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Probability and Computing – Classic Hash Tables

Stefan Walzer | WS 2024/2025

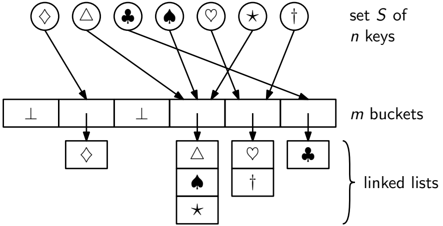
Content

Hash Table with Chaining

e.g. `std::unordered_set`, `java.util.HashMap`

Terminology

D:	*Universe* (or domain) of *keys* (strings, integers, game states in chess)
$S \subseteq D$:	set of n keys (possibly with associated data)
$h : D \to R$:	hash function, range usually $R = [m]$
$\alpha = \frac{n}{m}$:	*load factor*, $\alpha = \mathcal{O}(1)$



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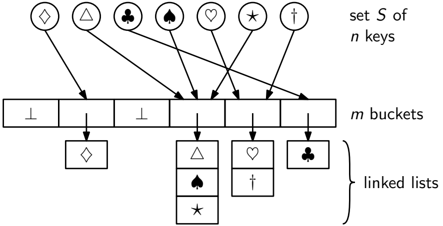

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Operations in time t with $\mathbb{E}[t] = \mathcal{O}(1)$.
Randomness comes from the hash function.

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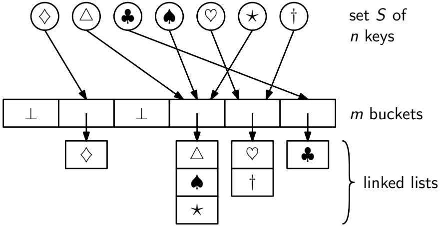

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Ideal Hash Functions

Every function from D to R is equally likely to be h.

Ideal Hash Functions are Impractical

Naive Idea

- Let R^D denote all functions from D to R. We pick $h \sim \mathcal{U}(R^D)$.
- There are $|R|$ options for the hash of each $x \in D$
- Hence: $|R^D| = |R|^{|D|}$

$x \in D$	x_1	x_2	x_3	$\dots$	$x_{\|D\|}$
$h(x) \in R$	?	?	?	$\dots$	?

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Why $h \sim \mathcal{U}(R^D)$ is desirable

- $h \sim \mathcal{U}(R^D) \Leftrightarrow \forall x_1, \dots, x_n \in D : h(x_1), h(x_2), \dots, h(x_n)$ are *independent* and uniformly random in R.
  $\hookrightarrow$ independence is very useful in an analysis
- In particular: $\forall x_1, \dots, x_n \in D, \forall i_1, \dots, i_n : \Pr_{h \sim \mathcal{U}(R^D)}[h(x_1) = i_1 \wedge \dots \wedge h(x_n) = i_n] = |R|^{-n}$.

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Why $h \sim \mathcal{U}(R^D)$ is unwieldy

$\log_2(|R|^{|D|}) = |D| \cdot \log_2(|R|)$ bits to store $h \sim \mathcal{U}(R^D)$ $\rightsquigarrow$ for $D = \{0, 1\}^{64}$: more than 2^{64} bits.

Content

What is a Hash Function?
(it depends on who you ask)

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Cryptographic Hash Function

A **collision resistant** function such as $h =$ `sha256sum`

```
$ sha256sum myfile.txt
018a7eaee8a...3e79043e21ab4  myfile.txt
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Range $R = \{0,1\}^{256}$. It is hard to find x, y with $h(x) = h(y)$.
$\hookrightarrow$ Files with equal hashes are likely the same.

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Cryptographic Pseudorandom Function

A function f : Seeds $\times D \to R$ where $\log_2 |\text{Seeds}|$ is small and no efficient algorithm can distinguish

- $f(s, \cdot)$ for $s \sim \mathcal{U}(\text{Seeds})$ and
- $h(\cdot)$ for $h \sim \mathcal{U}(R^D)$,

except with negligible probability.

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Hash Function in Algorithm Engineering

- typically small range $|R| = \mathcal{O}(n)$
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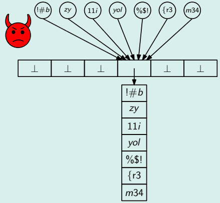
⚠ HashDoS is a thing.



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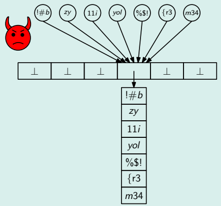

⚠ HashDoS is a thing.
However: Hash function and hash values need not be public.

High-Speed Hashing in Practical Data Structures

Black Magic, do not touch!

MurmurHash

Bitshifts, Magic Constants, ...

```
uint32_t murmur3_32(const uint8_t* key,
            size_t len, uint32_t seed) {
    uint32_t h = seed;
    uint32_t k;
    for (size_t i = len >> 2; i; i--) {
        memcpy(&k, key, sizeof(uint32_t));
        key += sizeof(uint32_t);
        h ^= murmur_32_scramble(k);
        h = (h << 13) | (h >> 19);
        h = h * 5 + 0xe6546b64;
    }
    [...]
    return h;
}
static inline uint32_t murmur_32_scramble(uint32_t k) {
    k *= 0xcc9e2d51;
    k = (k << 15) | (k >> 17);
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Usage

For $R = [m]$, pick seed $\sim \mathcal{U}(\{0,1\}^{32})$ and use

$$h(x) = \texttt{murmur3\_32}(x, \text{seed}) \bmod m.$$

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- **NO,** HashDoS attacks are known.
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- **MAYBE,** for your favourite application.

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What should a Theorist do?

Approach 1: Ignore the Problem

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Simple Uniform Hashing Assumption (SUHA)

- We have access to $h \sim \mathcal{U}(R^D)$ for any R and D.
- h takes $\mathcal{O}(1)$ time to evaluate.
- h takes no space to store.

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3. *Hope* that algorithm still works with real hash functions.

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- *Modelling* assumption.
  $\hookrightarrow$ like e.g. ideal gas law in physics
- Excellent track record in non-adversarial settings.

What should a Theorist do?

