Karlsruhe Institute of Technology

Text Indexing

Lecture 08: FM-Index and *r*-Index

Florian Kurpicz

Recap: Wavelet Trees

$[0, 7]$

0	1	6	7	1	5	4	2	6	3
0	0	1	1	0	1	1	0	1	0

	0	1	6	7	1	5	4	2	6	3
→	0	0	1	1	0	1	1	0	1	0
	0	0	1	1	0	0	0	1	1	1
	0	1	0	1	1	1	0	0	0	1

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0	1	0	1	1	1	0	0	0	1

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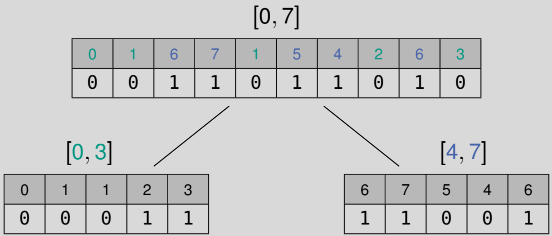

0	1	6	7	1	5	4	2	6	3
0	0	1	1	0	1	1	0	1	0
0	0	1	1	0	0	0	1	1	1
0	1	0	1	1	1	0	0	0	1

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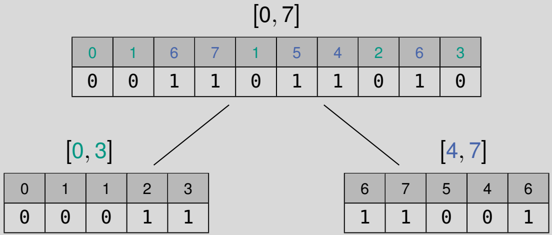

0	1	6	7	1	5	4	2	6	3
0	0	1	1	0	1	1	0	1	0
0	0	1	1	0	0	0	1	1	1
0	1	0	1	1	1	0	0	0	1

Recap: Wavelet Trees

[0, 7]

0	1	6	7	1	5	4	2	6	3
0	0	1	1	0	1	1	0	1	0

[0, 3]

0	1	1	2	3
0	0	0	1	1

[4, 7]

6	7	5	4	6
1	1	0	0	1

	0	1	6	7	1	5	4	2	6	3
	0	0	1	1	0	1	1	0	1	0
$\rightarrow$	0	0	1	1	0	0	0	1	1	1
	0	1	0	1	1	1	0	0	0	1

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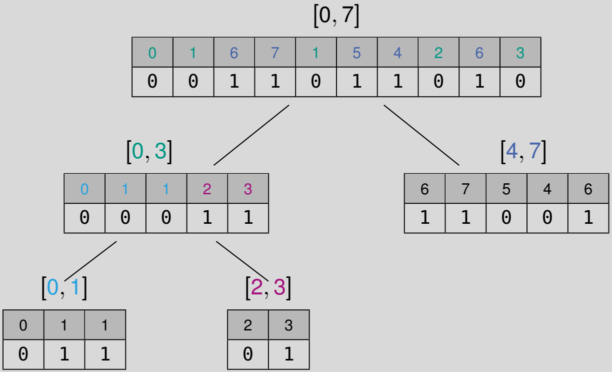

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0	0	1	1	0	1	1	0	1	0
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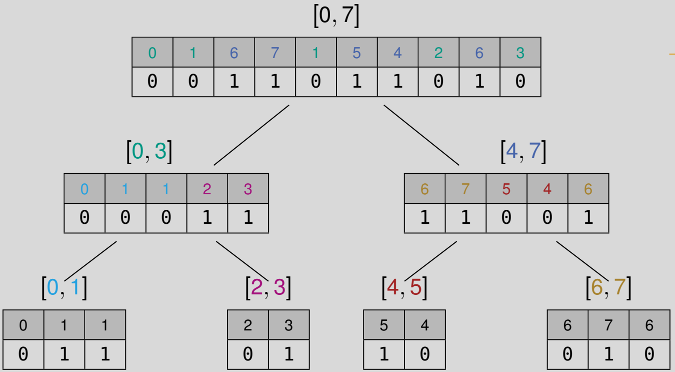

