# Probability and Computing – The Peeling Algorithm

Stefan Walzer | WS 2024/2025

# Content

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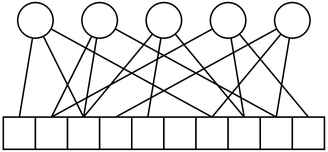
# Cuckoo Hashing with one table and $k$ hash functions



$$\begin{array}{rl} n \in \mathbb{N} & \text{keys} \\ m \in \mathbb{N} & \text{table size} \\ \alpha = \frac{n}{m} & \text{load factor} \\ h_1, \dots, h_k \sim \mathcal{U}([m]^D) & \text{hash functions} \end{array}$$

$\hookrightarrow$ Could also use a separate table per hash function.

## randomWalkInsert($x$)

**while** $x \neq \bot$ **do** // TODO: limit
  sample $i \sim \mathcal{U}([k])$
  swap($x, T[h_i(x)]$)

(some improvements possible)

## Theorem (without proof)

For each $k \in \mathbb{N}$ there is a **threshold** $c_k^*$ such that:

- if $\alpha < c_k^*$ all keys can be placed with probability $1 - \mathcal{O}(\frac{1}{m})$.
- if $\alpha > c_k^*$ **not** all keys can be placed with probability $1 - \mathcal{O}(\frac{1}{m})$.

$c_2^* = \frac{1}{2}, \quad c_3^* \approx 0.92, \quad c_4^* \approx 0.98, \dots$

## Theorem (Bell, Frieze, 2024)

If $k \geq 4$ and $\alpha < c_k^*$ then, conditioned on a high probability event[a], the expected insertion time is $\mathcal{O}(1)$.

[a] Without this conditioning, randomWalkInsert might be trapped in an infinite loop.

# Static Hash Tables

## *Static* Hash Table

`construct`($S$): builds table $T$ with key set $S$

`lookup`($x$): checks if $x$ is in $T$ or not

$\hookrightarrow$ no insertions or deletions after construction!

## Constructing cuckoo hash tables:

- solved by Khosla 2013: “Balls into Bins Made Faster”
- matching algorithm resembling preflow push
- expected running time $\mathcal{O}(n)$, finds placement whenever one exists
- not in this lecture

## *Greedily* constructing cuckoo hash tables

- Peeling: simple algorithm but sophisticated analysis
- interesting applications beyond hash tables (see “retrieval” in next lecture)

# Content

# The Peeling Algorithm

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## constructByPeeling($S \subseteq D,\ h_1, h_2, h_3 \in [m]^D$)

$T \leftarrow [\perp, \ldots, \perp]$ // empty table of size $m$

**while** $\exists i \in [m] : \exists$ *exactly one* $x \in S : i \in \{h_1(x), h_2(x), h_3(x)\}$ **do**

    // $x$ is only unplaced key that may be placed in $i$

    $T[i] \leftarrow x$

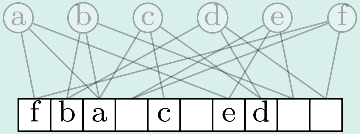
    $S \leftarrow S \setminus \{x\}$

**if** $S = \varnothing$ **then**

    **return** $T$

**else**

    **return** NOT-PEELABLE




## Exercise

- Success of constructByPeeling does not depend on choices for $i$ made by while.
- constructByPeeling can be implemented in linear time.

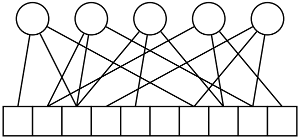
# Peelability and the Cuckoo Graph

## Cuckoo Graph and Peelability

- The **Cuckoo Graph** is the bipartite graph

$$G_{S,h_1,h_2,h_3} = (S, [m], \{(x, h_i(x)) \mid x \in S, i \in [3]\})$$

- Call $G_{S,h_1,h_2,h_3}$ **peelable** if constructByPeeling($S$, $h_1$, $h_2$, $h_3$) succeeds.
- If $h_1, h_2, h_3 \sim \mathcal{U}([m]^D)$ then the distribution of $G_{S,h_1,h_2,h_3}$ does not depend on $S$. We then simply write $G_{m,\alpha m}$.
  - $m$ □-nodes and $\lfloor \alpha m \rfloor$-○-nodes
  - think: $\alpha$ is constant and $m \to \infty$.



