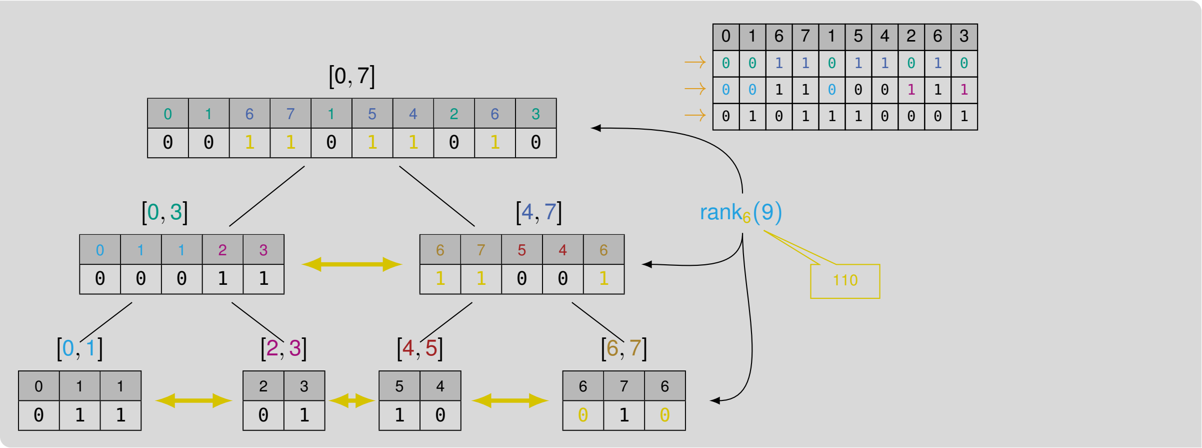
Karlsruhe Institute of Technology

Text Indexing

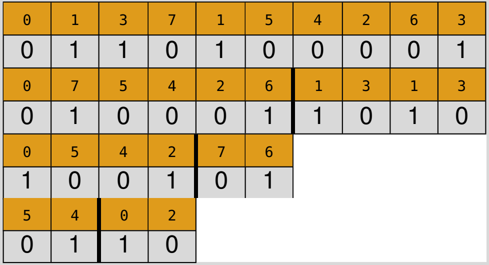
Lecture 08: FM-Index and *r*-Index

Florian Kurpicz

Recap: Wavelet Trees



Recap: Compressed Wavelet Trees



- intervals are only missing to the right (white space)
- no holes allow for easy querying

- build wavelet tree for compressed text
- compress text using bit-wise negated canonical Huffman-codes

- can a wavelet tree be compressed further?

Bit Vector Compression (1/2)

- compress (sparse) bit vectors
- bit vector contains k one bits
- use $O(k \lg \frac{n}{k}) + o(n)$ bits
- retrieve $\Theta(\lg n)$ bits at the same time
- similar to *rank* data structure

- split bit vector into (super-)blocks
- blocks of size $s = \frac{\lg n}{2}$
- super-blocks of size $s' = s^2$

Array C

- number of ones in i-th block

Lookup-Tables L_i

- for $i \in [0, s]$ store lookup-table containing all bit vectors with i one bits

- use variable-length codes to identify content of block
- concatenate all codes in bit vector V

Bit Vector V

- let k_i be number of ones in i-th block
- use $\lceil \lg \binom{s}{k_i} \rceil$ bits to encode block ⓘ position in lookup-table
- concatenate all codes

Bit Vector Compression (2/2)

Karlsruhe Institute of Technology

Array *SBlock*

- for every super-block i, $SBlock[i]$ contains position of encoding of first block in i-th super-block in V
- $\lceil \lg n \rceil$ bits per entry

Array *Block*

- for every block i, $Block[i]$ contains position of encoding of i-th block in V relative to its super-block
- $O(\lg \lg n)$ bits per entry

Lemma: Compressed Bit Vectors

A bit vector of size n containing k ones can be represented using $O(k \lg \frac{n}{k}) + o(n)$ bits allowing $O(1)$ time access to individual bits

Proof (Sketch space requirements)

- $|C| = O(\frac{n}{s} \lg s) = o(n)$ bits
- $|SBlock| = O(\frac{n}{s'} \lg n) = o(n)$ bits
- $|Block| = O(\frac{n}{s} \lg s) = o(n)$ bits
- $\sum_{k=0}^{s} |L_k| \leq (s+1)2^s s = o(n)$ bits
- $|V| = \sum_{i=1}^{\lceil n/s \rceil} \lceil \lg \binom{s}{k_i} \rceil \leq \lg \binom{n}{k} + n/s \leq \lg((n/k)^k) + n/s = k \lg \frac{n}{k} + O(\frac{n}{\lg n})$ bits