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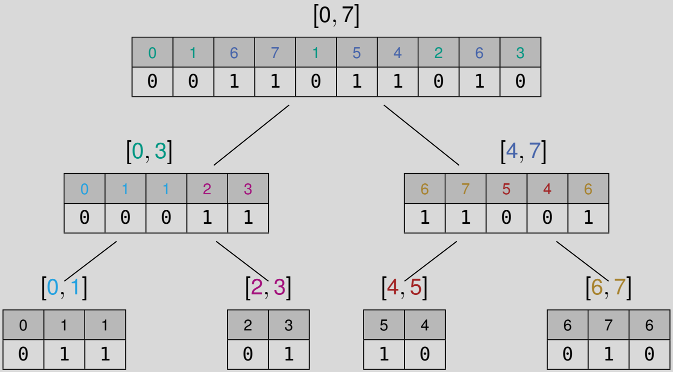

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$\mathrm{rank}_6(9)$

110

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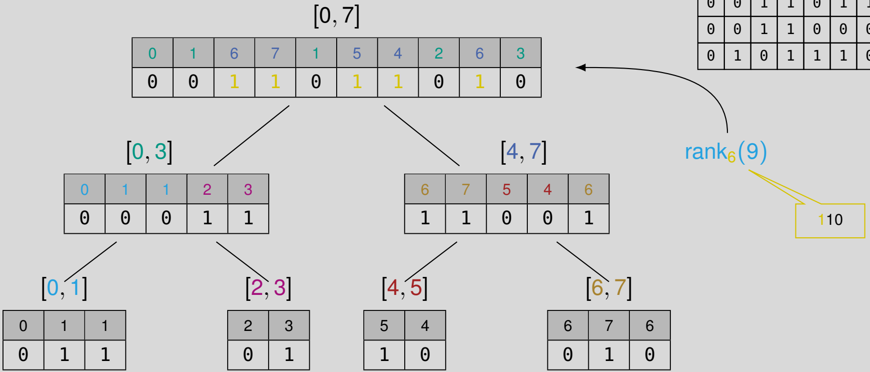

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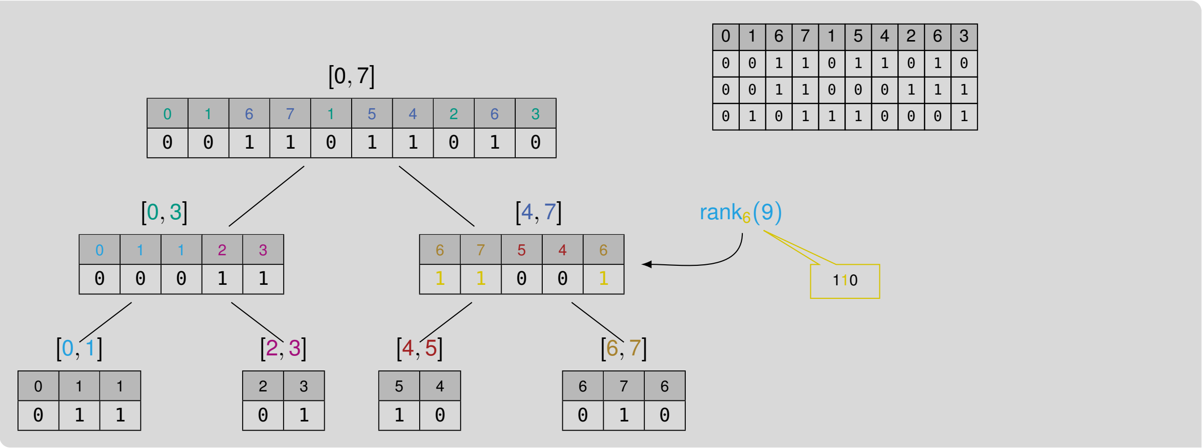

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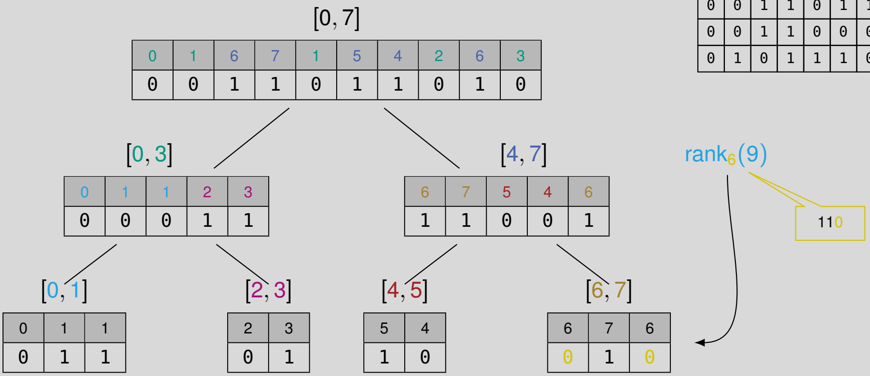

