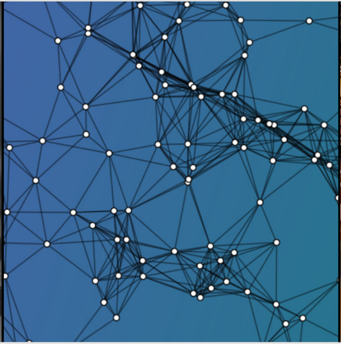
Probability and Computing – Concentration Bounds

Stefan Walzer | WS 2024/2025

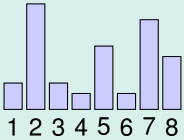




Content

How to describe the shape of a real-valued distribution?

Karlsruhe Institute of Technology

In general: Cannot be summarise in a few numbers



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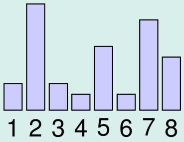

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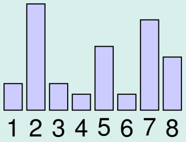

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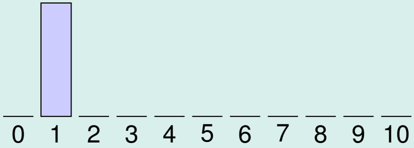

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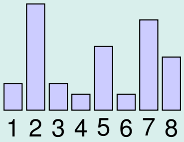

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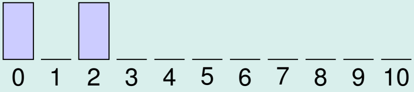

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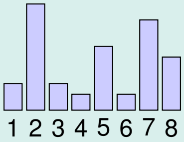

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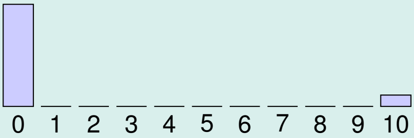

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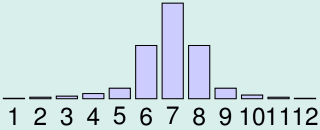
Many distributions: Unimodal, i.e. one “bump”



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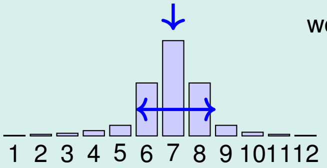

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- where is the bump?
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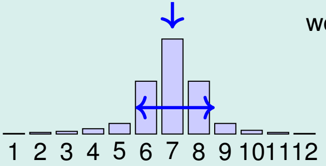

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Concentration Bound

Statement of the form:

$$\Pr[X \notin [a, b]] \le \varepsilon.$$

Bound is strong if $b - a$ and ε are small.

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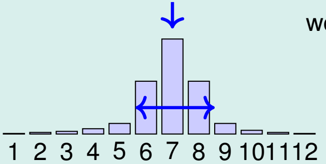

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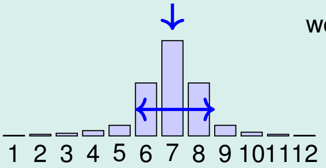

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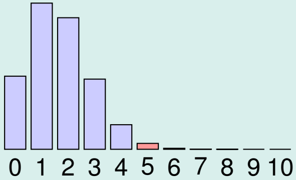
Special Cases

Concentration around Expectation: $\Pr[|X - \mathbb{E}[X]| > c] \leq \varepsilon$ $([a, b] = [\mathbb{E}[X] - c, \mathbb{E}[X] + c])$

Tail bound: $\Pr[X > b] \leq \varepsilon$ $(a = -\infty)$

Benchmark for What Follows

Running Example



Let $X \sim Bin(10, \frac{1}{6})$ the number of sixes when rolling a fair die 10 times.

$$\Pr[X \geq 5] \approx 0.016.$$

- 0.016 is calculated explicitly // only feasible for tiny examples
- later: we use general methods for obtaining bounds on $\Pr[X \geq 5]$

Markov: Tail bound for non-negative random variables

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Markov Inequality

Let X be a non-negative random variable and $b \geq 0$. Then $\Pr[X \geq b] \leq \mathbb{E}[X]/b$.

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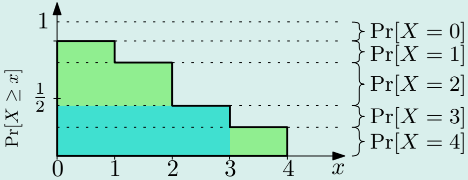
Proof by Picture



From the picture ($b = 3$):
$b \cdot \Pr[X \geq b] \leq \mathbb{E}[X]$

Rearranging gives:
$\Pr[X \geq b] \leq \mathbb{E}[X]/b$.

