Karlsruhe Institute of Technology

# Text Indexing

## Lecture 08: Wavelet Trees

Florian Kurpicz

# PINGO

https://pingo.scc.kit.edu/345678

# Recap: Rank-Queries

- for a bit vector of size $n$
- blocks of size $s = \lfloor \frac{\lg n}{2} \rfloor$
- super blocks of size $s' = s^2 = \Theta(\lg^2 n)$
- information for 0s or 1s enough
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- query in $O(1)$ time using three subqueries
  - one in super-block
  - one in block
  - one for remaining bitvector smaller than $s$

# Select in $o(n)$ Space and $O(1)$ Time

- $select_0$ in a bit vector of size $n$ that contains $k$ zeros
- naive solutions
  - scan bit vector: $O(n)$ time and no space overhead
  - store $k$ solutions in $S[1..k]$ and $select_0(i) = S[i]$ if $k \in O(n/lgn)$ this suffice

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- $|B_{\lfloor i/b \rfloor}| \geq \lg^4 n$: store answers naively
  - requires $O(b \lg n) = O(\lg^3 n)$ bits of space
  - there are at most $O(n/\lg^4 n)$ such blocks
  - total $O(n/\lg n) = o(n)$ bits of space

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  - there are at most $O(n/\lg^4 n)$ such blocks
  - total $O(n/\lg n) = o(n)$ bits of space
- $|B_{\lfloor i/b \rfloor}| < \lg^4 n$: divide super-block into blocks
  - same idea: variable-sized blocks containing $b' = \sqrt{\lg n}$ zeros
  - (prefix) sum $O((k \lg \lg n)/b') = o(n)$ bits
  - if size $\geq \lg n$ store all answers
  - if size $< \lg n$ store lookup table

# Rank- and Select-Queries on Bit Vectors

**Lemma: Binary Rank- and Select-Queries**

Given a bit vector of size $n$, there exists data structures that can be computed in time $O(n)$ of size $o(n)$ bits that can answer rank and select queries on the bit vector in $O(1)$ time

# Preliminaries

## Definition: Bit Representation

Given a text $T$ over an alphabet of size $\sigma$, each character can be represented using $\lceil \lg \sigma \rceil$ bits.

- the leftmost bit is the **most significant bit** and
- the rightmost bit is the **least significant bit**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
|---|---|---|---|---|---|---|---|---|
| $(0$ | $(0$ | $(0$ | $(0$ | $(1$ | $(1$ | $(1$ | $(1$ | MSB |
| $0$ | $0$ | $1$ | $1$ | $0$ | $0$ | $1$ | $1$ | |
| $0)_2$ | $1)_2$ | $0)_2$ | $1)_2$ | $0)_2$ | $1)_2$ | $0)_2$ | $1)_2$ | LSB |

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- bit representation is integer in binary

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| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | |
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## Definition: Bit Prefix

A bit prefix of length $k$ are the $k$ MSBs of a characters bit representation

# Wavelet Trees [GGV03] (1/2)

## Definition: Wavelet Tree

Given a text $T$ of length $n$ over an alphabet $\Sigma = [1, \sigma]$, a **wavelet tree** is a binary tree, where

- each node represents characters in $[\ell, r] \subseteq [1, \sigma]$,
- if a node represents characters in $[\ell, r]$, then its left and right child
- represent characters in $[\ell, (\ell + r)/2)$ and $[(\ell + r)/2, r]$
- a node is a leaf if $\ell + 2 \geq r$
- characters are represented using a bit vector
- an entry is 1 if the character is represented in the right child and 0 otherwise

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- in practice, level-wise wavelet trees have less overhead
- navigation still easy

# Wavelet Trees (2/2)

$[0, 7]$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 6 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 6 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
| $\rightarrow$ | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

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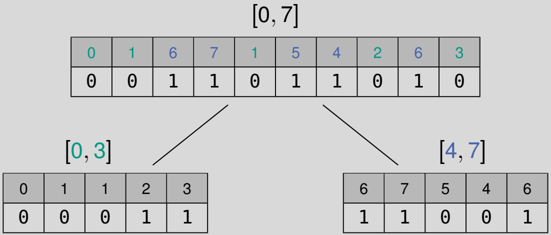
|---|---|---|---|---|---|---|---|---|---|---|
| → | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

# Wavelet Trees (2/2)




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 6 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

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[0, 7]

| 0 | 1 | 6 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |

[0, 3]

| 0 | 1 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 |

[4, 7]

| 6 | 7 | 5 | 4 | 6 |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 |

| 0 | 1 | 6 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

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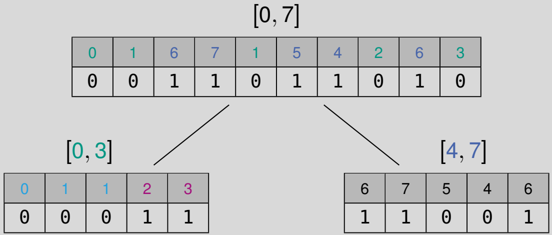


| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 6 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
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| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

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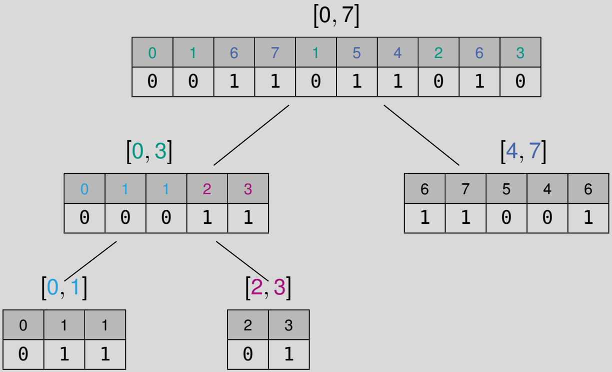


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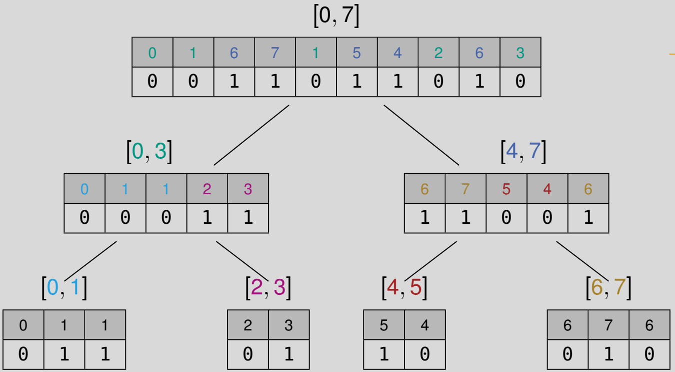