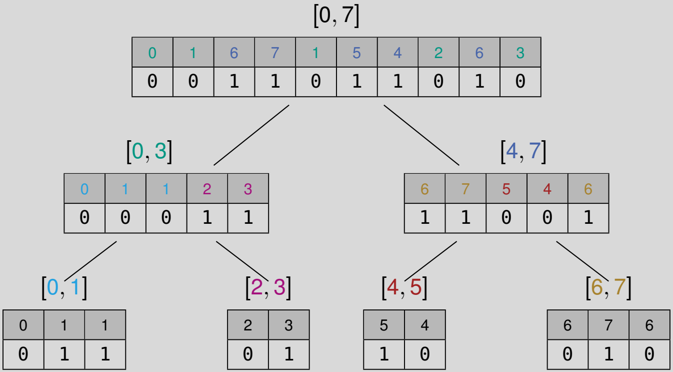

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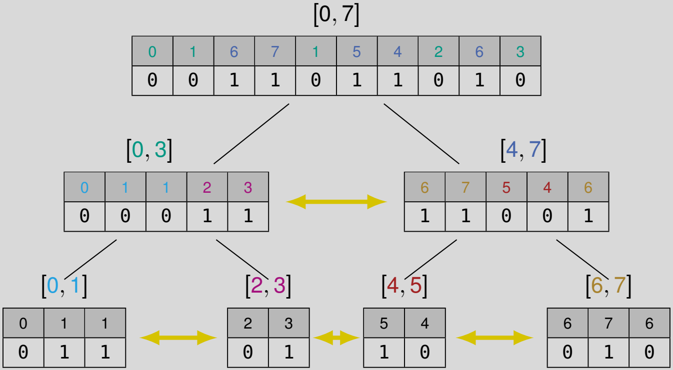

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Recap: Compressed Wavelet Trees

0	1	3	7	1	5	4	2	6	3
0	1	1	0	1	0	0	0	0	1
0	7	5	4	2	6	1	3	1	3
0	1	0	0	0	1	1	0	1	0
0	5	4	2	7	6				
1	0	0	1	0	1				
5	4	0	2						
0	1	1	0						

- intervals are only missing to the right (white space)
- no holes allow for easy querying

- build wavelet tree for compressed text
- compress text using bit-wise negated canonical Huffman-codes

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Karlsruhe Institute of Technology

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- compress text using bit-wise negated canonical Huffman-codes

- can a wavelet tree be compressed further?

Bit Vector Compression (1/2)

- compress (sparse) bit vectors
- bit vector contains k one bits
- use $O(k \lg \frac{n}{k}) + o(n)$ bits
- retrieve $\Theta(\lg n)$ bits at the same time
- similar to *rank* data structure

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- blocks of size $s = \frac{\lg n}{2}$
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- for $i \in [0, s]$ store lookup-table containing all bit vectors with i one bits

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Bit Vector V

- let k_i be number of ones in i-th block
- use $\lceil \lg \binom{s}{k_i} \rceil$ bits to encode block position in lookup-table
- concatenate all codes

Bit Vector Compression (2/2)

Array *SBlock*

- for every super-block i, $SBlock[i]$ contains position of encoding of first block in i-th super-block in V
- $\lceil \lg n \rceil$ bits per entry

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Proof (Sketch space requirements)

- $|C| = O(\frac{n}{s} \lg s) = o(n)$ bits
- $|SBlock| = O(\frac{n}{s'} \lg n) = o(n)$ bits
- $|Block| = O(\frac{n}{s} \lg s) = o(n)$ bits
- $\sum_{k=0}^{s} |L_k| \leq (s+1)2^s s = o(n)$ bits
- $|V| = \sum_{i=1}^{\lceil n/s \rceil} \lceil \lg \binom{s}{k_i} \rceil \leq \lg \binom{n}{k} + n/s \leq \lg((n/k)^k) + n/s = k \lg \frac{n}{k} + O(\frac{n}{\lg n})$ bits