Approach 2: Bring your own Hash Functions

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Analyse Algorithm using Universal Hashing

1. Define family $\mathcal{H} \subseteq R^D$ of hash functions with $\log(|\mathcal{H}|)$ not too large.
   $\hookrightarrow$ sampling and storing $h \in \mathcal{H}$ is cheap
2. Proof that algorithm with $h \sim \mathcal{U}(\mathcal{H})$ has good expected behaviour.

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Remarks

- Mathematical structure of $\mathcal{H}$ must be amenable to analysis.
- *Rigorously* covers non-adversarial settings.
- Proofs often difficult.
  $\hookrightarrow$ Wider theory practice gap than with SUHA.

What should a Theorist do?

Approach 3: Let the Cryptographers do the Work

How to Analyse your Algorithm using Cryptographic Assumptions

1. Analyse algorithm under *SUHA*.
2. Actually use *cryptographic pseudorandom function* f.
   - **Case 1:** Everything still works. Great! :-)
   - **Case 2:** Something fails.
     $\Rightarrow$ Your use case can tell the difference between f and true randomness.
     $\hookrightarrow$ The cryptographers said this is impossible. ↯

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Should we use cryptographic pseudorandom functions?

- **YES.** Algorithms become robust even in some adversarial settings.
  $\hookrightarrow$ e.g. Python, Haskell, Ruby, Rust use **SipHash** by default

  `https://en.wikipedia.org/wiki/SipHash`

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   - **Case 1:** Everything still works. Great! :-)
   - **Case 2:** Something fails.
     $\Rightarrow$ Your use case can tell the difference between f and true randomness.
     $\hookrightarrow$ The cryptographers said this is impossible. $\lightning$

Should we use cryptographic pseudorandom functions?

- **YES.** Algorithms become robust even in some adversarial settings.
  $\hookrightarrow$ e.g. Python, Haskell, Ruby, Rust use **SipHash** by default
  `https://en.wikipedia.org/wiki/SipHash`
- **NO.** Too slow in high-performance settings.

Hash Function	MiB / sec
SipHash	944
Murmur3F	7623
xxHash64	12109

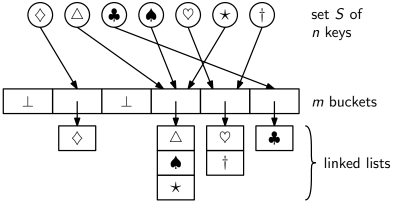
(source: `https://github.com/rurban/smhasher`)

Content

Hash Table with Chaining

Search Time under Chaining

$$\max_{\substack{S \subseteq D \\ |S|=n}} \max_{x \in D} \quad 1 + |\{y \in S \mid h(y) = h(x)\}|$$



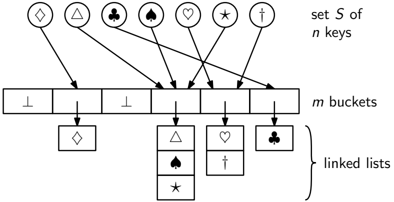
Hash Table with Chaining

Karlsruhe Institute of Technology

Search Time under Chaining

For $n, m \in \mathbb{N}$ and a family $\mathcal{H} \subseteq [m]^D$ of hash functions the *maximum expected search time* is at most

$$T_{\text{chaining}}(n, m, \mathcal{H}) = \max_{\substack{S \subseteq D \\ |S| = n}} \max_{x \in D} \mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})} \Big[1 + |\{y \in S \mid h(y) = h(x)\}| \Big]$$



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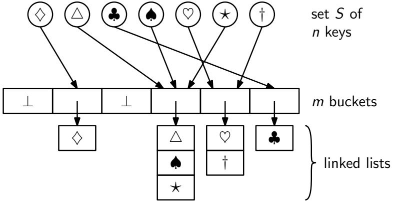
⚠ Key set is *worst case*. Only $h \in \mathcal{H}$ is random. Key set is fixed *before* h is chosen.



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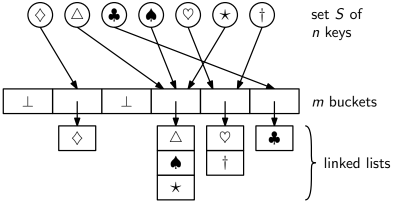

Theorem: Hash Table with Chaining under SUHA

If $\mathcal{H} = [m]^D$ then $T_{\text{chaining}}(n, m, \mathcal{H}) \leq 2 + \alpha = \mathcal{O}(1)$ if $\alpha \in \mathcal{O}(1)$.

Analysis of Hash Table with Chaining under SUHA

Karlsruhe Institute of Technology

Theorem: Hash Table with Chaining under SUHA

Let $\mathcal{H} = [m]^D$, $S \subseteq D$ with $|S| = n$ and $x \in D$ then

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$$\begin{aligned} &\mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})}\Big[1 + |\{y \in S \mid h(y) = h(x)\}|\Big] \\ = &\mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})}\Big[1 + \sum_{y \in S} [h(y) = h(x)]\Big] \end{aligned}$$

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$$= 1 + \sum_{y\in S}\Pr_{h\sim\mathcal{U}(\mathcal{H})}[h(y) = h(x)]$$

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&\leq 1 + 1 + \sum_{y\in S\setminus\{x\}}\Pr_{h\sim\mathcal{U}(\mathcal{H})}[h(y) = h(x)] \\
&= 2 + \sum_{y\in S\setminus\{x\}}\frac{1}{m} \leq 2 + \frac{n}{m} = 2 + \alpha. \quad \square
\end{aligned}$$

Content

A Universal Hash Family

Karlsruhe Institute of Technology

Definition: c-universal hash family

A class $\mathcal{H} \subseteq [m]^D$ is called *c-universal* if: $\forall x \neq y \in D : \Pr_{h \sim \mathcal{U}(\mathcal{H})}[h(x) = h(y)] \leq \frac{c}{m}$.

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Reminder (?): Finite Fields

Let $\mathbb{F}_p = \{0, \ldots, p-1\}$ for a prime number p. Then $(\mathbb{F}_p, \times, \oplus)$ is a field where

$$a \times b := (a \cdot b) \bmod p \qquad \textit{and} \qquad a \oplus b := (a + b) \bmod p.$$

In particular $(\mathbb{F}_p^* := \mathbb{F}_p \setminus \{0\}, \times)$ is a group.