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|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

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| | | | | | | | | | |
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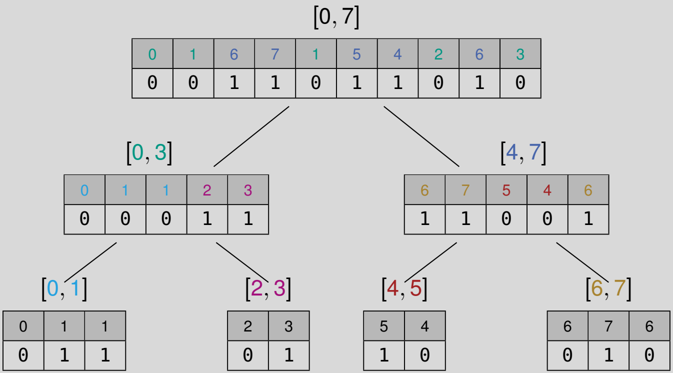
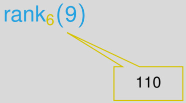
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |



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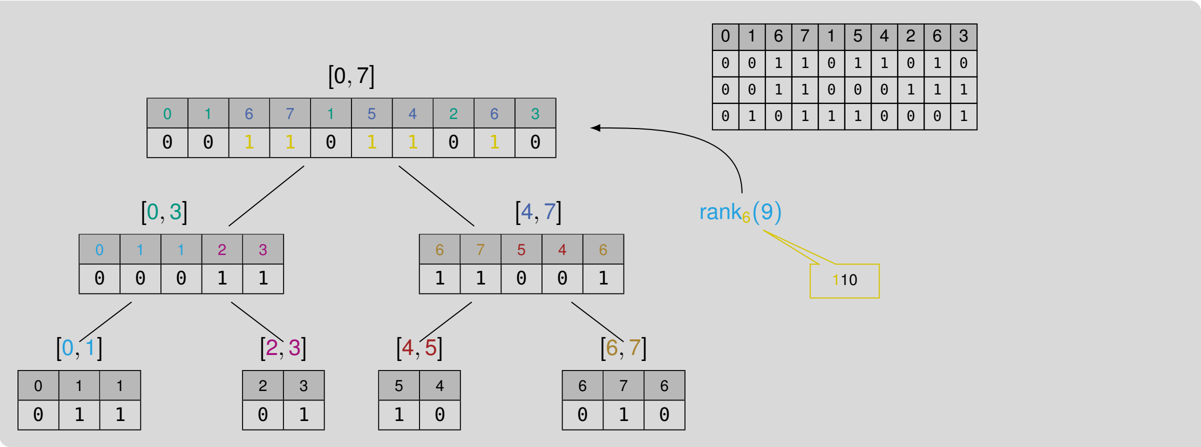

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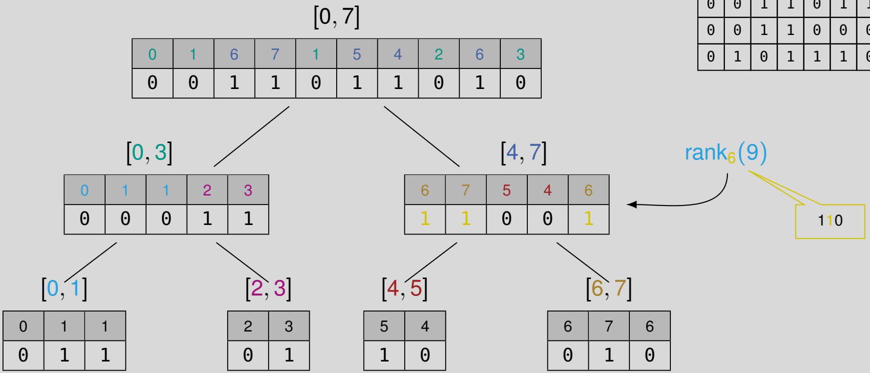
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |



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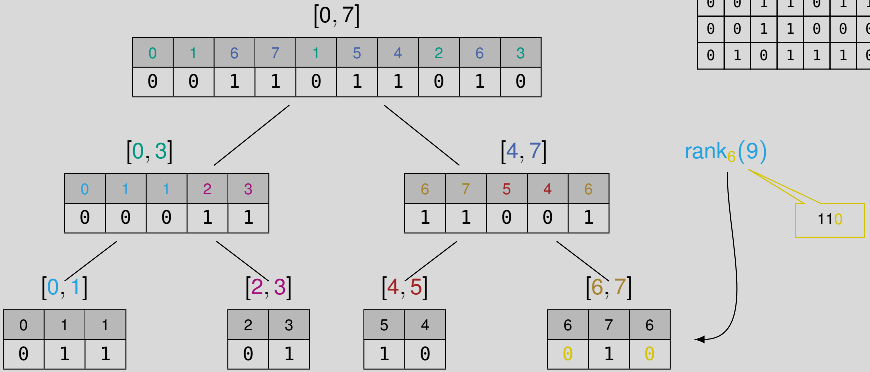

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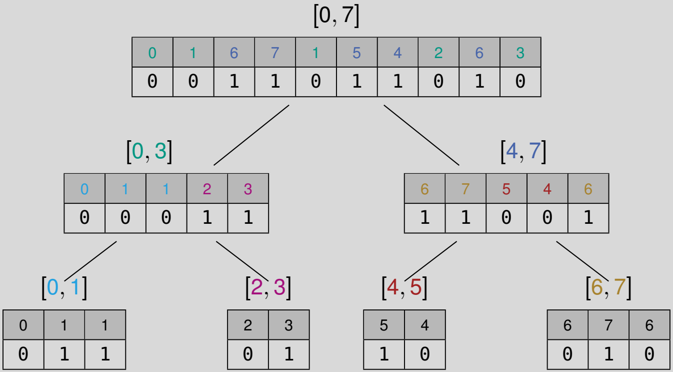


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| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