Note: Tight if and only if $\Pr[X \in \{0, b\}] = 1$.

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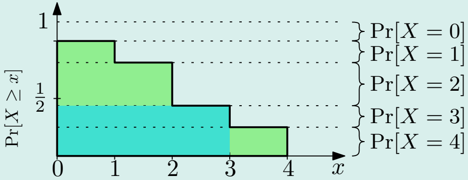

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Actual Proof

$$\mathbb{E}[X] \overset{\text{LTE}}{=} \underbrace{\mathbb{E}[X \mid X \geq b]}_{\geq b} \cdot \Pr[X \geq b] + \underbrace{\mathbb{E}[X \mid X < b]}_{\geq 0} \cdot \Pr[X < b] \geq b \cdot \Pr[X \geq b]. \quad \square$$

Markov's Inequality: Benchmark

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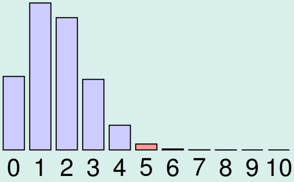
$\Pr[X \geq b] \leq \mathbb{E}[X]/b.$

Bound from Markov Inequality

$$\Pr[X \geq 5] \leq \frac{\mathbb{E}[X]}{5} = \frac{10/6}{5} \approx 0.333.$$

Running Example



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Let X be a non-negative random variable and $b \geq 0$. Then $\Pr[X \geq b] \leq \mathbb{E}[X]/b$.

Corollary for non-negative X and strictly increasing $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$\Pr[X \geq b] = \Pr[f(X) \geq f(b)] \leq \mathbb{E}[f(X)]/f(b)$.

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Corollary for possibly negative X

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Combining both Ideas with $f(x) = x^2$

$\Pr[|X - \mathbb{E}X| \geq b] = \Pr[|X - \mathbb{E}X|^2 \geq b^2] \leq \mathbb{E}[|X - \mathbb{E}X|^2]/b^2 = Var(X)/b^2$.

Chebyshev's inequality ~~Corollaries to Markov's Inequality~~

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Moments[1]

Definitions

Let X be real-valued random variable and $n \in \mathbb{N}$. Then

$\mathbb{E}[X^n]$ is the nth **(raw) moment**, $\mathbb{E}[(X - \mathbb{E}[X])^n]$ is the nth **central moment**,

$\mathbb{E}[|X|^n]$ is the nth absolute[a] moment, $\mathbb{E}[|X - \mathbb{E}[X]|^n]$ is the nth absolute central moment.

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What's so special about the second central moment $Var(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$?

Intuitive Meaning

It's the mean squared distance.

Exercise: Nice mathematical properties

If X, Y are independent, then
$Var(aX + bY) = a^2 \cdot Var(X) + b^2 \cdot Var(Y)$.

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Chebyshev's Inequality: Benchmark

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$\Pr[|X - \mathbb{E}X| \geq b] \leq Var(X)/b^2.$

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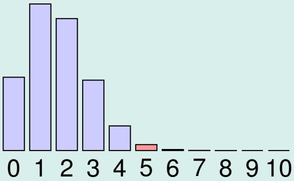

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- let $X_i = [i\text{th roll is a six}]$, $X = \sum_{i=1}^{10} X_i$
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- $Var(X) = 10 \cdot Var(X_1) = \frac{50}{36}$.

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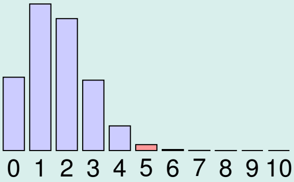

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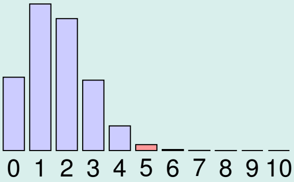

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$\Pr[X \geq 5] = \Pr[X - \mathbb{E}[X] \geq 5 - \frac{5}{3} = \frac{10}{3}] \leq \Pr[|X - \mathbb{E}[X]| \geq \frac{10}{3}] \leq Var(X)/(\frac{10}{3})^2 = \frac{50 \cdot 9}{36 \cdot 100} = \frac{1}{8} = 0.125.$

Moment Generating Function

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$$M_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}\Big[\sum_{i=0}^{\infty} \frac{(tX)^i}{i!}\Big] = \sum_{i=0}^{\infty} \frac{t^i}{i!} \cdot \mathbb{E}[X^i].$$ // generating function[a] for $(\mathbb{E}[X^0], \mathbb{E}[X^1], \mathbb{E}[X^2], \dots)$