Recap: Backwards Search in the BWT

Function *BackwardsSearch*$(P[1..n], C, rank)$**:**

- $s = 1, e = n$
- **for** $i = m, \ldots, 1$ **do**
  - $s = C[P[i]] + rank_{P[i]}(s-1) + 1$
  - $e = C[P[i]] + rank_{P[i]}(e)$
  - **if** $s > e$ **then**
    - **return** $\emptyset$
- **return** $[s, e]$

- no access to text or *SA* required
- no binary search
- existential queries are easy
- counting queries are easy
- reporting queries require additional information
- example on the board

The FM-Index [FM00]

Building Blocks of FM-Index

- wavelet tree on BWT providing *rank*-function
- C-array
- sampled suffix array with sample rate s
- bit vector marking sampled suffix array positions

Lemma: FM-Index

Given a text T of length n over an alphabet of size σ, the FM-index requires $O(n \lg \sigma)$ bits of space and can answer counting queries in $O(m \lg \sigma)$ time and reporting queries in $O(occ + m \lg \sigma)$ time

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Space Requirements

- wavelet tree: $n\lceil \lg \sigma \rceil(1 + o(1))$ bits
- C-array: $\sigma \lceil \lg n \rceil$ bits ℹ $n(1 + o(1))$ bits if $\sigma \geq \frac{n}{\lg n}$
- sampled suffix array: $\frac{n}{s}\lceil \lg n \rceil$ bits
- bit vector: $n(1 + o(1))$ bits

- space and time bounds can be achieved with $s = \lg_\sigma n$

Conclusion FM-Index

- FM-index is easy to compress
- wavelet tree on *BWT* can be compressed
- bit vector can be compressed

- very small in comparison with suffix tree or suffix array
- compression does not make use of structure of *BWT* ℹ wavelet trees are compressed using Huffman-codes

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Definition: Run (simplified, recap)

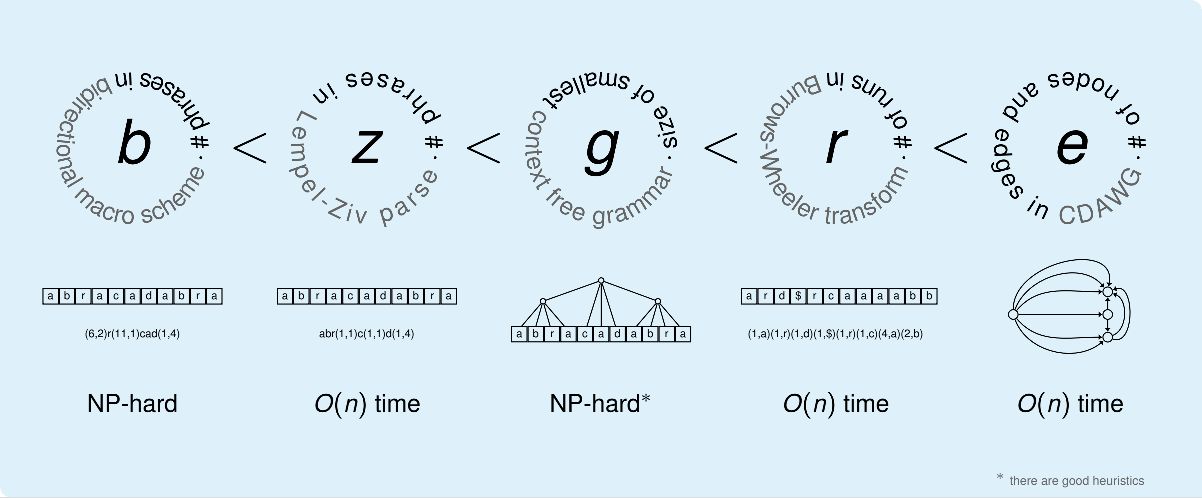
Given a text T of length n, we call its substring $T[i..j]$ a **run**, if

- $T[k] = T[\ell]$ for all $k, \ell \in [i, j]$ and
- $T[i-1] \neq T[i]$ and $T[j+1] \neq T[j]$

ℹ (To be more precise, this is a definition for a run of a periodic substring with smallest period 1, but this is not important for this lecture )

	1	2	3	4	5	6	7	8	9	0	11	12	13
L	a	b	$	c	c	b	b	a	a	a	a	b	b

Measures of Repetitiveness (Excerpt)