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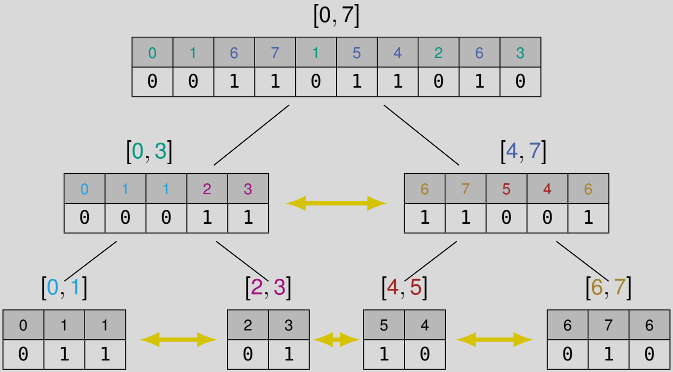


| | | | | | | | | | |
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| 0 | 1 | 6 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

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- in each node, all represented characters share a bit prefix
- on depth $\ell$ the longest common bit prefix has length $\ell - 1$
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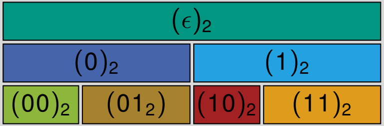

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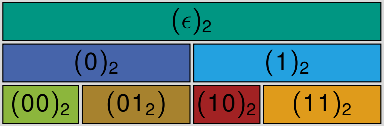


- finding characters in the wavelet tree requires finding the correct interval
- finding the position of a character requires finding the position in the last interval

# Rank-, Select-, and Access-Queries in Wavelet Trees (1/2)

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## Rank-Queries

- use rank queries on bit vectors
- at depth $\ell$ as for $\ell$-th MSB
- follow through tree according to bit

- as seen on a previous slide

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## Access-Queries

- follow bits through the wavelet tree
- return read bits

- same as rank but returning bit pattern instead of final rank
- see example on the board

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## Rank-Queries

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- **PINGO** what is the query time of rank queries in wavelet trees?

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- requires up and down traversal of the wavelet tree
- see example on the board

## Access-Queries

- follow bits through the wavelet tree
- return read bits

- same as rank but returning bit pattern instead of final rank
- see example on the board

# Rank-, Select-, and Access-Queries in Wavelet Trees (2/2)

Karlsruhe Institute of Technology

**Lemma: Query Times Wavelet Tree**

Given a text $T$ over an alphabet of size $\sigma$, the wavelet tree of the text can answer *rank*, *select*, and *access* queries in $O(\lg \sigma)$ time

**Proof (Sketch)**

All queries require

- just a constant number of rank and select queries on the bit vectors and
- at most one traversals from the root of the tree to a leaf and
- one traversal from a leaf to the root of the tree

# Bit Reversal Permutation

- given a bit representation of a character $\alpha$
- $\mathit{reverse}(\alpha)$ reverses the bits
- the MSB becomes the least significant bit

**Definition: Bit-Reversal Permutation**

The **bit-reversal permutation** $\rho_k$ is a permutation of the numbers $[0, 2^k)$ with

$$\rho_k(i) = \mathit{reverse}(i)$$

for $i \in [0, 2^k)$

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- $\rho_2 = (0, 2, 1, 3) = ((00)_2, (10)_2, (01)_2, (11)_2)$
- $\rho_{k+1} = (2\rho_k(0), \ldots, 2\rho_k(2^k - 1), 2\rho_k(0) + 1, \ldots, 2\rho_k(2^k - 1) + 1)$

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- same intervals as a wavelet tree
- used in the wavelet matrix

# Alternative Representation

- alternative representation of wavelet trees
- removing tree structure
- only two areas per level ℹ the intervals discussed before still exist

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- alternative representation of wavelet trees
- removing tree structure
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**Definition: Wavelet Matrix [CNP15]**

Given a text $T$ of length $n$ over an alphabet of size $\sigma$ a wavelet matrix consists of

- bit vectors $BV_\ell$ for $\ell \in [1, \lceil \lg \sigma \rceil]$ of size $n$ and
- an array $Z[1..\lceil \lg \sigma \rceil]$

Such that

- $Z[\ell]$ contains the number of zero bits in $BV_\ell$
- $BV_1$ contains all MSBs in text order
- $BV_\ell$ contains the $\ell$-th MSB the character at position $i$ in $BV_{\ell-1}$ at position
  - $rank_0(i)$ if $BV_{\ell-1} = 0$ and
  - $Z[\ell-1] + rank_1(i)$ if $BV_{\ell-1} = 1$

# Alternative Representation

- alternative representation of wavelet trees
- removing tree structure
- only two areas per level ℹ the intervals discussed before still exist

- better suited for large alphabets
- seemingly less structure
- retaining all important properties

**Definition: Wavelet Matrix [CNP15]**

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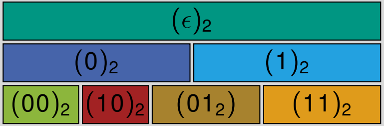
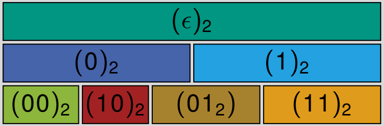
# Intervals of a Wavelet Matrix




- a wavelet matrix has the same intervals a wavelet tree has
- intervals not bounded by parent ℹ no tree structure

# Intervals of a Wavelet Matrix

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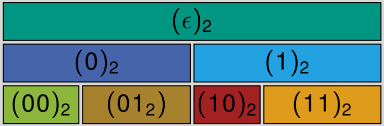


- intervals of a wavelet tree (for comparison)

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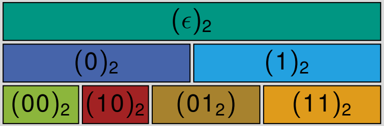


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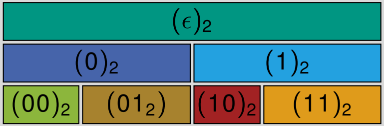


- intervals of a wavelet tree (for comparison)

- **PINGO** is answering queries with a wavelet matrix as simple as with a wavelet tree?