Recap: Backwards Search in the BWT

KIT – Karlsruhe Institute of Technology

Function *BackwardsSearch*($P[1..n], C, rank$):

$s = 1, e = n$
for $i = m, \ldots, 1$ **do**
  $s = C[P[i]] + rank_{P[i]}(s - 1) + 1$
  $e = C[P[i]] + rank_{P[i]}(e)$
  if $s > e$ **then**
    return $\emptyset$
return $[s, e]$

- no access to text or *SA* required
- no binary search
- existential queries are easy
- counting queries are easy
- reporting queries require additional information
- example on the board

The FM-Index [FM00]

Building Blocks of FM-Index

- wavelet tree on BWT providing *rank*-function
- C-array
- sampled suffix array with sample rate s
- bit vector marking sampled suffix array positions

Lemma: FM-Index

Given a text T of length n over an alphabet of size σ, the FM-index requires $O(n \lg \sigma)$ bits of space and can answer counting queries in $O(m \lg \sigma)$ time and reporting queries in $O(occ + m \lg \sigma)$ time

Space Requirements

- wavelet tree: $n\lceil \lg \sigma \rceil(1 + o(1))$ bits
- C-array: $\sigma\lceil \lg n \rceil$ bits ℹ $n(1 + o(1))$ bits if $\sigma \geq \frac{n}{\lg n}$
- sampled suffix array: $\frac{n}{s}\lceil \lg n \rceil$ bits
- bit vector: $n(1 + o(1))$ bits

- space and time bounds can be achieved with $s = \lg_\sigma n$

Conclusion FM-Index

- FM-index is easy to compress
- wavelet tree on *BWT* can be compressed
- bit vector can be compressed

- very small in comparison with suffix tree or suffix array
- compression does not make use of structure of *BWT* ℹ wavelet trees are compressed using Huffman-codes

Definition: Run (simplified, recap)

Given a text T of length n, we call its substring $T[i..j]$ a **run**, if

- $T[k] = T[\ell]$ for all $k, \ell \in [i, j]$ and
- $T[i-1] \neq T[i]$ and $T[j+1] \neq T[j]$

ℹ (To be more precise, this is a definition for a run of a periodic substring with smallest period 1, but this is not important for this lecture )

	1	2	3	4	5	6	7	8	9	0	11	12	13
L	a	b	$	c	c	b	b	a	a	a	a	b	b

Measures of Repetitiveness (Excerpt)

order of magnitude in practice

$b < z < g < r < e$

- b: # phrases in bidirectional macro scheme
- z: # phrases in Lempel-Ziv parse
- g: size of smallest context free grammar
- r: # of runs in Burrows-Wheeler transform
- e: # of nodes and edges in CDAWG

a	b	r	a	c	a	d	a	b	r	a

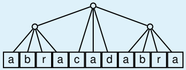
(6,2)r(11,1)cad(1,4)

NP-hard

a	b	r	a	c	a	d	a	b	r	a

abr(1,1)c(1,1)d(1,4)

$O(n)$ time

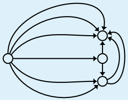


NP-hard*

a	r	d	$	r	c	a	a	a	a	b	b

(1,a)(1,r)(1,d)(1,$)(1,r)(1,c)(4,a)(2,b)

$O(n)$ time



$O(n)$ time

* there are good heuristics

Motivation: r-Index

Measure for Compressibility

- k-th order empirical entropy H_k
- number of LZ factors z
- number of *BWT* runs r

- z and r not blind to repetitions
- how do they relate?

Lemma: *BWT* runs and LZ factors [KK20]

Given a text T of length n. Let z be the number of LZ77 factors and r the number of runs in T's *BWT*, then

$$r \in O(z \lg^2 n)$$

- more details in next lecture

Main Part of Backwards-Search

Function *BackwardsSearch*$(P[1..n], C, rank)$**:**

- $s = 1, e = n$
- **for** $i = m, \ldots, 1$ **do**
  - $s = C[P[i]] + rank_{P[i]}(s - 1) + 1$
  - $e = C[P[i]] + rank_{P[i]}(e)$
  - **if** $s > e$ **then**
    - **return** $\emptyset$
- **return** $[s, e]$