## Peeling simplified (not computing placement)

**while** $\exists$ □*-node of degree* 1 **do**
  remove it and its incident ○

$G$ is peelable if and only if
this algorithm removes all ○-nodes.

# Content

# Peeling Theorem

## Peeling Threshold

Let $c_3^\Delta = \min_{y\in[0,1]} \frac{y}{3(1-e^{-y})^2} \approx 0.81$.

## Theorem (today's goal)

Let $\alpha < c_3^\Delta$. Then $\Pr[G_{m,\alpha m}$ is peelable$] = 1 - o(1)$.

## Remark: More is known.

- For "$\alpha < c_3^\Delta$" we get "peelable" with probability $1 - \mathcal{O}(1/m)$.
- For "$\alpha > c_3^\Delta$" we get "not peelable" with probability $1 - \mathcal{O}(1/m)$.
- Corresponding thresholds $c_k^\Delta$ for $k \geq 3$ hash functions are also known.

## Exercise: What about $k = 2$?

Peeling does not reliably work for $k = 2$ for any $\alpha > 0$.

# Peeling Theorem: Proof outline

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## Theorem (today's goal)

Let $\alpha < c_3^\Delta$. Then $\Pr[G_{m,\alpha m} \text{ is peelable}] = 1 - o(1)$.

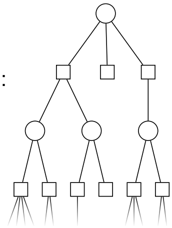
## Proof Idea

The random (possibly) infinite tree $T_\alpha$ can be peeled for $\alpha < c_3^\Delta$ and $T_\alpha$ is locally like $G_{m,\alpha m}$.

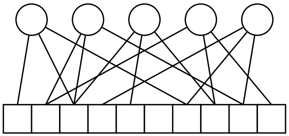
## Steps

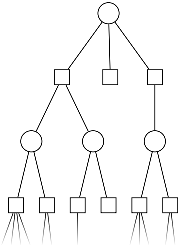
- I What is $T_\alpha$?
- II What does peeling mean in this setting?
- III What role does $c_3^\Delta$ play?
- IV What does it mean for $T_\alpha$ to be locally like $G_{m,\alpha m}$?
- V What is the probability that a fixed key of $G_{m,\alpha m}$ is peeled?
- VI What is the probability that *all* keys of $G_{m,\alpha m}$ are peeled?

$T_\alpha$ :



# i What is $T_\alpha$?





## Observations for the finite Graph $G_{m,\alpha m}$

- each ○ has 3 □ as neighbours (rare exception: $h_1(x), h_2(x), h_3(x)$ not distinct)
- each □ has random number $X$ of ○ as neighbours with $X \sim \text{Bin}(3n, \frac{1}{m}) = \text{Bin}(3\lfloor \alpha m \rfloor, \frac{1}{m})$. In an exercise you'll show

$$\Pr[X = i] \xrightarrow{m \to \infty} \Pr_{Y \sim \text{Pois}(3\alpha)}[Y = i].$$

## **Definition** of the (possibly) infinite random tree $T_\alpha$

- root is ○ and has three □ as children
- each □ has random number of ○ children, sampled Pois$(3\alpha)$ (independently for each □).
- each non-root ○ has two □ as children.

Remark: $T_\alpha$ is finite with positive probability $> 0$, e.g. when the first three Pois$(3\alpha)$ random variables come out as 0. But $T_\alpha$ is also infinite with positive probability.