# Example Wavelet Tree and Wavelet Matrix

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 3 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
| $BV_0$ | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| | 0 | 1 | 3 | 1 | 2 | 3 | 7 | 5 | 4 | 6 |
| $BV_1$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| | 0 | 1 | 1 | 3 | 2 | 3 | 5 | 4 | 7 | 6 |
| $BV_2$ | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |

- queries on the wavelet matrix work similar
- example on the board

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 3 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
| $BV_0$ | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| | 0 | 1 | 3 | 1 | 2 | 3 | 7 | 5 | 4 | 6 |
| $BV_1$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| | 0 | 1 | 1 | 5 | 4 | 3 | 2 | 3 | 7 | 6 |
| $BV_2$ | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |

$Z[0] = 6$ $Z[1] = 5$ $Z[2] = 4$

# Naive Wavelet Tree and Wavelet Matrix Construction (1/2)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 3 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
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| | 0 | 1 | 3 | 1 | 2 | 3 | 7 | 5 | 4 | 6 |
| $BV_1$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| | 0 | 1 | 1 | 3 | 2 | 3 | 5 | 4 | 7 | 6 |
| $BV_2$ | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |

## Wavelet Tree

- first level are MSBs of characters of text
- for each level $\ell > 1$
  - stably sort text using Radix sort by bit prefixes of length $\ell - 1$
  - take $\ell$-th MSB of sorted sequence
  - sorted sequence is new text

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|---|---|---|---|---|---|---|---|---|---|---|
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| | 0 | 1 | 3 | 1 | 2 | 3 | 7 | 5 | 4 | 6 |
| $BV_1$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
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| | 0 | 1 | 3 | 1 | 2 | 3 | 7 | 5 | 4 | 6 |
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| | 0 | 1 | 1 | 5 | 4 | 3 | 2 | 3 | 7 | 6 |
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$Z[0] = 6$ $Z[1] = 5$ $Z[2] = 4$

## Wavelet Matrix

- first level are MSBs of characters of text
- for each level $\ell > 1$
  - stably sort text by $\ell - 1$ MSB
  - take $\ell$-th MSB of sorted sequence
  - sorted sequence is new text

# Wavelet Tree and Wavelet Matrix Construction (2/2)

- to make both fully functional bit vectors are augmented with binary rank and select support

**Lemma: Running Time and Memory Requirements Wavelet Tree and Wavelet Matrix**

Given a text $T$ over an alphabet of size $\sigma$, the wavelet tree and wavelet matrix require $(1 + o(1))n\lceil\lg \sigma\rceil$ bits of space and can be constructed in $O(n \lg \sigma)$ time

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- **PINGO** is there a asymptotically faster construction method?

# Better Wavelet Tree Construction [Bab+15; MNV16]

- using requires broadword programming
- every $\tau$-th level is a big level
- big levels contain enough information to compute small levels below
- small levels computed by splitting big levels
- $O(b/\lg n)$ characters at a time with $b = o(\lg n)$
- sketch on board

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- can be implemented using AVX/SSE instructions [Din+23; Kan18]

# Huffman-shaped Wavelet Trees

- wavelet trees can be compressed
- more precise: the text can be compressed
- use Huffman codes
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## Huffman Codes (Recap)

- idea is to create a binary tree
- each character $\alpha$ is a leaf and has weight $\mathit{Hist}[\alpha]$
- create node for two nodes without parent with smallest weight
- give new node total weight of children
- repeat until only one node without parent remains
- label edges:
  - left edge: 0
  - right edge: 1
- path to children gives code for character

# Huffman-shaped Wavelet Trees

- wavelet trees can be compressed
- more precise: the text can be compressed
- use Huffman codes
- wavelet trees cannot handle holes
- use canonical Huffman codes

## Canonical Huffman Codes (Recap)

- start with Huffman codes, code word 0, and length 1
- to get canonical code for current length, then add 1 to code word
- to update length add 1 and append required amount of zeros to code word

## Huffman Codes (Recap)

- idea is to create a binary tree
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- give new node total weight of children
- repeat until only one node without parent remains
- label edges:
  - left edge: 0
  - right edge: 1
- path to children gives code for character

# Huffman-shaped Wavelet Trees

| $\alpha$ | $hc(\alpha)$ | $chc(\alpha)$ |
|---|---|---|
| 1 | $(11)_2$ | $(11)_2$ |
| 3 | $(01)_2$ | $(10)_2$ |
| 6 | $(100)_2$ | $(011)_2$ |
| 7 | $(101)_2$ | $(010)_2$ |
| 0 | $(0000)_2$ | $(0011)_2$ |
| 2 | $(0001)_2$ | $(0010)_2$ |
| 4 | $(0010)_2$ | $(0001)_2$ |
| 5 | $(0011)_2$ | $(0000)_2$ |

- Huffman codes (hc)
- canonical Huffman codes (chc) that are bit-wise negated

# Huffman-shaped Wavelet Trees

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| 3 | $(01)_2$ | $(10)_2$ |
| 6 | $(100)_2$ | $(011)_2$ |
| 7 | $(101)_2$ | $(010)_2$ |
| 0 | $(0000)_2$ | $(0011)_2$ |
| 2 | $(0001)_2$ | $(0010)_2$ |
| 4 | $(0010)_2$ | $(0001)_2$ |
| 5 | $(0011)_2$ | $(0000)_2$ |

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 7 | 1 | 5 | 4 | 2 | 6 | 3 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 7 | 5 | 4 | 2 | 6 | 1 | 3 | 1 | 3 |
| 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 5 | 4 | 2 | 7 | 6 | | | | |
| 1 | 0 | 0 | 1 | 0 | 1 | | | | |
| 5 | 4 | 0 | 2 | | | | | | |
| 0 | 1 | 1 | 0 | | | | | | |