Measures of Repetitiveness (Excerpt)

order of magnitude in practice

# phrases in bidirectional macro scheme		# phrases in Lempel-Ziv parse		size of smallest context free grammar		# of runs in Burrows-Wheeler transform		# of nodes and edges in CDAWG
b	$<$	z	$<$	g	$<$	r	$<$	e

a	b	r	a	c	a	d	a	b	r	a

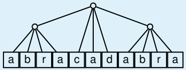
(6,2)r(11,1)cad(1,4)

NP-hard

a	b	r	a	c	a	d	a	b	r	a

abr(1,1)c(1,1)d(1,4)

$O(n)$ time

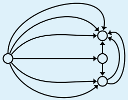


NP-hard*

a	r	d	$	r	c	a	a	a	a	b	b

(1,a)(1,r)(1,d)(1,$)(1,r)(1,c)(4,a)(2,b)

$O(n)$ time



$O(n)$ time

* there are good heuristics

Motivation: *r*-Index

Measure for Compressibility

- k-th order empirical entropy H_k
- number of LZ factors z
- number of *BWT* runs r

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- how do they relate?

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Lemma: *BWT* runs and LZ factors [KK20]

Given a text T of length n. Let z be the number of LZ77 factors and r the number of runs in T's *BWT*, then

$$r \in O(z \lg^2 n)$$

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- more details in next lecture

Main Part of Backwards-Search

```
Function BackwardsSearch(P[1..n], C, rank):
    s = 1, e = n
    for i = m, ..., 1 do
        s = C[P[i]] + rank_{P[i]}(s − 1) + 1
        e = C[P[i]] + rank_{P[i]}(e)
        if s > e then
            return ∅
    return [s, e]
```

Goals

- simulate *BWT* and *rank* on *BWT* in
- $O(r \lg n)$ bits of space

The r-Index [GNP20] (1/3)

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- build wavelet tree for L'

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- accumulate for each character by performing exclusive prefix sum over run lengths'

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Array C'

- $C'[\alpha]$ stores the start of the run lengths in R for each character $\alpha \in \Sigma$ starting at 0

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Bit Vector B

- compressed bit vector of length n containing ones at positions where BWT runs start and rank-support

The r-Index (2/3)

$rank_\alpha(BWT, i)$ with r-Index

- compute number j of run ($j = rank_1(B, i)$)
- compute position k in R ($k = C'[\alpha]$)
- compute number ℓ of α runs before the j-th run ($\ell = rank_\alpha(L', j - 1)$)
- compute number k of αs before the j-th run ($k = R[k + \ell]$)
- compute character β of run ($\beta = L'[j]$)
- if $\alpha \neq \beta$ return k ℹ i is not in the run
- else return $k + i - I[j] + 1$ ℹ i is in the run

The r-Index (3/3)

Lemma: Space Requirements r-Index

Given a text T of length n over an alphabet of size σ that has r *BWT* runs, then its r-index requires

$$O(r \lg n) \text{ bits}$$

and can answer *rank*-queries on the *BWT* in $O(\lg \sigma)$. Given a pattern of length m, the r-index can answer pattern matching queries in time

$$O(m \lg \sigma)$$

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Given a pattern of length m, the r-index can answer pattern matching queries in time

$$O(m \lg \sigma)$$

- what about reporting queries?

Locating Occurrences (Sketch)

- modify backwards-search that it maintains $SA[e]$
- after backwards-search output $SA[e], SA[e-1], \ldots, SA[s]$
- in $O(r \lg n)$ bits and $O(occ \cdot \lg\lg r)$ time

Maintaining $SA[e]$

- sample SA positions at ends of runs
- if next character is $BWT[e]$, then next $SA[e']$ is $SA[e]-1$
- otherwise locate end of run and extract sample

Output Result

- following LF not possible unbounded
- deduce $SA[i-1]$ from $SA[i]$
- character in L and F in same order
- only beginning of runs complicated
- for every character build predecessor data structure over sampled SA-values at end of runs
- associate with $\langle i, SA[i]\rangle$

Now: OptBWTR

	Time (locate)	Time (count)	Space (words)
r-index [GNP20]	$O(\|P\| \log\log_w(\sigma + n/r) + occ)$ $O(\|P\| + occ)$	$O(\|P\| \log\log_w(\sigma + n/r))$ $O(\|P\|)$	$O(r)$ $O(r \log\log(\sigma + n/r))$
OptBWTR [NT21]	$O(\|P\| \log\log_w \sigma + occ)$	$O(\|P\| \log\log_w \sigma)$	$O(r)$