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Chernoff Bound

Let X be a real-valued RV and $b \geq 0$. Then

- $\Pr[X \geq b] \leq \inf_{t>0} \mathbb{E}[e^{tX}]/e^{tb}$ and
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Proof of first variant (second works simlarly)

Let $t > 0$ be arbitrary. Then $x \mapsto e^{tx}$ is increasing.

$$\Pr[X \geq b] = \Pr[e^{tX} \geq e^{tb}] \overset{\text{Markov}}{\leq} \mathbb{E}[e^{tX}]/e^{tb}.$$

Done, since $t > 0$ was arbitrary.

Generic Chernoff Bound: Benchmark

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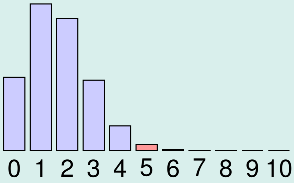

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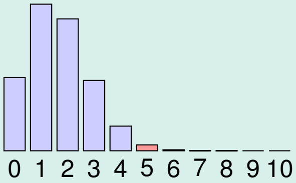

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Note: For independent $X_1, \ldots, X_n$

$\mathbb{E}[e^{t(X_1+\cdots+X_n)}] = \mathbb{E}[e^{tX_1} \cdot \ldots \cdot e^{tX_n}] = \mathbb{E}[e^{tX_1}] \cdot \ldots \cdot \mathbb{E}[e^{tX_n}].$

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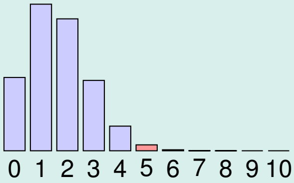
Running Example



Let $X \sim Bin(10, \frac{1}{6})$ the number of sixes when rolling a fair die 10 times.

$\Pr[X \geq 5] \approx 0.016.$

Note: For independent $X_1, \ldots, X_n$

$\mathbb{E}[e^{t(X_1+\cdots+X_n)}] = \mathbb{E}[e^{tX_1} \cdot \ldots \cdot e^{tX_n}] = \mathbb{E}[e^{tX_1}] \cdot \ldots \cdot \mathbb{E}[e^{tX_n}].$

Resulting Chernoff Bound

$$\Pr[X \geq 5] \leq \inf_{t>0} \mathbb{E}[e^{tX}]/e^{5t} = \inf_{t>0} \left(\frac{e^t+5}{6}\right)^{10} / e^{5t} \overset{\text{Wolfram Alpha}}{\approx} 0.053. \quad \text{// with } t = \ln(5)$$

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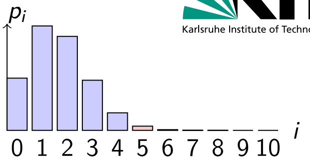

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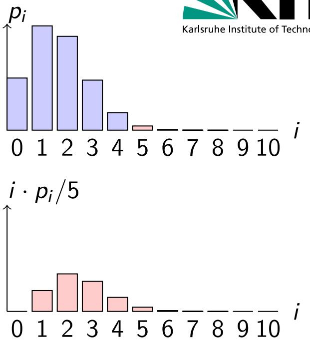

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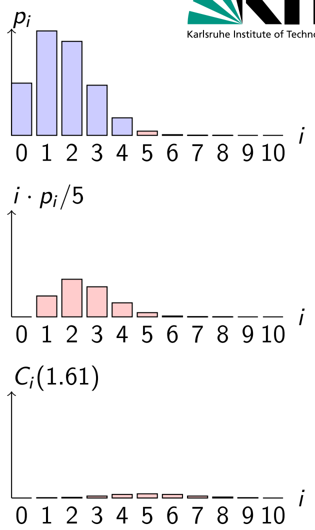

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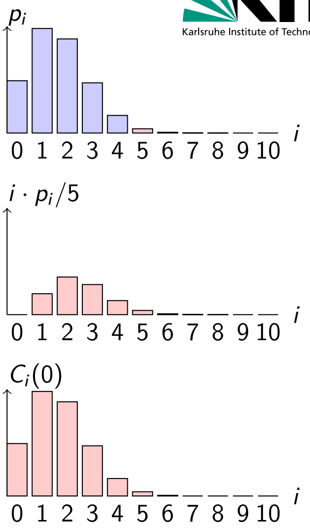

