- $MTF = 2$ and $X = \texttt{a, \$, b, c}$
- $MTF = 2\,3$ and $X = \texttt{b, a, \$, c}$
- $MTF = 2\,3\,3$ and $X = \texttt{\$, b, a, c}$
- $MTF = 2\,3\,3\,4$ and $X = \texttt{c, \$, b, a}$
- $MTF = 2\,3\,3\,4\,1$ and $X = \texttt{c, \$, b, a}$
- $MTF = 2\,3\,3\,4\,1\,1$ and $X = \texttt{c, \$, b, a}$
- $\ldots$
- $MTF = 2\,3\,3\,4\,1\,1\,3\,1\,4\,1\,1\,1\,2\,1$

Pattern Matching using the BWT

Recap

Given a text T of length n over an alphabet Σ, $\alpha \in \Sigma$, and $i \in [1, n]$ then

$$C[\alpha] = |i \in [1, n] \colon T[i] < \alpha|$$

and

$$rank_\alpha(i) = |\{j \in [1, i] \colon T[j] = \alpha\}|$$

- find interval of occurrences in *SA* using *BWT*
- *SA* is divided into intervals based on first character of suffix ℹ as seen during SAIS
- text from *BWT* is backwards
- search pattern backwards

- interval for α is $[C[\alpha - 1], C[\alpha + 1]]$
- find sub-interval using $rank_\alpha$

- example on the board

Backwards Search in the BWT

Function *BackwardsSearch*($P[1..n], C, rank$)**:**

$s = 1, e = n$
for $i = m, \ldots, 1$ **do**
  $s = C[P[i]] + rank_{P[i]}(s - 1) + 1$
  $e = C[P[i]] + rank_{P[i]}(e)$
  if $s > e$ **then**
    return $\emptyset$
return $[s, e]$

- no access to text or *SA* required
- no binary search
- existential queries are easy
- counting queries are easy
- reporting queries require additional information
- example on the board

Sampling the Suffix Array

- reporting queries require *SA*
- storing whole *SA* requires too much space
- better: sample every s-th *SA* position in SA'

- how to find sampled position?
- mark sampled positions in bit vector of size n
- if match occurs check if position is sampled
- otherwise find sample using *LF*
- if $SA[i] = j$ then $SA[LF(i)] = j - 1$

- $rank_1(i)$ in bit vector is number of sample
- $SA'[rank_1(i)]$ is sampled value
- $SA'[rank_1(i)] + \#$steps till sample found is correct *SA* value

- finding a sample requires $O(s \cdot t_{rank})$ time

Efficient Bit Vectors in Practice (1/3)

Karlsruhe Institute of Technology

std::vector<char/int/...>

- easy access
- very big: $1, 4, \ldots$ bytes per bit

std::vector<bool>

- bit vector in C++ (1 bit per byte)
- easy access
- layout depending on implementation

std::vector<uint64_t>

- requires 8 bytes per bit(?)
- store 64 bits in 8 bytes
- how to access bits

- $i/64$ is position in 64-bit word
- $i\%64$ is position in word

0	1	2	3	4	5	6	7	8	9
64 bits	64 bits	64 bits	64 bits	64 bits	64 bits	64 bits	64 bits	64 bits	64 bits

	63	0	1	2	3	4	5	...	62	63	0	
...	0	1	1	1	0	1	0	...	1	0	0	...

Efficient Bit Vectors in Practice (2/3)

```
// There is a bit vector
std::vector<uint64_t> bit_vector;

// access i-th bit
uint64_t block = bit_vector[i/64];
bool bit = (block >> ( 63 - (i % 64)) )  & 1ULL;
```

shift bits right **# bits** **logical and 1**

0	1	2	3	4	5	...	62	63
1	1	1	0	1	0	...	1	0

>> 60

0	1	2	3	4	5	...	62	63
0	0	0	0	0	0	...	1	0
							and	1

Efficient Bit Vectors in Practice (3/3)

Karlsruhe Institute of Technology

```
(block >> (63-(i%64))) & 1ULL;
```

- fill bit vector from left to right

0	1	2	3	4	5	...	62	63
1	1	1	0	1	0	...	1	0
0	0	0	0	0	0	...	1	0

- assembler code:

```
mov  ecx, edi
not  ecx
shr  rsi, cl
mov  eax, esi
and  eax, 1
```

```
(block >> (i%64)) & 1ULL;
```