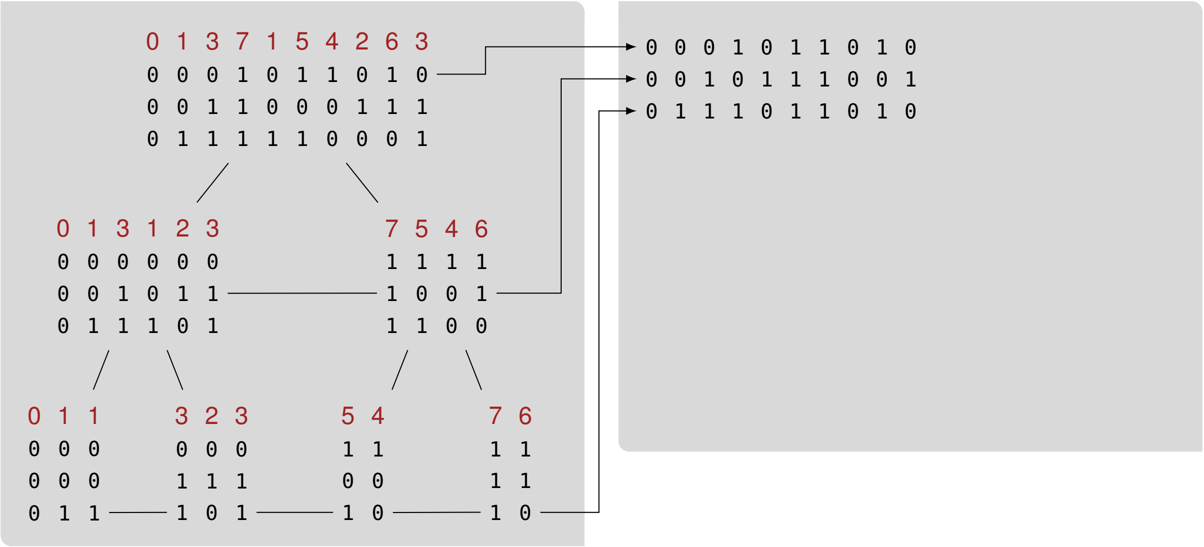
- intervals are only missing to the right (white space)
- no holes allow for easy querying

# Practical Sequential Wavelet Tree Construction

**Bottom-Up Construction [FKL18]**

- scan the text and create histogram
- while scanning compute first level
- use histogram to compute borders of intervals
- scan text again and fill bit vectors

- example on the next slide


0 1 3 7 1 5 4 2 6 3
0 0 0 1 0 1 1 0 1 0
0 0 1 1 0 0 0 1 1 1
0 1 1 1 1 1 0 0 0 1
0 1 3 1 2 3
0 0 0 0 0 0
0 0 1 0 1 1
0 1 1 1 0 1
7 5 4 6
1 1 1 1
1 0 0 1
1 1 0 0
0 1 1
0 0 0
0 0 0
0 1 1
3 2 3
0 0 0
1 1 1
1 0 1
5 4
1 1
0 0
1 0
7 6
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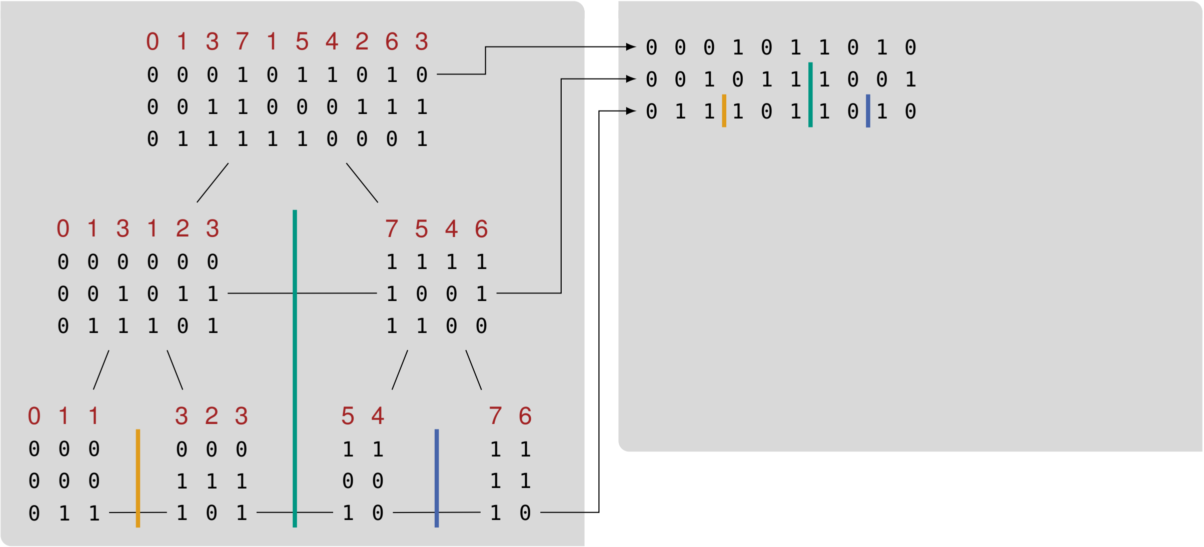
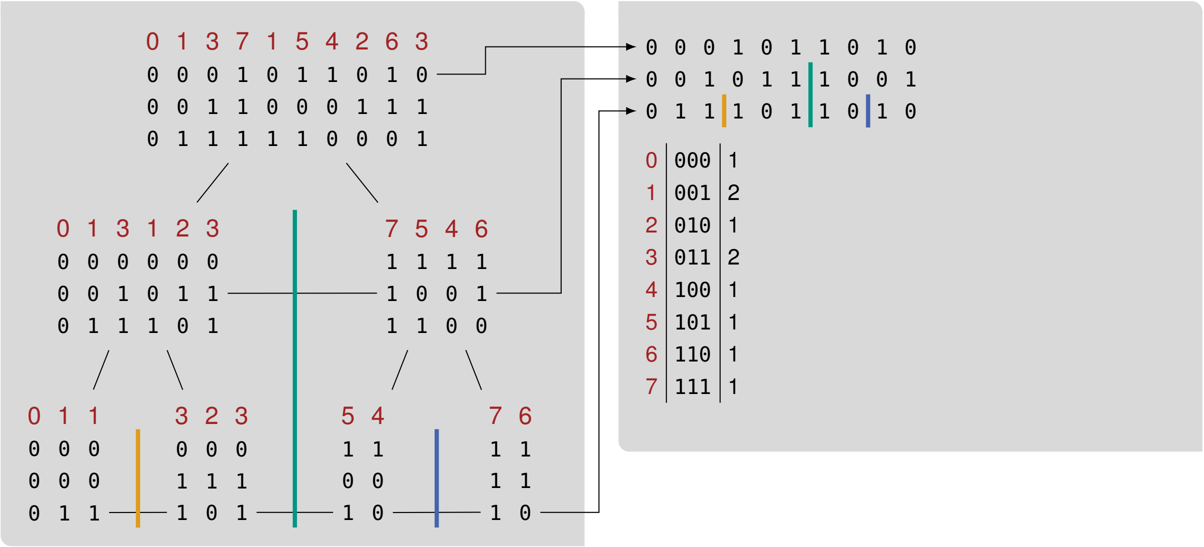
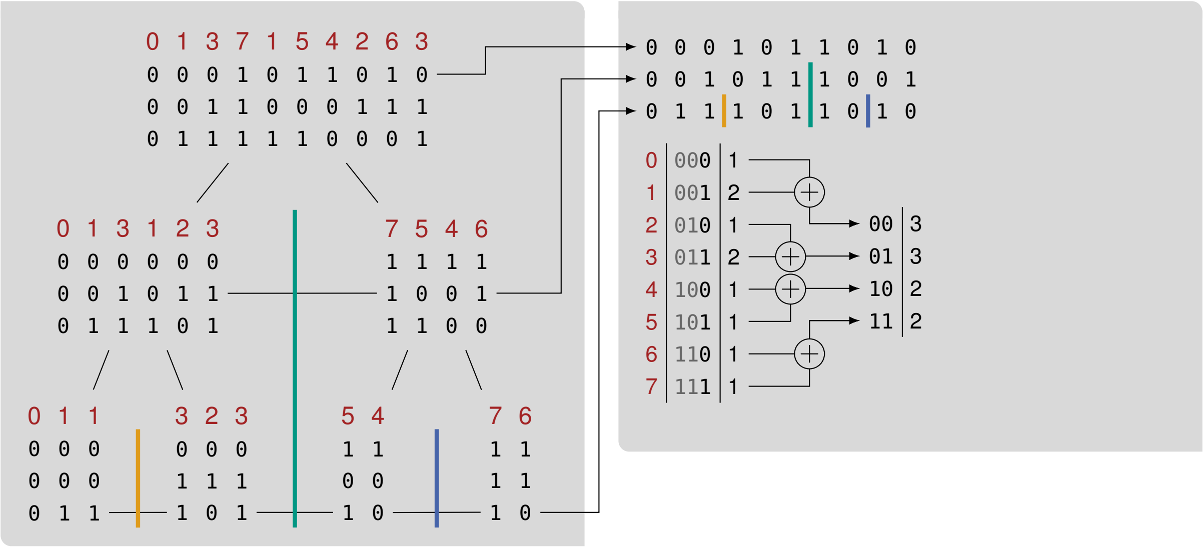
0 0 0 1 0 1 1 0 1 0
0 0 1 0 1 1 1 0 0 1
0 1 1 1 0 1 1 0 1 0