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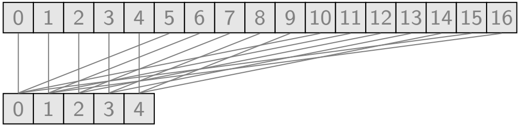
The class of Linear Hash Functions

Assume $D \subseteq \mathbb{F}_p$ for prime p. Then the following class is 1-universal:

$$\mathcal{H}^{\text{lin}}_{p,m} := \{x \mapsto ((a \times x) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}.$$

Can’t we use a simpler class?



$\bmod m$

$p = 17$

$m = 5$

What about just $x \mapsto x \bmod m$?

Nothing is random. We have $h(0) = h(5)$ but this should hold only with probability $1/5$.

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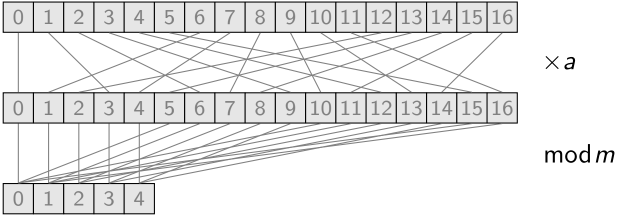
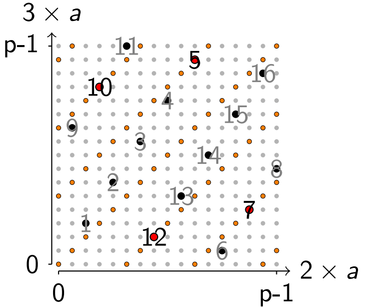


$p = 17$
$m = 5$
$a = 3 \; // \sim \mathcal{U}(\mathbb{F}_p^*)$



What about just $x \mapsto (a \times x) \bmod m$?

- Example: Do 2 and 3 collide? Picture $\{(a \times 2, a \times 3) \mid a \in \mathbb{F}_p^*\}$.
  $\Pr_{a \sim \mathcal{U}(\mathbb{F}_p^*)}[h(2) = h(3)] = \Pr[a \in \{5, 7, 10, 12\}] = \frac{4}{16} > \frac{3}{15} = \frac{1}{5}$.
  $\Rightarrow$ not 1-universal (but 2-universal)
- Also note: $h(0) = 0$ is not random.

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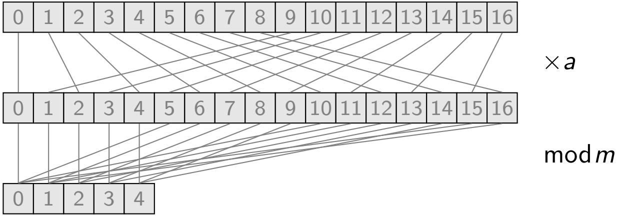

$p = 17$
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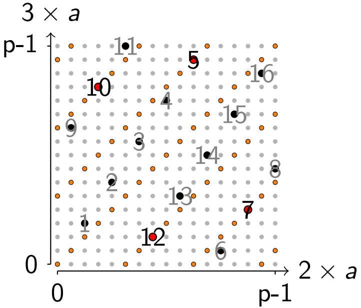
$a = 2 \; // \sim \mathcal{U}(\mathbb{F}_p^*)$



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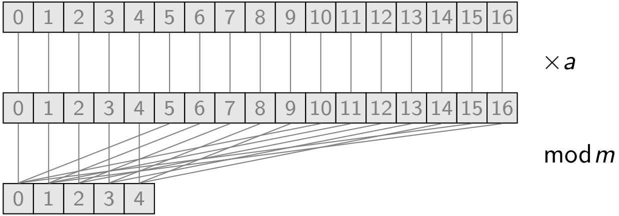

$p = 17$
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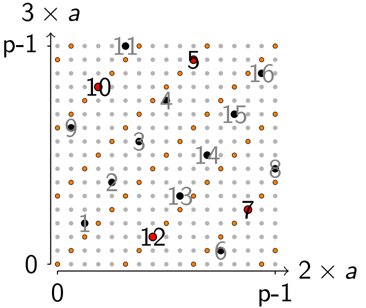
$a = 1 \; // \sim \mathcal{U}(\mathbb{F}_p^*)$



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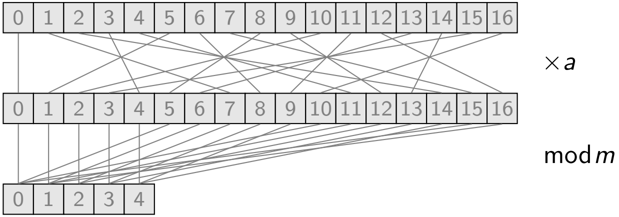
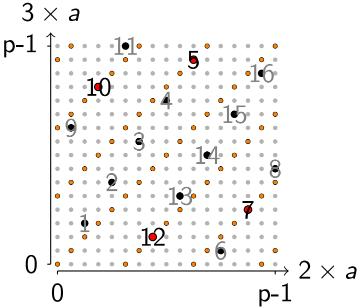


$p = 17$
$m = 5$
$a = 7 \;// \sim \mathcal{U}(\mathbb{F}_p^*)$



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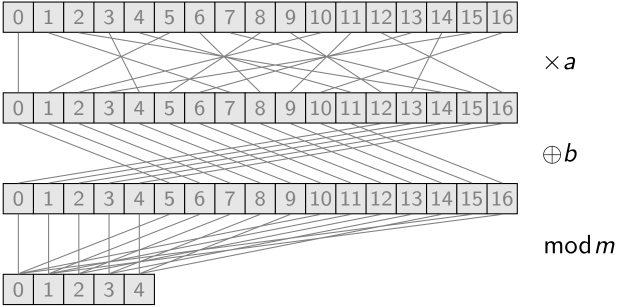
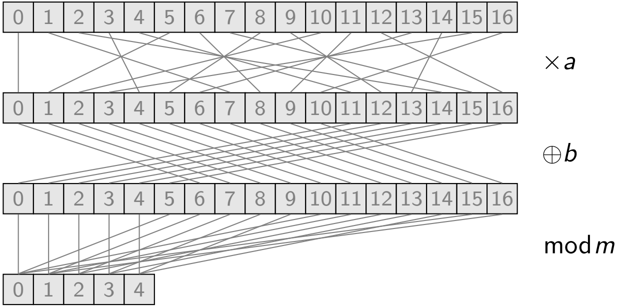


$p = 17$
$m = 5$
$a = 7 \; // \sim \mathcal{U}(\mathbb{F}_p^*)$
$b = 5 \; // \sim \mathcal{U}(\mathbb{F}_p)$

Back to $x \mapsto ((a \times x) \oplus b) \bmod m$

Mathematically "cleaner". Proof of 1-universality on next slide.