- fill bit vector right to left

63	62	...	5	4	3	2	1	0
0	1	...	0	1	0	1	1	1
0	0	...	1	1	0	0	1	0

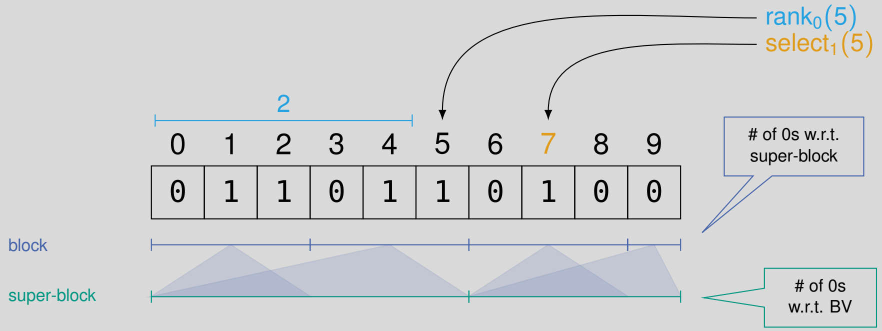
- assembler code:

```
mov  ecx, edi
shr  rsi, cl
mov  eax, esi
and  eax, 1
```

Rank Queries in Bit Vectors (1/2)

$\text{rank}_\alpha(i)$ # of αs before i

$\text{select}_\alpha(j)$ position of j-th α



Rank Queries in Bit Vectors (2/2)

- for a bit vector of size n
- blocks of size $s = \lfloor \frac{\lg n}{2} \rfloor$
- super blocks of size $s' = s^2 = \Theta(\lg^2 n)$

- for all $\lfloor \frac{n}{s'} \rfloor$ super blocks, store number of 0s from beginning of bit vector to end of super-block
- $n/s' \cdot \lg n = O(\frac{n}{\lg n}) = o(n)$ bits of space

- PINGO how fast can rank queries be answered?

- for all $\lfloor \frac{n}{s} \rfloor$ blocks, store number of 0s from beginning of super block to end of block
- $n/s \cdot \lg s' = O(\frac{n \lg \lg n}{\lg n}) = o(n)$ bits of space

- for all length-s bit vectors, for every position i store number of 0s up to i
- $2^{\frac{\lg n}{2}} \cdot s \cdot \lg s = O(\sqrt{n} \lg n \lg \lg n) = o(n)$ bits of space

- query in $O(1)$ time
- $rank_0(i) = i - rank_1(i)$

The FM-Index (First Look) [FM00]

Building Blocks of FM-Index

- wavelet tree on BWT providing *rank*-function
  wavelet trees are topic of next lecture!
- C-array
- sampled suffix array with sample rate s
- bit vector marking sampled suffix array positions

Lemma: FM-Index Space Requirements

Given a text T of length n over an alphabet of size σ, the FM-index requires $O(n \lg \sigma)$ bits of space

Space Requirements

- wavelet tree: $n\lceil \lg \sigma \rceil(1 + o(1))$ bits
- C-array: $\sigma \lceil \lg n \rceil$ bits $n(1 + o(1))$ bits if $\sigma \geq \frac{n}{\lg n}$
- sampled suffix array: $\frac{n}{s}\lceil \lg n \rceil$ bits
- bit vector: $n(1 + o(1))$ bits

- space and time bounds can be achieved with $s = \lg_\sigma n$

Conclusion and Outlook

This Lecture

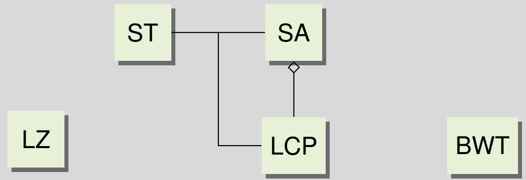
- Burrows-Wheeler transform
- introduction to FM-index

- efficient bit vectors
- rank queries on bit vectors

Next Lecture

- wavelet trees
- more on FM-index

Linear Time Construction



Bibliography I

[BW94] Michael Burrows and David J. Wheeler. *A Block-Sorting Lossless Data Compression Algorithm*. Technical report. 1994.

[FM00] Paolo Ferragina and Giovanni Manzini. “Opportunistic Data Structures with Applications”. In: *FOCS*. IEEE Computer Society, 2000, pages 390–398. DOI: 10.1109/SFCS.2000.892127.

[KK19] Dominik Kempa and Tomasz Kociumaka. “String Synchronizing Sets: Sublinear-Time BWT Construction and Optimal LCE Data Structure”. In: *STOC*. ACM, 2019, pages 756–767.

[ZL77] Jacob Ziv and Abraham Lempel. “A Universal Algorithm for Sequential Data Compression”. In: *IEEE Trans. Inf. Theory* 23.3 (1977), pages 337–343. DOI: 10.1109/TIT.1977.1055714.

[ZL78] Jacob Ziv and Abraham Lempel. “Compression of Individual Sequences via Variable-Rate Coding”. In: *IEEE Trans. Inf. Theory* 24.5 (1978), pages 530–536. DOI: 10.1109/TIT.1978.1055934.