# ii What does peeling mean in this setting?

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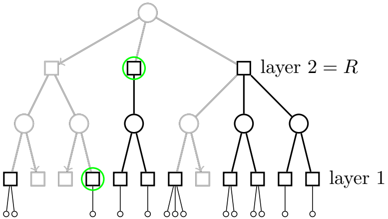
## Peeling Algorithm

**while** $\exists$ *childless* $\square$*-node* **do**
remove it and its incident $\bigcirc$

$\hookrightarrow$ not well defined outcome on $T_\alpha$!
$\hookrightarrow$ but well defined on $T_\alpha^R$!




## Peel only the first $R \in \mathbb{N}$ layers

- Let $T_\alpha^R$ be the first $2R + 1$ levels of $T_\alpha$.
- *R layers* of $\square$-nodes, labeled bottom to top.
- Run peeling on $T_\alpha^R$ (later $R \to \infty$).

$\hookrightarrow$ Why not consider the first $2R$ levels? (without $+1$)

## Only care whether root is removed (root represents arbitrary node in $G_{m,\alpha m}$)

We may then simplify the peeling algorithm.

- replace "$\square$-node of degree 1" condition with stronger "childless $\square$-node".
  - prevents peeling of $\square$-nodes with one child and no parent
  - no matter: such nodes are disconnected from the root anyway
- whether node is peeled only depends on subtree
  $\hookrightarrow$ one bottom up pass suffices for peeling

# What does peeling mean in this setting? (2)

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## Observation

Let $q_R = \Pr[\text{root survives when peeling } T_\alpha^R]$.
The values $q_R$ are decreasing in $R$.

## Peeling Algorithm

**while** $\exists$ *childless* $\square$*-node* **do**
remove it and its incident $\bigcirc$

## Proof.

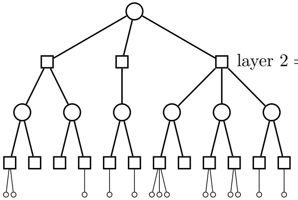
Assume when peeling $T_\alpha^R$ the sequence $\vec{x} = (x_1, \ldots, x_k)$ is a valid sequence of $\square$-node choices. Then $\vec{x}$ is also valid when peeling $T_\alpha^{R+1}$.

$$\text{peeling } T_\alpha^R \text{ removes the root} \Rightarrow \text{peeling } T_\alpha^{R+1} \text{ removes the root}$$

$$\text{root survives when peeling } T_\alpha^{R+1} \Rightarrow \text{peeling } T_\alpha^R \text{ removes the root}$$

$$q_{R+1} \le q_R$$

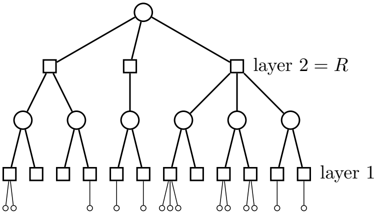
$\square$



# What does peeling mean in this setting? (3)

## Peeling $T_\alpha^R$ bottom up

**for** $i = 1$ **to** $R$ **do** // □-layers bottom to top
  **for** *each* □*-node* $v$ *in layer* $i$ **do**
    **if** $v$ *has no children* **then**
      remove $v$ and its parent ○




## Survival probabilities $p_i :=$ Pr[□-node in layer $i$ is *not* peeled]

$$p_1 = \Pr[\square\text{-node has} \geq 1 \text{ child}] = \Pr_{Y \sim \mathrm{Pois}(3\alpha)}[Y > 0] = 1 - e^{-3\alpha}.$$

$$p_i = \Pr[\text{layer } i\ \square\text{-node } v \text{ has} \geq 1 \textit{ surviving} \text{ child}] = \Pr_{X \sim \mathrm{Pois}(3\alpha p_{i-1}^2)}[X > 0] = 1 - e^{-3\alpha p_{i-1}^2}.$$

$Y :=$ number of (initial) children of $v$
$X :=$ number of surviving children of $v$
each child ○-node survives if both its □-children from layer $i-1$ survive $\rightsquigarrow$ probability $p_{i-1}^2$.
$\Rightarrow Y \sim \mathrm{Pois}(3\alpha)$ and $X \sim \mathrm{Bin}(Y, p_{i-1}^2)$.
$\Rightarrow X \sim \mathrm{Pois}(3\alpha p_{i-1}^2)$. $\rightsquigarrow$ exercise!