Goals

- simulate *BWT* and *rank* on *BWT* in
- $O(r \lg n)$ bits of space

The r-Index [GNP20] (1/3)

Given a text T of length n over an alphabet Σ and its BWT, the r-index of this text consists of the following data structures

Array I

- $I[i]$ stores position of i-th run in BWT

Array L'

- $L'[i]$ stores character of i-th run in BWT
- build wavelet tree for L'

Array R

- lengths of BWT runs stably sorted by runs' characters
- accumulate for each character by performing exclusive prefix sum over run lengths'

Array C'

- $C'[\alpha]$ stores the start of the run lengths in R for each character $\alpha \in \Sigma$ starting at 0

Bit Vector B

- compressed bit vector of length n containing ones at positions where BWT runs start and rank-support

The r-Index (2/3)

$rank_\alpha(BWT, i)$ with r-Index

- compute number j of run ($j = rank_1(B, i)$)
- compute position k in R ($k = C'[\alpha]$)
- compute number ℓ of α runs before the j-th run ($\ell = rank_\alpha(L', j-1)$)
- compute number k of αs before the j-th run ($k = R[k + \ell]$)
- compute character β of run ($\beta = L'[j]$)
- if $\alpha \neq \beta$ return k ℹ i is not in the run
- else return $k + i - I[j] + 1$ ℹ i is in the run

The r-Index (3/3)

Lemma: Space Requirements r-Index

Given a text T of length n over an alphabet of size σ that has r *BWT* runs, then its r-index requires

$$O(r \lg n) \text{ bits}$$

and can answer *rank*-queries on the *BWT* in $O(\lg \sigma)$. Given a pattern of length m, the r-index can answer pattern matching queries in time

$$O(m \lg \sigma)$$

- what about reporting queries?

Locating Occurrences (Sketch)

- modify backwards-search that it maintains $SA[e]$
- after backwards-search output $SA[e], SA[e-1], \ldots, SA[s]$
- in $O(r \lg n)$ bits and $O(occ \cdot \lg\lg r)$ time

Maintaining $SA[e]$

- sample SA positions at ends of runs
- if next character is $BWT[e]$, then next $SA[e']$ is $SA[e] - 1$
- otherwise locate end of run and extract sample

Output Result

- following LF not possible unbounded
- deduce $SA[i-1]$ from $SA[i]$
- character in L and F in same order
- only beginning of runs complicated
- for every character build predecessor data structure over sampled SA-values at end of runs
- associate with $\langle i, SA[i]\rangle$

Now: OptBWTR

	Time (locate)	Time (count)	Space (words)
r-index [GNP20]	$O(\|P\| \log\log_w(\sigma + n/r) + occ)$ $O(\|P\| + occ)$	$O(\|P\| \log\log_w(\sigma + n/r))$ $O(\|P\|)$	$O(r)$ $O(r \log\log(\sigma + n/r))$
OptBWTR [NT21]	$O(\|P\| \log\log_w \sigma + occ)$	$O(\|P\| \log\log_w \sigma)$	$O(r)$

RLBWT

- partition *BWT* into r substrings
- $BWT = L_1 L_2 \dots L_r$
- L_i is maximal repetition of same character
- $\ell_1 = 1$ and $\ell_i = \ell_{i-1} + |L_{i-1}|$
- $RLBWT = (L_1[1], \ell_1)(L_2[1], \ell_2) \dots (L_r[1], \ell_r)$

- let δ be permutation of $[1, r]$ such that

$$LF(\ell_{\delta[1]}) < LF(\ell_{\delta[2]}) < \cdots < LF(\ell_{\delta[r]})$$

Lemma: LF and RLBWT

- Let $\ell_x < i < \ell_{x+1}$ for some $i \in [1, n]$, then

$$LF(i) = LF(\ell_x) + (i - \ell_x)$$

- $LF(\ell_{\delta[1]}) = 1$ and
  $LF(\ell_{\delta[i]}) = LF(\ell_{\delta[i-1]}) + |L_{\delta[i-1]}|$

$T =$ ababcabcabba$

BWT	a	b	$	c	c	b	b	a	a	a	a	b	b
	a	b	$	c^2		b^2		a^4				b^2	
LF	2	7	1	12	13	8	9	3	4	5	6	10	11

Input and Output Intervals

T = ababcabcabba$

BWT	a	b	$	c	c	b	b	a	a	a	a	b	b
	a	b	$	c^2		b^2		a^4				b^2	
LF	2	7	1	12	13	8	9	3	4	5	6	10	11
in	1	2	3	4	5	6	7	8	9	10	11	12	13
out	1	2	3	4	5	6	7	8	9	10	11	12	13