0 1 3 7 1 5 4 2 6 3
0 0 0 1 0 1 1 0 1 0
0 0 1 1 0 0 0 1 1 1
0 1 1 1 1 1 0 0 0 1
0 1 3 1 2 3
0 0 0 0 0 0
0 0 1 0 1 1
0 1 1 1 0 1
7 5 4 6
1 1 1 1
1 0 0 1
1 1 0 0
0 1 1
0 0 0
0 0 0
0 1 1
3 2 3
0 0 0
1 1 1
1 0 1
5 4
1 1
0 0
1 0
7 6
1 1
1 1
1 0
0 0 0 1 0 1 1 0 1 0
0 0 1 0 1 1 1 0 0 1
0 1 1 1 0 1 1 0 1 0


0 1 3 7 1 5 4 2 6 3
0 0 0 1 0 1 1 0 1 0
0 0 1 1 0 0 0 1 1 1
0 1 1 1 1 1 0 0 0 1
0 1 3 1 2 3
0 0 0 0 0 0
0 0 1 0 1 1
0 1 1 1 0 1
7 5 4 6
1 1 1 1
1 0 0 1
1 1 0 0
0 1 1
0 0 0
0 0 0
0 1 1
3 2 3
0 0 0
1 1 1
1 0 1
5 4
1 1
0 0
1 0
7 6
1 1
1 1
1 0
0 0 0 1 0 1 1 0 1 0
0 0 1 0 1 1 1 0 0 1
0 1 1 1 0 1 1 0 1 0
0 000 1
1 001 2
2 010 1
3 011 2
4 100 1
5 101 1
6 110 1
7 111 1