# ii What does peeling mean in this setting? (3)

## Peeling $T_\alpha^R$ bottom up

**for** $i = 1$ **to** $R$ **do** // □-layers bottom to top
  **for** *each* □*-node* $v$ *in layer* $i$ **do**
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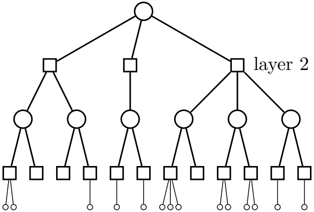



## Survival probabilities $p_i :=$ Pr[□-node in layer $i$ is *not* peeled]

$$p_1 = \Pr[\square\text{-node has} \geq 1 \text{ child}] = \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y > 0] = 1 - e^{-3\alpha}.$$

$$p_i = \Pr[\text{layer } i\ \square\text{-node } v \text{ has} \geq 1 \textit{ surviving} \text{ child}] = \Pr_{X\sim\mathrm{Pois}(3\alpha p_{i-1}^2)}[X > 0] = 1 - e^{-3\alpha p_{i-1}^2}.$$

□-survival probabilities. With $p_0 := 1$ we have

$$p_i = \begin{cases} 1 & \text{if } i = 0 \\ 1 - e^{-3\alpha p_{i-1}^2} & \text{if } i = 1, 2, \ldots \end{cases}$$

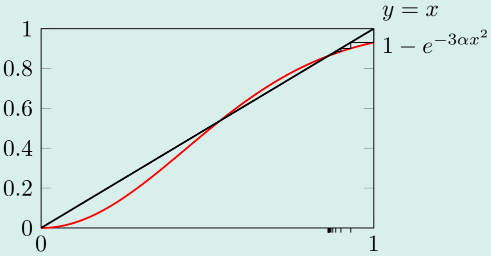
Moreover: $q_R :=$ Pr[root survives] $= p_R^3$.

# iii What role does $c_3^\Delta \approx 0.81$ play?

$$p_i = \begin{cases} 1 & \text{if } i = 0 \\ 1 - e^{-3\alpha p_{i-1}^2} & \text{if } i = 1, 2, \ldots \end{cases}$$

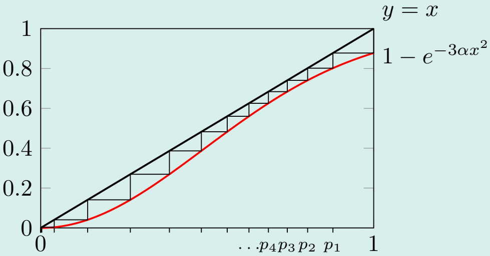
$\hookrightarrow$ consider $f(x) = 1 - e^{-3\alpha x^2}$

## Case 1: $\exists x > 0 : f(x) = x$.




$$\Rightarrow \lim_{i\to\infty} p_i = x^* = \max\{x \in [0,1] \mid f(x) = x\}.$$

## Case 2: $\forall x \in (0,1] : f(x) < x$



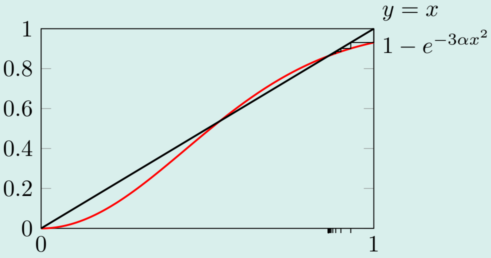

$$\Rightarrow \lim_{i\to\infty} p_i = 0.$$

# $\boxed{\text{iii}}$ What role does $c_3^\Delta \approx 0.81$ play?

$$p_i = \begin{cases} 1 & \text{if } i = 0 \\ 1 - e^{-3\alpha p_{i-1}^2} & \text{if } i = 1, 2, \ldots \end{cases}$$