- there are r intervals
- represent domain of *LF* by intervals

- solve LF without predecessor queries ℹ we did not use predecessor queries
- predecessor queries are bottleneck

Disjoint Interval Sequence & Move Query

Definition: Disjoint Interval Sequence

Let $I = (p_1, q_1), (p_2, q_2), \ldots, (p_k, q_k)$ be a sequence of k pairs of integers. We introduce a permutation π of $[1, k]$ and sequence $d_1, d_2, \ldots, d_k$ for I. π satisfies $q_{\pi[1]} \leq q_{\pi[2]} \leq \cdots \leq q_{\pi[k]}$, and $d_i = p_{i+1} - p_i$ for $i \in [1, k]$, where $p_{k+1} = n + 1$. We call the sequence I a disjoint interval sequence if it satisfies the following three conditions:

- $p_1 = 1 < p_2 < \cdots < p_k \leq n$
- $q_{\pi[1]} = 1$,
- $q_{\pi[i]} = q_{\pi[i-1]} + d_{\pi[i-1]}$ for each $i \in [2, k]$.

$T = \texttt{ababcabcabba\$}$

in	1	2	3	4	5	6	7	8	9	10	11	12	13
out	1	2	3	4	5	6	7	8	9	10	11	12	13

Move Query

$$\text{move}(i, x) = (i', x')$$

- i position in input interval
- x input interval
- i' position in output interval
- x' input interval covering i'

Answering Move Query

Karlsruhe Institute of Technology

- $D_{pair} = (p_i, q_i)$ for every interval
- $D_{index}[i]$ index of input interval containing q_i

example on the board

Lemma: LF and RLBWT

- Let $\ell_x < i < \ell_{x+1}$ for some $i \in [1, n]$, then

$$LF(i) = LF(\ell_x) + (i - \ell_x)$$

- $LF(\ell_{\delta[1]}) = 1$ and
  $LF(\ell_{\delta[i]}) = LF(\ell_{\delta[i-1]}) + |L_{\delta[i-1]}|$

- $Move(i, x) = (i', x')$
  - i position in input sequence
  - x index of interval containing i
- $i' = q_x + (i - p_x)$
- x' initially $D_{index}[x]$
- scan D_{pair} from x' until $p'_x \geq I'$
- x' index satisfying condition

Moving for LF

LF Query

- input: interval containing an integer i
- output: interval containing $LF(i)$

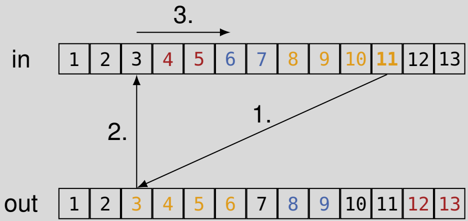
- 1. move to corresponding output interval
- 2. move to input interval containing position j
- 3. linear search on at most four intervals

- worst-case intervals

- balance intervals

$T = \texttt{ababcabcabba\$}$

BWT	a	b	$	c	c	b	b	a	a	a	a	b	b
	a	b	$	c^2		b^2		a^4				b^2	
LF	2	7	1	12	13	8	9	3	4	5	6	10	11



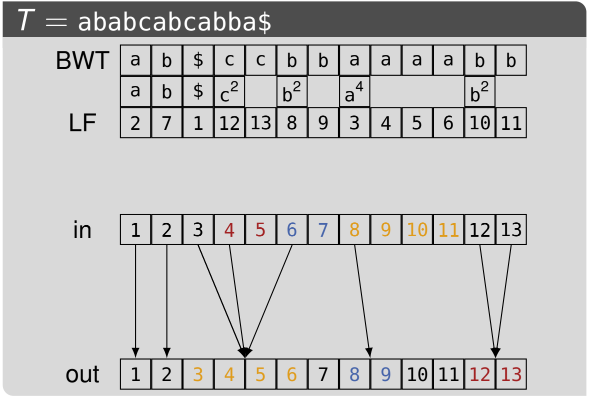
Balance the Move Data Structure (1/2)

Karlsruhe Institute of Technology

Definition: Permutation Graph

- each interval in the input and output sequence is a node
- each input interval $[p_i, p_i + d_i - 1]$ has a single outgoing edge pointing to output interval that contains p_i
- resulting graph $G(I)$ has k edges

- $G(I)$ is out-balanced if each output interval has at most three incoming edges



Balance Move Data Structure (2/2)