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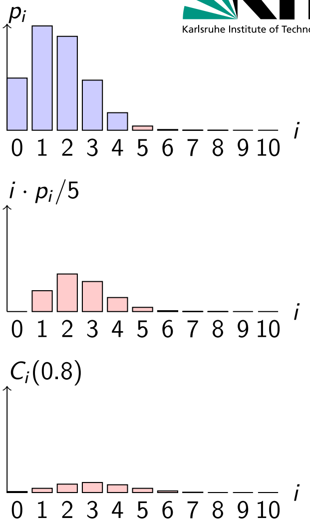

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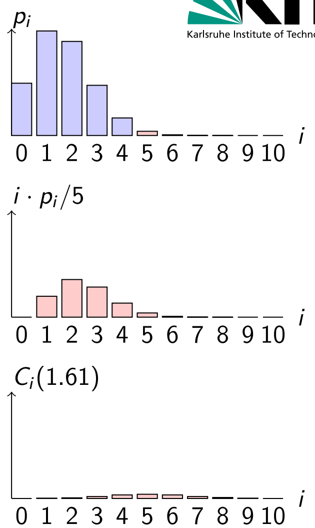

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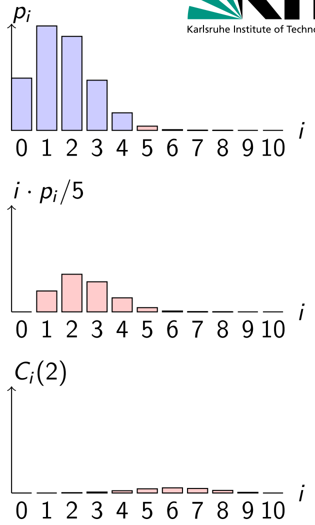

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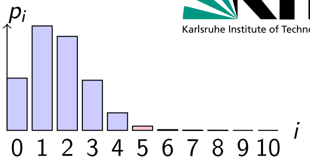
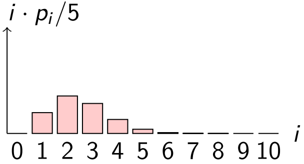
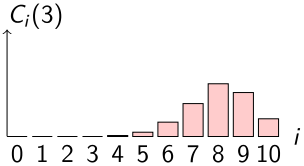
Remark: Finding the best t numerically is easy

$C_i''(t) > 0$ for all t, so $t \mapsto \mathbb{E}[e^{tX}]/e^{tb}$ is convex.







The Chernoff Bound for *your* Favourite Random Variable

Maybe someone already did the work...?

- The general Chernoff bound is hard to use
  - How to compute $\mathbb{E}[e^{tX}]$?
  - How to choose t?
- Many variants for special cases exist.

Multiplicative Form for Sums of Bernoulli RV

Theorem

Let $X = X_1 + \cdots + X_n$ with $X_i \sim Ber(p_i)$ and $\mu := \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then for any $\delta > 0$:

$$\Pr[X \geq (1+\delta)\mu] \leq \Bigg(\underbrace{\frac{e^{\delta}}{(1+\delta)^{1+\delta}}}_{f(\delta)}\Bigg)^{\mu}.$$

Note: $f(0) = 1$ and f is decreasing on $(0, \infty)$.

Hence for constant $\delta > 0$ the bound is exponentially small in μ.

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- $\Pr[X \geq (1+\delta)\mu] \overset{\text{Chernoff}}{\leq} \frac{\mathbb{E}[e^{tX}]}{e^{(1+\delta)\mu t}} \leq \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}} = \Big(\frac{e^{e^t-1}}{e^{(1+\delta)t}}\Big)^{\mu} \overset{t=\ln(1+\delta)}{=} \Big(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\Big)^{\mu}$.

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Consider $\ln(f(\delta)) = \delta - (1+\delta)\ln(1+\delta) =: g(\delta)$. Note that $g'(\delta) = 1 - \frac{1+\delta}{1+\delta} - \ln(1+\delta) = -\ln(1+\delta) < 0$ for $\delta > 0$.

Hence $g(\delta)$ is decreasing on $\mathbb{R}_{\geq 0}$ and so $f(\delta)$ is also decreasing on $\mathbb{R}_{\geq 0}$. □

Multiplicative Form for Sums of Bernoulli RV (ii)

The function $f(\delta) = \frac{e^\delta}{(1+\delta)^{1+\delta}}$ is still inconvenient to work with...