0 1 3 7 1 5 4 2 6 3
0 0 0 1 0 1 1 0 1 0
0 0 1 1 0 0 0 1 1 1
0 1 1 1 1 1 0 0 0 1
0 1 3 1 2 3
0 0 0 0 0 0
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0 1 1 1 0 1
7 5 4 6
1 1 1 1
1 0 0 1
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0 1 1
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3 2 3
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5 4
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7 6
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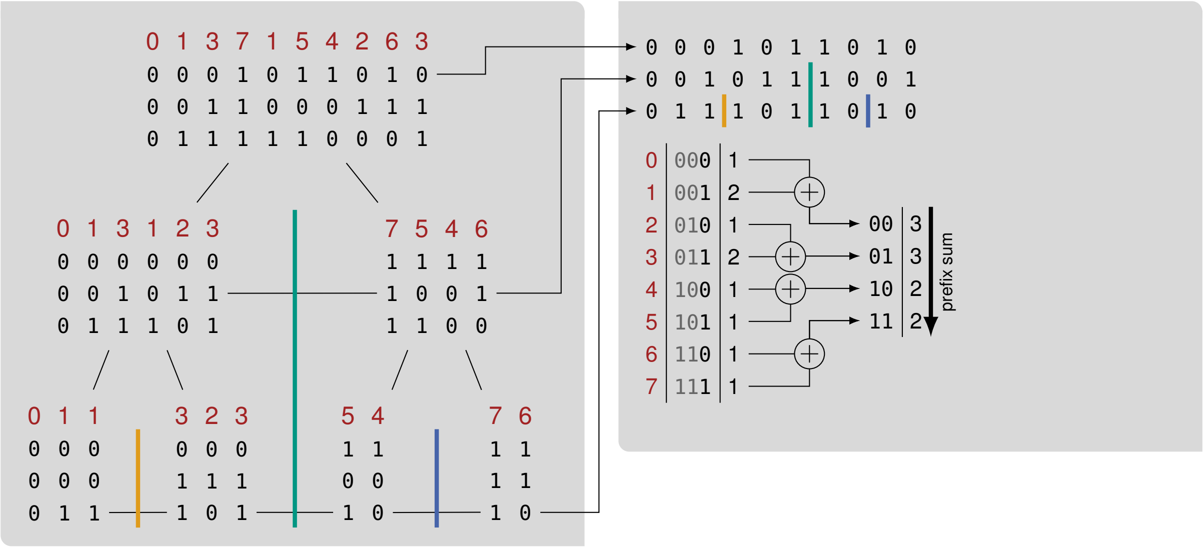
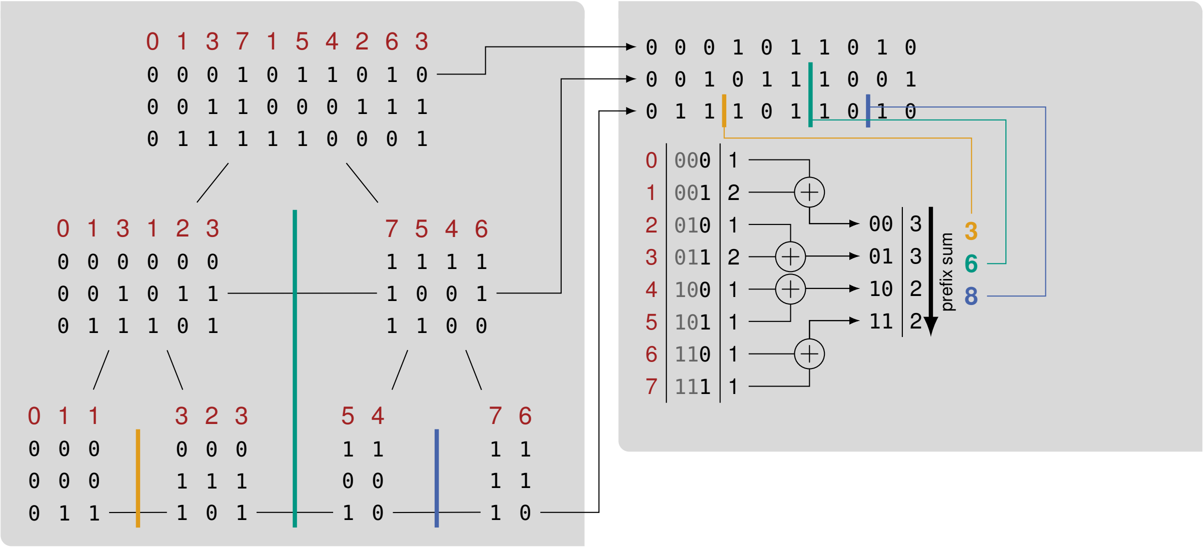
1 0
0 0 0 1 0 1 1 0 1 0
0 0 1 0 1 1 1 0 0 1
0 1 1 1 0 1 1 0 1 0
0 000 1
1 001 2
2 010 1
3 011 2
4 100 1
5 101 1
6 110 1
7 111 1
00 3
01 3
10 2
11 2


0 1 3 7 1 5 4 2 6 3
0 0 0 1 0 1 1 0 1 0
0 0 1 1 0 0 0 1 1 1
0 1 1 1 1 1 0 0 0 1
0 1 3 1 2 3
7 5 4 6
0 1 1
3 2 3
5 4
7 6
0 0 0 1 0 1 1 0 1 0
0 0 1 0 1 1 1 0 0 1
0 1 1 1 0 1 1 0 1 0
0 000 1
1 001 2
2 010 1
3 011 2
4 100 1
5 101 1
6 110 1
7 111 1
00 3
01 3
10 2
11 2
prefix sum


0 1 3 7 1 5 4 2 6 3
0 0 0 1 0 1 1 0 1 0
0 0 1 1 0 0 0 1 1 1
0 1 1 1 1 1 0 0 0 1
0 1 3 1 2 3
0 0 0 0 0 0
0 0 1 0 1 1
0 1 1 1 0 1
7 5 4 6
1 1 1 1
1 0 0 1
1 1 0 0
0 1 1
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1 1
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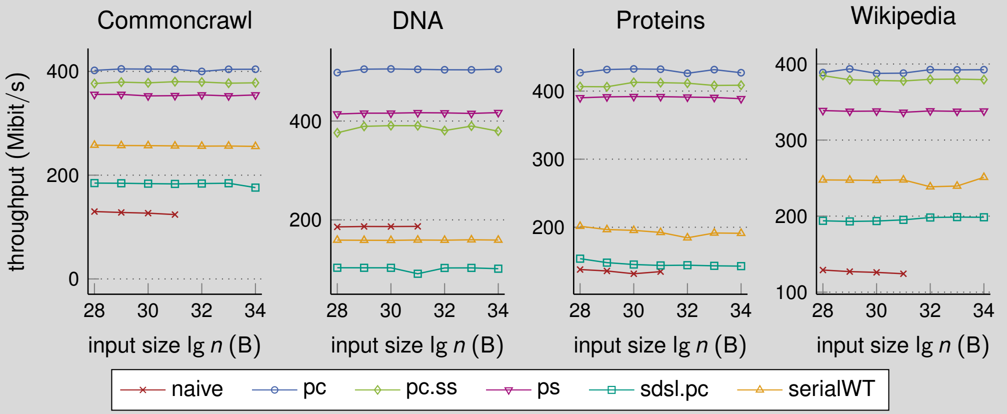
0 0 0 1 0 1 1 0 1 0
0 0 1 0 1 1 1 0 0 1
0 1 1 1 0 1 1 0 1 0
0 000 1
1 001 2
2 010 1
3 011 2
4 100 1
5 101 1
6 110 1
7 111 1
00 3
01 3
10 2
11 2
prefix sum
3
6
8

# Experimental Setup

- 64 GB RAM
- two Intel Xeon E5-2640v4 CPUs (10 cores at 2.4 GHz base frequency, 3.4 GHz maximum turbo frequency, and cache sizes: 32 KB L1D and L1I, 256 KB L2, 25.6 MB L3)