RLBWT

- partition BWT into r substrings
- $BWT = L_1 L_2 \ldots L_r$
- L_i is maximal repetition of same character
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- $RLBWT = (L_1[1], \ell_1)(L_2[1], \ell_2) \ldots (L_r[1], \ell_r)$

RLBWT

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Lemma: LF and RLBWT

- Let $\ell_x < i < \ell_{x+1}$ for some $i \in [1, n]$, then

$$LF(i) = LF(\ell_x) + (i - \ell_x)$$

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Input and Output Intervals

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- there are r intervals
- represent domain of *LF* by intervals

- solve LF without predecessor queries ℹ we did not use predecessor queries
- predecessor queries are bottleneck

Disjoint Interval Sequence & Move Query

Definition: Disjoint Interval Sequence

Let $I = (p_1, q_1), (p_2, q_2), \ldots, (p_k, q_k)$ be a sequence of k pairs of integers. We introduce a permutation π of $[1, k]$ and sequence $d_1, d_2, \ldots, d_k$ for I. π satisfies $q_{\pi[1]} \leq q_{\pi[2]} \leq \cdots \leq q_{\pi[k]}$, and $d_i = p_{i+1} - p_i$ for $i \in [1, k]$, where $p_{k+1} = n + 1$. We call the sequence I a disjoint interval sequence if it satisfies the following three conditions:

- $p_1 = 1 < p_2 < \cdots < p_k \leq n$
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Move Query

$$\text{move}(i, x) = (i', x')$$

- i position in input interval
- x input interval
- i' position in output interval
- x' input interval covering i'

Answering Move Query

- $D_{pair} = (p_i, q_i)$ for every interval
- $D_{index}[i]$ index of input interval containing q_i

example on the board

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Moving for LF

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LF Query

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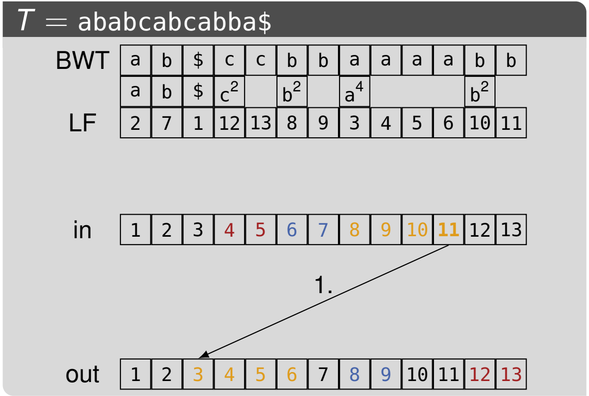

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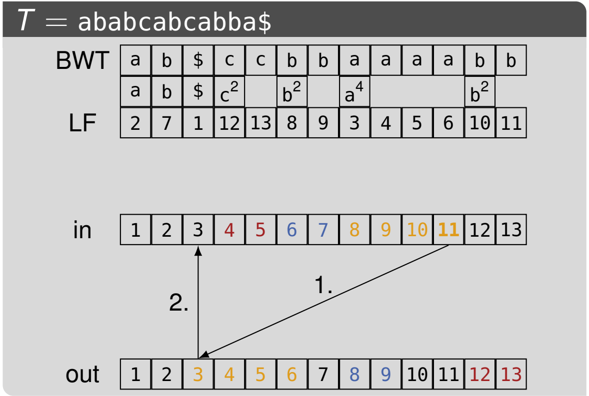

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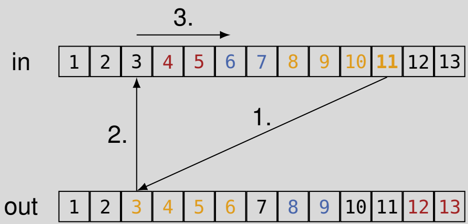

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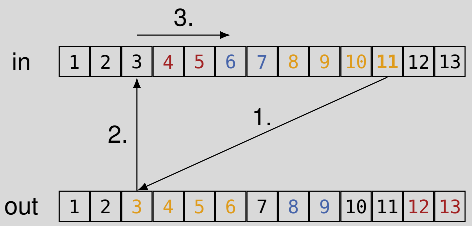