Can’t we use a simpler class?



$p = 17$
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$a = 7 \; // \sim \mathcal{U}(\mathbb{F}_p^*)$
$b = 6 \; // \sim \mathcal{U}(\mathbb{F}_p)$

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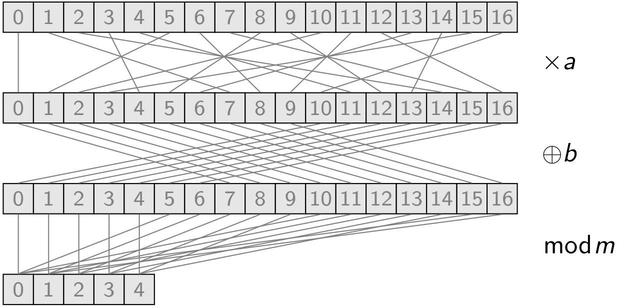

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Proof that $\mathcal{H}^{\text{lin}}_{p,m} := \{x \mapsto ((a \times x) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}$ is 1-universal.

Let $x \neq y \in \mathbb{F}_p$. (To show: $\Pr_{h \sim \mathcal{H}^{\text{lin}}_{p,m}}[h(x) = h(y)] \leq 1/m$.)

Proof that $\mathcal{H}^{\text{lin}}_{p,m} := \{x \mapsto ((a \times x) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}$ is 1-universal.

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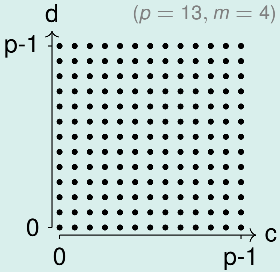
- The mapping $(a, b) \mapsto (c, d)$ is a bijection (for every $x \neq y$) from $\mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p \times \mathbb{F}_p$



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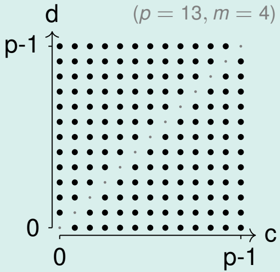
- The mapping $(a, b) \mapsto (c, d)$ is a bijection (for every $x \neq y$) from $\mathbb{F}_p^{*} \times \mathbb{F}_p \to \mathbb{F}_p \times \mathbb{F}_p \setminus \{(b, b) \mid b \in \mathbb{F}_p\}$



$P := \mathbb{F}_p \times \mathbb{F}_p \setminus \{(b, b) \mid b \in \mathbb{F}_p\}$

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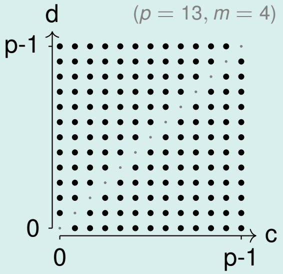
- The mapping $(a, b) \mapsto (c, d)$ is a bijection (for every $x \neq y$) from $\mathbb{F}_p^* \times \mathbb{F}_p \to P$. // $(a, b) \sim \mathcal{U}(\mathbb{F}_p^* \times \mathbb{F}_p) \Rightarrow (c, d) \sim \mathcal{U}(P)$



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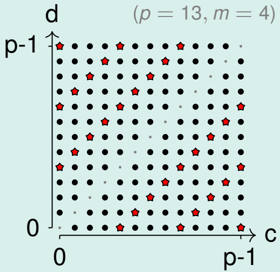

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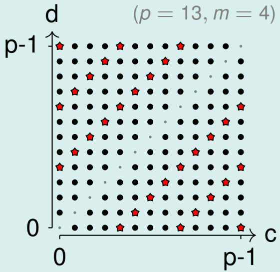

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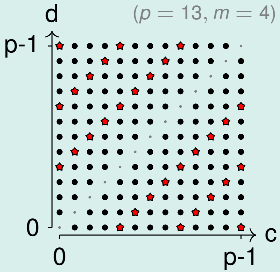

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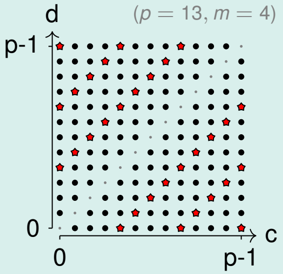

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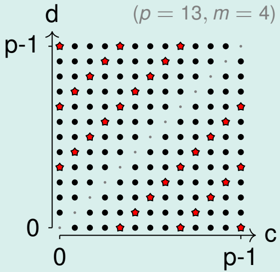

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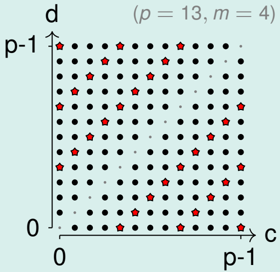

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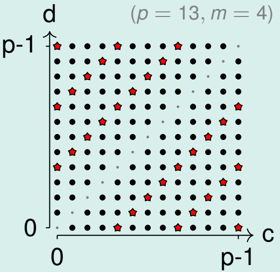

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Analysis of Hash Table with Chaining

. . . using a Universal Hash Family

Karlsruhe Institute of Technology

Theorem

If $\mathcal{H} \subseteq [m]^D$ is a c-universal hash family then $T_{\text{chaining}}(n, m, \mathcal{H}) \leq 2 + c\alpha = \mathcal{O}(1)$ if $\alpha \in \mathcal{O}(1)$ and $c \in \mathcal{O}(1)$.