Theorem (from last slide, slightly generalised)

Let $X = X_1 + \cdots + X_n$ with $X_i \sim \mathit{Ber}(p_i)$ and $\mu := \mathbb{E}[X]$.
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Corollary (in the same setting)

- $\Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\delta^2}{2+\delta}\mu}$ for $\delta \geq 0$,
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See also `https://en.wikipedia.org/wiki/Chernoff_bound`

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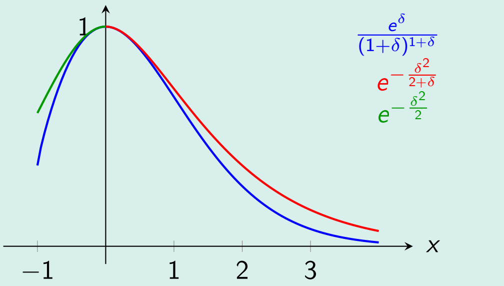
"Proof by Plot"



Simplified Chernoff Bounds: Benchmark

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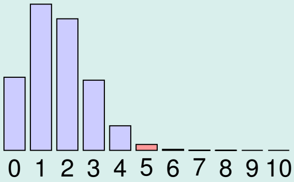
A Simplified Chernoff Bound

$\Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\delta^2}{2+\delta}\mu}$ for $0 \leq \delta$.

Running Example



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Calculations

- $\mu = \mathbb{E}[X] = \frac{10}{6}$
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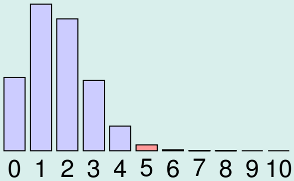

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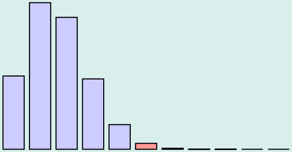

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Resulting Chernoff Bound

$$\Pr[X \geq 5] = \Pr[X \geq (1+2)\mu] \overset{\text{Chernoff}}{\leq} e^{-\frac{2^2}{2+2}\mu} = e^{-\frac{10}{6}} \approx 0.189.$$

How do the methods compare?

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Exercise

When throwing a fair die n times, bound the probability for seeing twice as many sixes as expected...

- ... using Markov,
- ... using Chebyshev,
- ... using Chernoff (simplified).

Compare the results.

Conclusion

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Summary: Methods for Obtaining Concentration Bounds for real-valued X

Method	Assumptions	Formula	Challenges	$\Pr[X \geq 5]$ in Benchmark
—	—	$\Pr[X \geq b] = \sum_{i=b}^{\infty} \Pr[X = i]$	tedious calculation	0.016
Markov	$X \geq 0$	$\Pr[X \geq b] \leq \frac{\mathbb{E}[X]}{b}$	compute $\mathbb{E}[X]$	0.333
Chebyshev	—	$\Pr[\lvert X - \mathbb{E}[X] \rvert \geq b] \leq \frac{Var(X)}{b^2}$	compute $Var(X)$	0.125
Chernoff	—	$\Pr[X \geq b] \leq \inf_{t>0} \frac{\mathbb{E}[e^{tX}]}{e^{tb}}$	compute $\mathbb{E}[e^{tX}]$, choose t	0.053
simpl. Ch.	X is sum of Bernoulli	$\Pr[X \geq (1+\delta)\mathbb{E}[X]] \leq e^{-\frac{\delta^2}{2+\delta}\mathbb{E}[X]}$	compute $\mathbb{E}[X]$	0.189

Further Chernoff-flavoured concentration bounds (X aggregates independent $(X_i)_{i \in \mathbb{N}}$)

Hoeffding's inequality, McDiarmid's inequality, Bernstein inequalities.

Anhang: Mögliche Prüfungsfragen I

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- Was versteht man unter einen Konzentrationsschranke?
- Was besagt die Markov-Ungleichung / Tschebyscheffsche Ungleichung / Chernoff-Ungleichung?
  - Was sind die jeweiligen Vorraussetzungen?
  - Wie schwierig sind die Schranken anzuwenden?
  - Wie gut oder schlecht sind die resultierenden Konzentrationsschranken im Vergleich?
- Detailfragen:
  - Beweise die Markov-Ungleichung.
  - Beweise die Tschebyscheffsche Ungleichung (aus der Markov-Ungleichung).
    - Bieten sich statt des zweiten auch höhere Momente zur Ableitung einer Ungleichung an?
  - Beweise die Chernoff-Ungleichung (aus der Markov-Ungleichung).
  - Was ist die momenterzeugende Funktion und warum heißt sie so?
  - Wie konnten wir $\mathbb{E}[e^{tX}]$ beschränken, als X Summe von Bernoulli Zufallsvariablen war?