- identify intervals with ≥ 5 incoming edges
- split it “equally”
- each new interval covers at least two input intervals

- number r' of balanced input intervals is $k + r$
- k is number of split operations
- r is number of runs in BWT

Lemma: Size of Out-Balanced Sequence

$k \leq r$ and $r' \leq 2r$

Proof

- output contains at least k big intervals, therefore $r' \geq 2k$
- $r' = r + k$, therefore $2k \leq r + k$
- this gives us $k \leq r$

Data Structures for Backwards Search

- r' balanced input & output intervals for LF queries
- rank & select data structure build on the BWT
  - rank in $O(\log\log_w \sigma)$ time
  - select in $O(1)$ time

- $O(r') = O(r)$ space
- $O(|P| \log\log_w \sigma)$ running time

- $F(I_{LF})$: move data structure for LF
- L_{first}: character of each run
- $R(L_{first})$: rank and select support on L_{first}

- current interval is $[b, e]$ for $P[i+1..m]$
- look if $P[i]$ occurs in $[b, e]$
  - $rank(L_{first}, c, j) - rank(L_{first}) \geq 1$
- find $\hat{b}, \hat{e}$ marking first/last occurrence of $P[i]$ in $[b, e]$
  - $\hat{b} = select(L_{first}, c, rank(L_{first}, c, i-1) + 1)$
  - $\hat{e} = select(L_{first}, c, rank(L_{first}, c, j))$
- use move data structure to find new b, e for $P[i..m]$

Φ and Its Inverse

- use Φ^{-1} to compute *occ*s of $SA[b..b+occ-1]$
- $\Phi^{-1}(SA[i]) = SA[i+1]$
- $SA[b..b+occ-1] =$ $SA[b], \Phi^{-1}(SA[b]), \Phi^{-1}(\Phi^{-1}(SA[b])),$ $\Phi^{-1}(\Phi^{-1}(\Phi^{-1}(SA[b]))), \ldots$

- Φ^{-1} can be represented by r input & output intervals [GNP20]
- use move data structure on balanced intervals
- keep track of $SA[b]$

$T =$ ababcabcabba$

BWT	a	b	$	c	c	b	b	a	a	a	a	b	b
	a	b	$	c^2		b^2		a^4				b^2	
LF	2	7	1	12	13	8	9	3	4	5	6	10	11
SA	13	12	1	9	6	3	11	2	10	7	4	8	5
Φ^{-1}	9	10	11	8	13	3	4	5	6	7	2	1	12
in	1	2	3	4	5	6	7	8	9	10	11	12	13
out	1	2	3	4	5	6	7	8	9	10	11	12	13

Conclusion and Outlook

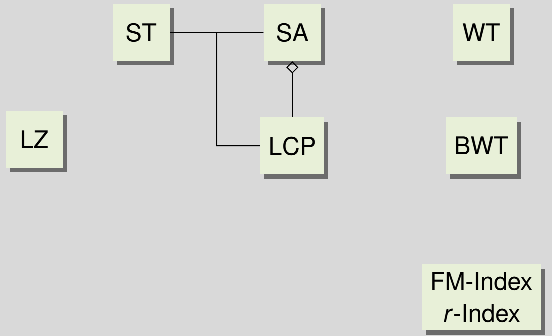
This Lecture

- move data structure
- optimal $O(r)$ space full-text index

Next Lecture

- r vs. z

Linear Time Construction



Bibliography I

[FM00] Paolo Ferragina and Giovanni Manzini. “Opportunistic Data Structures with Applications”. In: *FOCS*. IEEE Computer Society, 2000, pages 390–398. DOI: 10.1109/SFCS.2000.892127.

[GNP20] Travis Gagie, Gonzalo Navarro, and Nicola Prezza. “Fully Functional Suffix Trees and Optimal Text Searching in BWT-Runs Bounded Space”. In: *J. ACM* 67.1 (2020), 2:1–2:54. DOI: 10.1145/3375890.

[KK20] Dominik Kempa and Tomasz Kociumaka. “Resolution of the Burrows-Wheeler Transform Conjecture”. In: *FOCS*. IEEE, 2020, pages 1002–1013. DOI: 10.1109/FOCS46700.2020.00097.

[NT21] Takaaki Nishimoto and Yasuo Tabei. “Optimal-Time Queries on BWT-Runs Compressed Indexes”. In: *ICALP*. Volume 198. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021, 101:1–101:15. DOI: 10.4230/LIPIcs.ICALP.2021.101.