Proof: Mostly the same.

$$\forall S \subseteq [D], \forall x \in D : \qquad \mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})}\Big[1 + |\{y \in S \mid h(y) = h(x)\}|\Big]$$

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$$= 2 + \sum_{y \in S \setminus \{x\}} \frac{c}{m} \leq 2 + \frac{cn}{m} = 2 + c\alpha. \quad \square$$

More Universal Families

Examples for Universal Hash Families

- "**$((ax + b) \bmod p) \bmod m$**" is 1-universal

as discussed: $D = \mathbb{F}_p, \qquad R = [m],$

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More Universal Families

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- "$(\boldsymbol{ax}$ **mod** $\boldsymbol{p})$ **mod** $\boldsymbol{m}$" is only 2-universal:

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More Universal Families

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- **Multiply-Shift** is 1-universal:

  $$D = \{0, \ldots, 2^w - 1\}, \qquad R = \{0, \ldots, 2^\ell - 1\}$$

  $$\mathcal{H} = \{x \mapsto \lfloor ((a \cdot x + b) \text{ mod } 2^{2w}) / 2^{2w-\ell} \rfloor \mid a, b \in \{0, \ldots, 2^{2w} - 1\}\}.$$

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Selling point of multiply shift:

- "$x \bmod 2^{2w}$" drops some higher order bits
- "$\lfloor x/2^{2w-\ell}\rfloor$" drops some lower order bits
- No division or modulo operation needed!

For $w = 32$ (taken from Thorup 2015):

```
uint32_t hash(uint32_t x, uint32_t l,
              uint64_t a, uint64_t b) {
        return (a * x + b) >> (64-l);
}
```

Content

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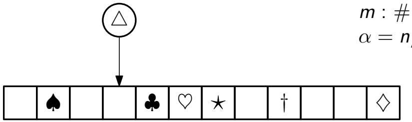
Hash Table with Linear Probing



S : set of n keys
m : # of buckets
$\alpha = n/m$

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	♠		△	♣	♡	⋆		†			♢

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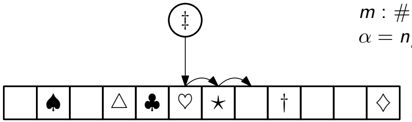

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Operations

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Insert. Put x into first empty bucket.

Hash Table with Linear Probing

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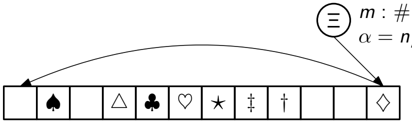

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Ξ	$\spadesuit$		$\triangle$	$\clubsuit$	$\heartsuit$	$\star$	$\ddagger$	$\dagger$			$\diamondsuit$

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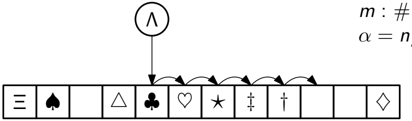

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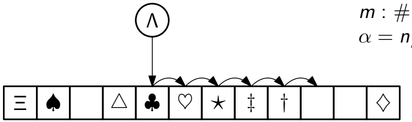

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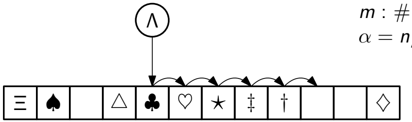

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↪ For details see `https://en.wikipedia.org/wiki/Linear_probing`

[a] Alternative implementations leaves special *tombstones* markers.

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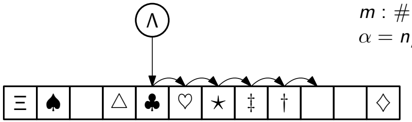

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Running Times

- Lookup($x \in S$): At most x's insertion time.

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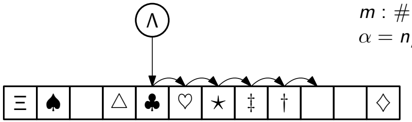

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[a] Alternative implementations leaves special *tombstones* markers.

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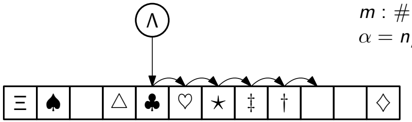
Hash Table with Linear Probing



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m : # of buckets
$\alpha = n/m$

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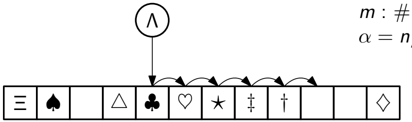

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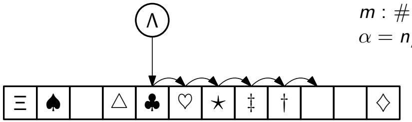

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Theorem: Linear Probing under SUHA

Let $T_{n,m}$ be the random insertion time into a linear probing hash table. If $\frac{1}{2} \le \alpha < 1$ then under SUHA we have

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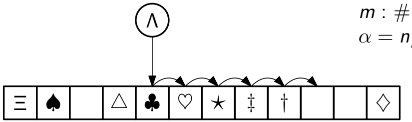

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Content

Preparation: A concentration bound

Chernoff

For $X \sim \mathit{Bin}(n, p)$ and $\delta \in [0, 1]$ we have $\Pr[X \geq (1 + \delta)\mathbb{E}[X]] \leq \exp(-\delta^2 \mathbb{E}[X]/3)$.

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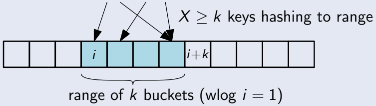
Lemma: $\Pr[\geq k$ hits in segment of length $k]$



Let $k \in \mathbb{N}$ and $X = |\{y \in S \mid h(y) \in \{1,\dots,k\}\}|$.