- same texts as in chapter 04
- results are average of 5 runs

# Experiments: Sequential Wavelet Tree Construction



# Experiments: Vectorized Wavelet Tree Construction [Din+23]

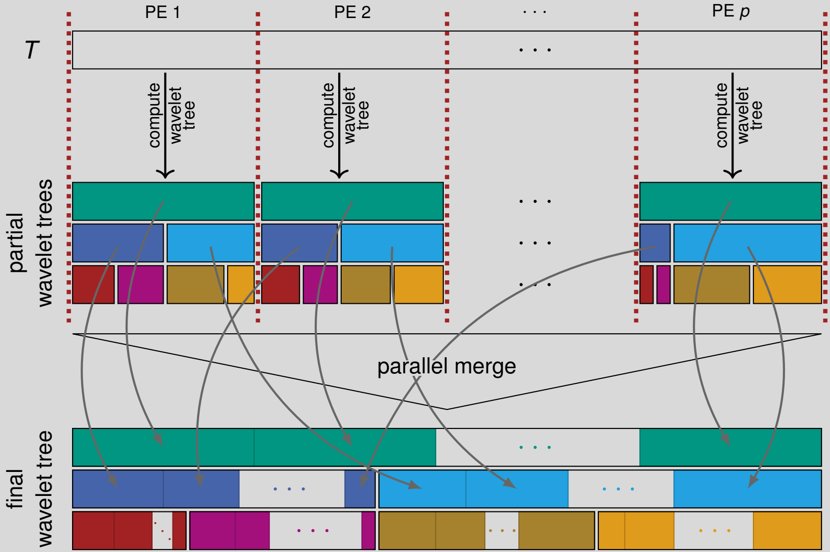
| File | `lut` | `ext` | `shuf64` | `shuf128` | `shuf256` | `shuf512` | `pc` | `pc-ss` |
|---|---|---|---|---|---|---|---|---|
| dblp.xml | 433.44 | 722.21 | 614.24 | 834.92 | 1 197.80 | **1 477.77** | 608.43 | 752.48 |
| dna | 529.32 | 883.00 | 563.11 | 668.93 | 862.49 | **1 011.45** | 594.02 | 745.68 |
| english | 456.91 | 770.55 | 677.96 | 906.42 | 1 304.80 | **1 642.69** | 623.08 | 704.90 |
| pitches | 448.02 | 749.24 | 686.88 | 886.62 | 1 276.36 | **1 584.19** | 578.70 | 328.47 |
| proteins | 375.73 | 575.99 | 565.63 | 707.23 | 985.35 | **1 178.02** | 633.58 | 761.41 |
| sources | 451.24 | 757.75 | 650.22 | 882.45 | 1 296.80 | **1 632.85** | 594.22 | 754.72 |
| cc.16gib | 453.97 | 729.58 | 653.25 | 875.61 | 1 265.27 | **1 604.84** | 628.46 | 752.97 |
| dna.16gib | 436.89 | 644.08 | 483.45 | 451.33 | 537.36 | 593.96 | **669.70** | 650.33 |
| wiki.16gib | 447.95 | 714.42 | 634.91 | 871.14 | 1 267.69 | **1 604.39** | 591.01 | 753.05 |
| ru.8gib | 317.20 | 642.51 | 506.04 | 660.23 | 938.68 | **1 121.03** | 346.96 | 170.44 |

# Parallel Wavelet Tree Construction in Practice

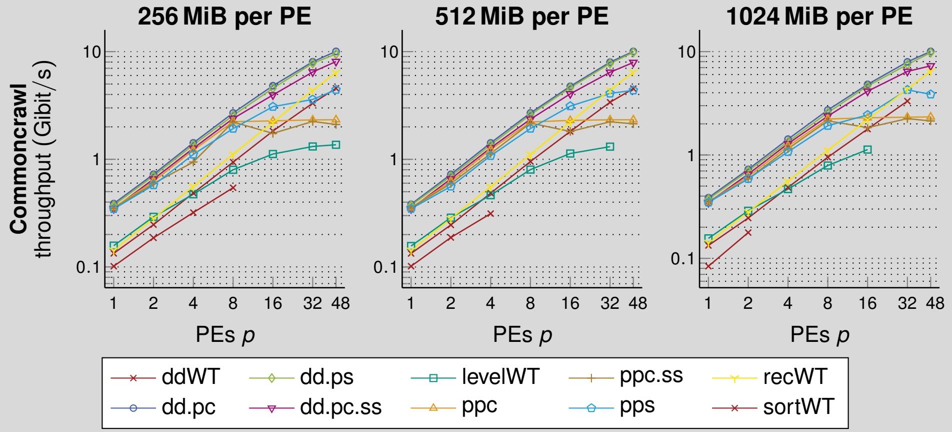
**Domain Decomposition [Fue+17]**

- create wavelet tree in parallel using $p$ PEs
- each PE gets a consecutive slice of text
- each PE builds partial wavelet tree for its text
- merge partial wavelet trees in parallel

- can utilize any sequential algorithm
- very fast in practice
- $O(n \lg \sigma / \sqrt{\lg n})$ work and $O(\sigma + \lg n)$ time [Shu20]


PE 1
PE 2
· · ·
PE $p$
$T$
compute wavelet tree
partial wavelet trees
parallel merge
final wavelet tree

# Experiments: Parallel Wavelet Tree Construction

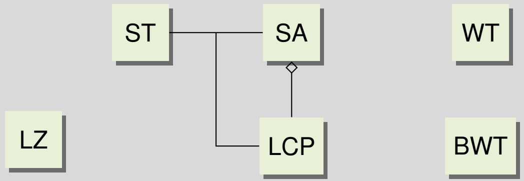


# Conclusion and Outlook

## This Lecture

- wavelet tree and wavelet matrix
- Huffman-shaped wavelet trees

## Linear Time Construction



# Conclusion and Outlook

Karlsruhe Institute of Technology

## This Lecture

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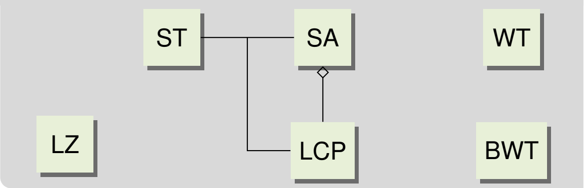
- Huffman-shaped wavelet trees

- select on bit vectors
- practical algorithms for wavelet tree construction

## Linear Time Construction



# Conclusion and Outlook

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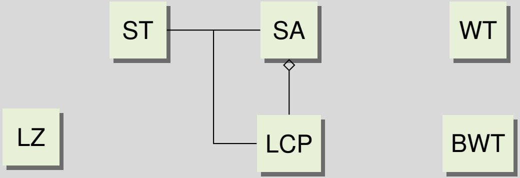


## Next Lecture

- FM-index
- r-Index

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