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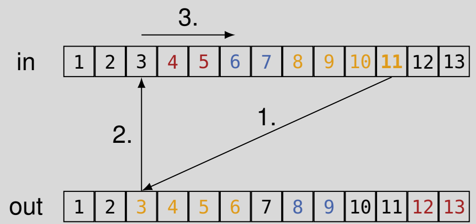

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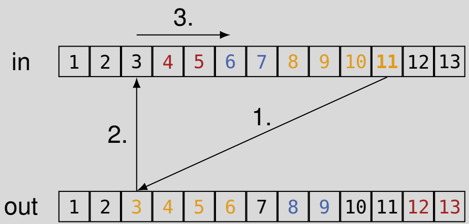

Balance the Move Data Structure (1/2)

Definition: Permutation Graph

- each interval in the input and output sequence is a node
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- resulting graph $G(I)$ has k edges

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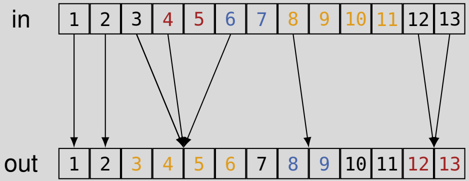

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Lemma: Size of Out-Balanced Sequence

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Proof

- output contains at least k big intervals, therefore $r' \geq 2k$
- $r' = r + k$, therefore $2k \leq r + k$
- this gives us $k \leq r$

Data Structures for Backwards Search

- r' balanced input & output intervals for LF queries
- rank & select data structure build on the BWT
  - rank in $O(\log \log_w \sigma)$ time
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- L_{first}: character of each run
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- current interval is $[b, e]$ for $P[i+1..m]$
- look if $P[i]$ occurs in $[b, e]$
  - $rank(L_{first}, c, j) - rank(L_{first}) \geq 1$
- find $\hat{b}, \hat{e}$ marking first/last occurrence of $P[i]$ in $[b, e]$
  - $\hat{b} = select(L_{first}, c, rank(L_{first}, c, i-1) + 1)$
  - $\hat{e} = select(L_{first}, c, rank(L_{first}, c, j))$
- use move data structure to find new b, e for $P[i..m]$

Φ and Its Inverse

- use Φ^{-1} to compute *occ*s of $SA[b..b+occ-1]$
- $\Phi^{-1}(SA[i]) = SA[i+1]$
- $SA[b..b+occ-1] =$
  $SA[b], \Phi^{-1}(SA[b]), \Phi^{-1}(\Phi^{-1}(SA[b])),$
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$T =$ ababcabcabba\$

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SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Φ^{-1}	9	10	11	8	13	3	4	5	6	7	2	1	12
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- Φ^{-1} can be represented by r input & output intervals [GNP20]
- use move data structure on balanced intervals
- keep track of $SA[b]$

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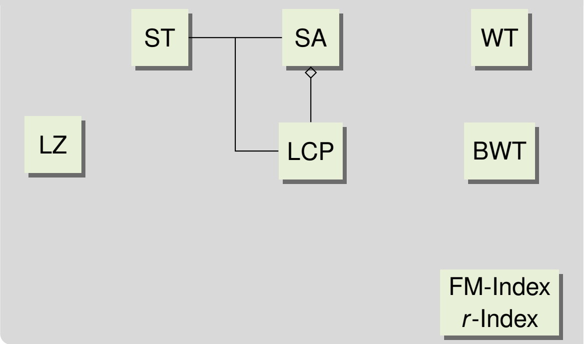
Conclusion and Outlook

This Lecture

- move data structure
- optimal $O(r)$ space full-text index

Linear Time Construction



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Next Lecture

- r vs. z

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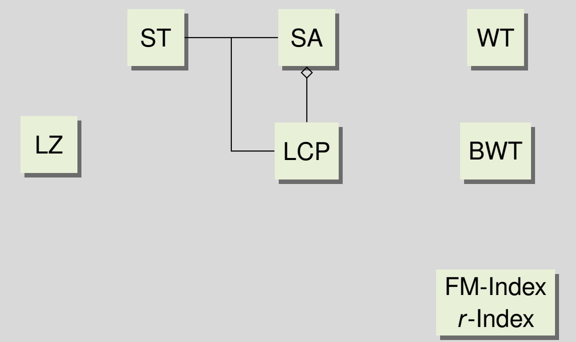

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