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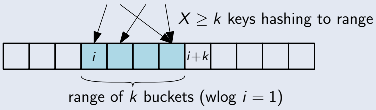

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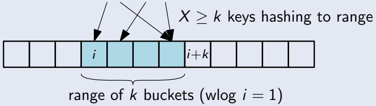

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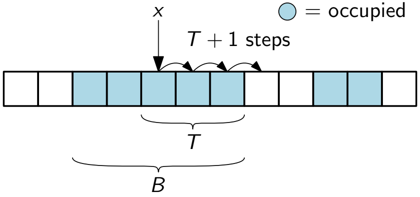
$$\begin{aligned}\Pr[X \geq k] &= \Pr[X \geq \tfrac{1}{\alpha}\mathbb{E}[X]] \\ &= \Pr[X \geq (1 + \tfrac{1-\alpha}{\alpha})\mathbb{E}[X]] \\ &\leq \exp(-(\tfrac{1-\alpha}{\alpha})^2 \alpha k/3) \\ &\leq \exp(-(1-\alpha)^2 k/3).\end{aligned}$$

Proof: Expected LP-Insertion Time under SUHA is $\mathcal{O}(1)$

$\mathbb{E}[T]$



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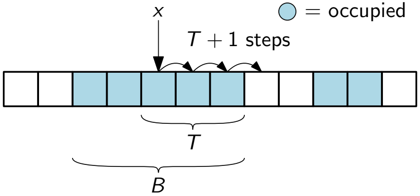

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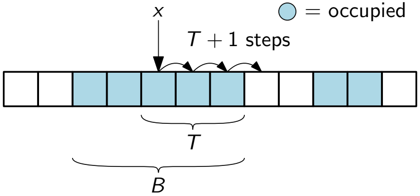

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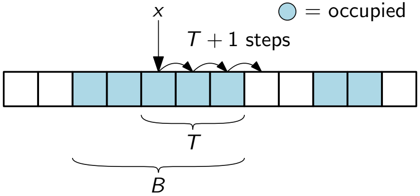

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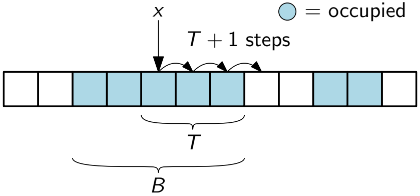

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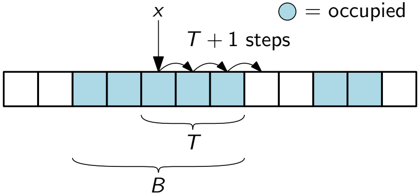

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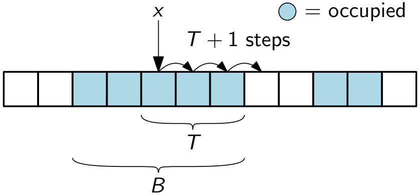

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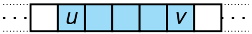

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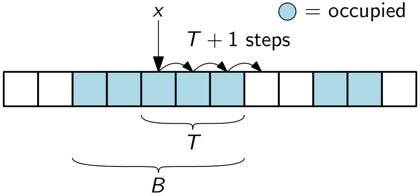

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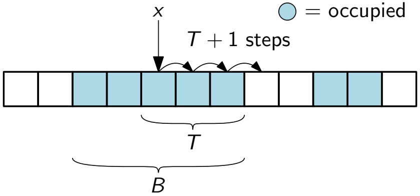
Wolfram Alpha gives: $\int_0^\infty k^2 \exp(-(1-\alpha)^2 k/3) = \frac{54}{(1-\alpha)^6}$.



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Content

Degrees of Independence

(Mutual / Collective) Independence

A family $\mathcal{E}$ of **events** is **independent** if $\forall k \in \mathbb{N}$ and distinct $E_1, \ldots, E_k \in \mathcal{E}$ we have

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d-Independent Hash Family

Definition: d-Independent Hash Family

A family $\mathcal{H} \subseteq R^D$ of hash functions is *d-independent* if for distinct $x_1, \ldots, x_d \in D$ and any $i_1, \ldots, i_d \in R$: (grey is implied by black)

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Karlsruhe Institute of Technology

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Theorem

Let $D = R = \mathbb{F}$ be a finite field. Then

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is a d-independent family.

Note: $\mathcal{H} \subseteq \mathbb{F}^{\mathbb{F}} \rightsquigarrow$ not yet useful.

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Corollary: Smaller Ranges (proof omitted)

- If m divides $|\mathbb{F}|$, then adding "mod m" gives a d-independent family $\mathcal{H}' \subseteq [m]^{\mathbb{F}}$.
- If m does not divide $|\mathbb{F}|$, then adding "mod m" gives a family $\mathcal{H}' \subseteq [m]^{\mathbb{F}}$ such that for $h \sim \mathcal{U}(\mathcal{H}')$ the family $(h(x))_{x \in \mathbb{F}}$ is d-independent but only *approximately* uniformly distributed in $[m]$.

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$$\Rightarrow \Pr_{h \sim \mathcal{U}(\mathcal{H})}[\forall j : h(x_j) = i_j] = \Pr_{a_0, \dots, a_{d-1} \sim \mathcal{U}(\mathbb{F})}[\vec{a} = M^{-1} \cdot \vec{i}\,] = |\mathbb{F}|^{-d}. \quad \square$$

Concentration Bound for d-Independent Variables

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(Tricky) Exercise

Let d be even and $X_1, \dots, X_n \sim Ber(p)$ a d-independent family of random variables with $p = \Omega(1/n)$.
Let $X = \sum_{i=1}^{n} X_i$. Then for any $\delta > 0$ we have

$$\Pr[X - \mathbb{E}[X] \geq \delta \mathbb{E}[X]] = \mathcal{O}(\delta^{-d} \mathbb{E}[X]^{-d/2}).$$

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Remark: Weaker than Chernoff, stronger than Chebyshev

- Chebycheff gives $\Pr[X - \mathbb{E}[X] \geq \delta\mathbb{E}[X]] \leq \frac{1-p}{\delta^2\mathbb{E}[X]}$. (requires $d = 2$)
  $\hookrightarrow$ uses that $Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n)$ for pairwise independent $X_1, \dots, X_n$.
- Chernoff gave $\Pr[X - \mathbb{E}[X] \geq \delta\mathbb{E}[X]] \leq \exp(-\delta^2\mathbb{E}[X]/3)$. (requires $d = n$).

Preparation: A Concentration Bound
again for *d*-independence

Lemma (last slide)

For d-independent $X_1, \ldots, X_n \sim Ber(p)$ and $X = \sum_{i \in [n]} X_i$ we have $\Pr[X \geq (1+\delta)\mathbb{E}[X]] = \mathcal{O}(\delta^{-d}\mathbb{E}[X]^{-d/2})$.

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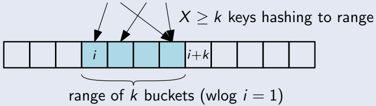
Lemma: $\geq k$ hits in segment of length k



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Let $k \in \mathbb{N}$ and $X = |\{y \in S \mid h(y) \in \{1, \dots, k\}\}|$.

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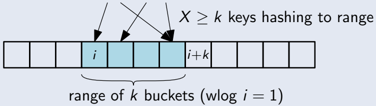

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Let $S = \{x_1, \ldots, x_n\}$ and
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Then $X = \sum_{i \in [n]} X_i$ fits the Lemma with $\mathbb{E}[X] = \frac{kn}{m} = \alpha k$.

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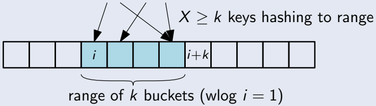

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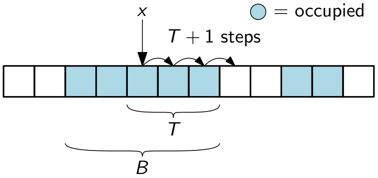
$$\begin{aligned}
\Pr[X \geq k] &= \Pr[X \geq \tfrac{1}{\alpha}\mathbb{E}[X]] \\
&= \Pr[X \geq (1 + \tfrac{1-\alpha}{\alpha})\mathbb{E}[X]] \\
&= \mathcal{O}\Big(\Big(\frac{1-\alpha}{\alpha}\Big)^{-d}(\alpha k)^{-d/2}\Big) \\
&\leq \mathcal{O}((1-\alpha)^{-d} k^{-d/2}). \quad \text{(using } \alpha \leq 1\text{)}
\end{aligned}$$

Theorem: Linear Probing with d-independence

Under the same conditions as before, except with 9-independent hash functions, the insertion time $T_{n,m}$ for linear probing satisfies:

$$\mathbb{E}[T_{n,m}] = \mathcal{O}(1)$$



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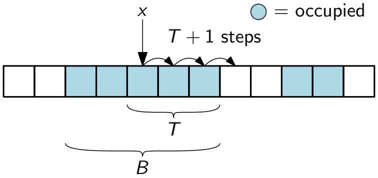
Proof Sketch

$\mathbb{E}[T]$



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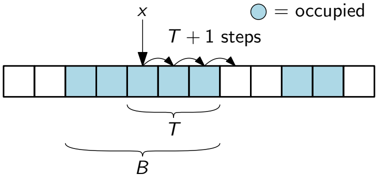

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$$\mathbb{E}[T] \le \mathbb{E}[B]$$

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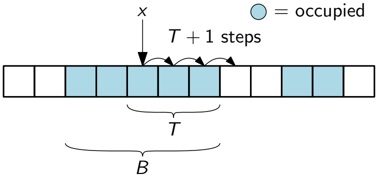

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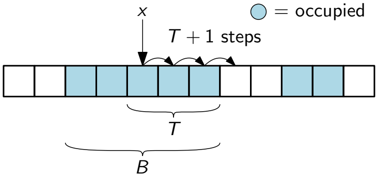

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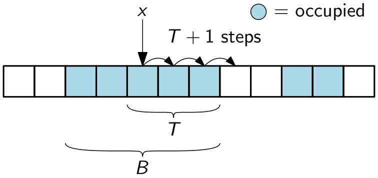

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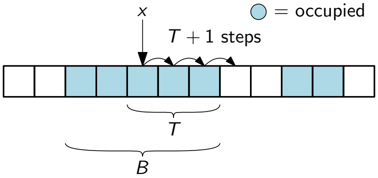
$$\overset{(3)}{=} \frac{\pi^2}{6} \mathcal{O}((1-\alpha)^{-8}) = \mathcal{O}(1). \quad \square$$



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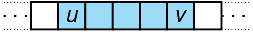

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(3) If interested, see 3Blue1Brown video: `https://www.youtube.com/watch?v=d-o3eB9sfls`

Final Remarks on Linear Probing + Universal Hashing

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Much more is known about insertion times of linear probing:

- Any 5-independent family gives $\mathcal{O}(\frac{1}{(1-\alpha)^2})$.
  $\hookrightarrow$ A. Pagh, R. Pagh, and Ruzic 2011
- An (artificially bad) 4-independent family gives $\Omega(\log n)$.
  $\hookrightarrow$ Pătraşcu and Thorup 2016
- A (well-designed) 4-independent family gives $\mathcal{O}(\frac{1}{(1-\alpha)^2})$.
  $\hookrightarrow$ Puatracscu and Thorup 2013

We Glossed over many Modern Insights

Storing Elements Naively is Inefficient (Cleary 1984)

Example: If $D = [2n]$ and $|S| = n$, then bitvector of $2n$ bits suffices.

$\hookrightarrow$ Much smaller than Array with $\mathcal{O}(n)$ entries of $\log_2 |D|$ bits!

In General: Should aim for $\log_2 \binom{|D|}{n}$ bits rather than $n \cdot \log_2 |D|$.

Tabulation Hashing offers Chernoff-type Concentration Bounds (Pătraşcu and Thorup 2012)

Pairs good practical performance with rigorous mathematical guarantees.

In Practice: Linear Probing can be Supercharged

E.g. using SIMD instructions. *Don't implement high-performance hash tables yourself.*

In Theory: Go Beyond Linear Probing (Bender, Farach-Colton, et al. 2022; Bender, Kuszmaul, and Zhou 2024)

Goals: Improve worst-case access times, improve running time for $\alpha \to 1$.

Conclusion

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Technical Takeaway: Performance of Hash Tables

For both an **ideal hash function** (SUHA) and a random hash function from a suitable **universal class**, a hash table using **linear probing** or **chaining** provably has an expected running time of $\mathcal{O}(1)$ per operation.

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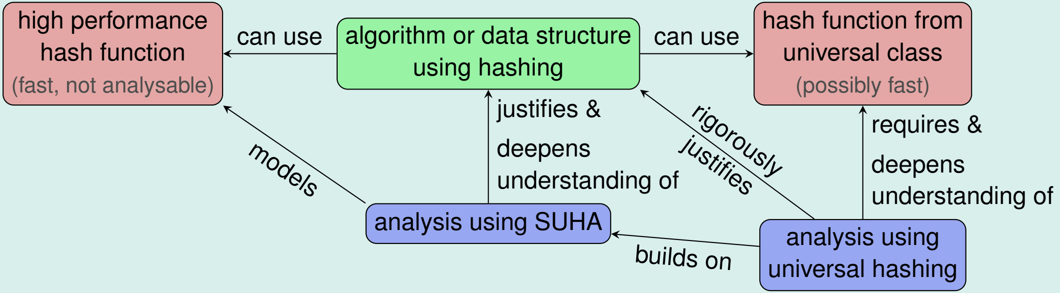

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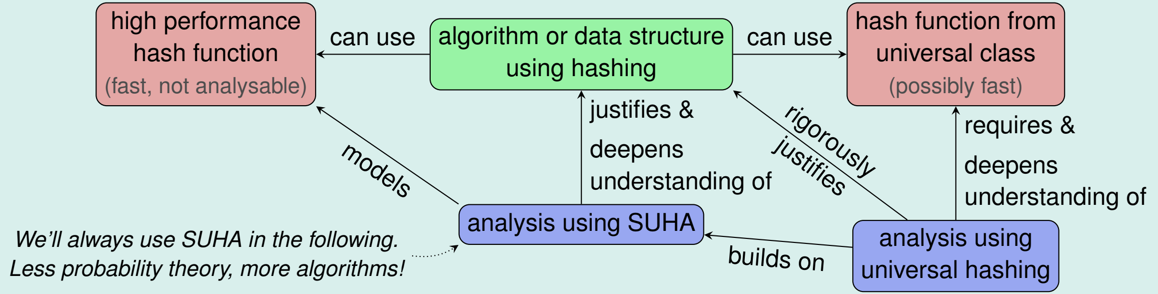

Anhang: Mögliche Prüfungsfragen I

- Was könnte eine Idealvorstellung einer Hashfunktion sein? Inwiefern wäre eine ideale Hashfunktion nützlich? Was ist das Problem an dieser Vorstellung?
- Was ist die Simple Uniform Hashing Assumption (SUHA)? Was spricht dafür diese Annahme zu treffen? Welche Alternativen gibt es?
- Inwiefern ist eine pseudozufällige Funktion mit kryptographischen Ununterscheidbarkeitsgarantien nützlich für uns? Wie ist der Zusammenhang zur SUHA?*
- Universelles Hashing:
  - Wie ist c-Universalität definiert?
  - Welche c-universellen Hashklasse haben wir kennengelernt? Wie haben wir die c-Universalität bewiesen?
  - Wie ist d-Unabhängigkeit für eine Hashklasse definiert?
  - Welche d-universelle Hashklasse haben wir kennengelernt?
  - Welcher Zusammenhang besteht zwischen d-Unabhängigkeit und c-Universalität? (Übungsaufgabe)
  - Chernoff Schranken sind für Summen unabhängiger Zufallsvariablen gedacht. Was kann man machen, wenn die Zufallsvariablen nur d-unabhängig sind?*

Anhang: Mögliche Prüfungsfragen II

- Betrachten wir Hashing mit verketteten Listen:
  - Welche Schranke an die erwartete Einfügezeit haben wir bewiesen? Wie?
  - An welcher Stelle spielt die Verteilung der Hashfunktion eine Rolle?
  - Nenne eine hinreichende Eigenschaft, die eine universelle Hashklasse haben sollte, damit der Beweis funktioniert.
- Betrachten wir Hashing mit linearem Sondieren:
  - Welche Schranke an die erwartete Laufzeit haben wir bewiesen? Wie?
  - An welcher Stelle spielt die Verteilung der Hashfunktion eine Rolle?
  - Nenne eine hinreichende Eigenschaft, die eine universelle Hashklasse haben sollte, damit der Beweis funktioniert.
  - Wie wir diese Eigenschaft ausgenutzt?*

References I

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[1] Michael A. Bender, Martin Farach-Colton, et al. “On the optimal time/space tradeoff for hash tables”. In: *54th STOC*. 2022, pp. 1284–1297. DOI: 10.1145/3519935.3519969.

[2] Michael A. Bender, William Kuszmaul, and Renfei Zhou. “Tight Bounds for Classical Open Addressing”. In: *CoRR* abs/2409.11280 (2024). DOI: 10.48550/ARXIV.2409.11280.

[3] John G. Cleary. “Compact Hash Tables Using Bidirectional Linear Probing”. In: *IEEE Trans. Computers* 33.9 (1984), pp. 828–834. DOI: 10.1109/TC.1984.1676499.

[4] Anna Pagh, Rasmus Pagh, and Milan Ruzic. “Linear Probing with 5-wise Independence”. In: *SIAM Rev.* 53.3 (2011), pp. 547–558. DOI: 10.1137/110827831. URL: https://doi.org/10.1137/110827831.

[5] Mihai Pătraşcu and Mikkel Thorup. “On the *k*-Independence Required by Linear Probing and Minwise Independence”. In: *ACM Trans. Algorithms* 12.1 (2016), 8:1–8:27. DOI: 10.1145/2716317. URL: https://doi.org/10.1145/2716317.

[6] Mihai Pătraşcu and Mikkel Thorup. “The Power of Simple Tabulation Hashing”. In: *J. ACM* (2012). DOI: 10.1145/2220357.2220361.

References II

[7] Mihai Puatracscu and Mikkel Thorup. "Twisted Tabulation Hashing". In: *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2013, New Orleans, Louisiana, USA, January 6-8, 2013*. Ed. by Sanjeev Khanna. SIAM, 2013, pp. 209–228. DOI: 10.1137/1.9781611973105.16. URL: https://doi.org/10.1137/1.9781611973105.16.

[8] Mikkel Thorup. "High Speed Hashing for Integers and Strings". In: *CoRR* abs/1504.06804 (2015). arXiv: 1504.06804. URL: http://arxiv.org/abs/1504.06804.