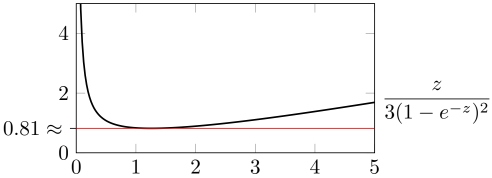
$\hookrightarrow$ consider $f(x) = 1 - e^{-3\alpha x^2}$

**Case 1: $\exists x > 0 : f(x) = x$.**




$$\Rightarrow \lim_{i\to\infty} p_i = x^* = \max\{x \in [0,1] \mid f(x) = x\}.$$

$$\begin{aligned}
\text{Case 1} &\Leftrightarrow \exists x > 0 : x = 1 - e^{-3\alpha x^2} \\
&\Leftrightarrow \exists x > 0 : x^2 = (1 - e^{-3\alpha x^2})^2 \\
&\Leftrightarrow \exists z > 0 : \frac{z}{3\alpha} = (1 - e^{-z})^2 \quad // \ z = 3\alpha x^2 \\
&\Leftrightarrow \exists z > 0 : \alpha = \frac{z}{3(1 - e^{-z})^2} \\
&\Leftrightarrow \alpha \geq \min_{z>0} \frac{z}{3(1 - e^{-z})^2} =: c_3^\Delta \approx 0.81
\end{aligned}$$



# Interim Conclusion: What we learned about peeling $T_\alpha$

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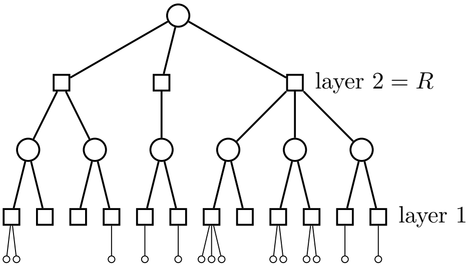
**Lemma**

For $\alpha < c_3^\Delta \approx 0.81$ we have

- $\lim\limits_{i\to\infty} p_i = 0$.
- $\lim\limits_{R\to\infty} q_R = \lim\limits_{R\to\infty} p_R^3 = 0$.

"Root rarely survives for large $R$."



# iv What does it mean for $T_\alpha$ to be locally like $G_{m,\alpha m}$?

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## Neighbourhoods in $T_\alpha$ and $G$

Let $R \in \mathbb{N}$. We consider

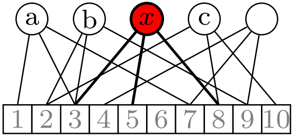
- $T_\alpha^R$ as before and
- for any fixed $x \in S$ the subgraph $G_{m,\alpha m}^{x,R}$ of $G_{m,\alpha m}$ induced by all nodes with distance at most $2R$ from $x$.

## Lemma

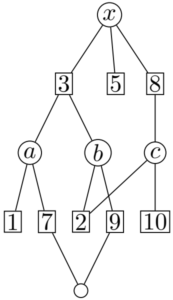
For any $R \in \mathbb{N}$, the **distribution** of $G_{m,\alpha m}^{x,R}$ converges the distribution of $T_\alpha^R$, i.e.

$$\forall T : \lim_{m \to \infty} \Pr[G_{m,\alpha m}^{x,R} = T] = \Pr[T_\alpha^R = T].$$

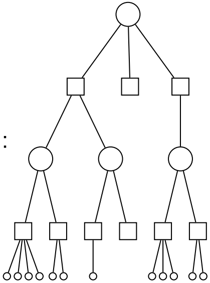
$G_{m,\alpha m}$ :




$G_{m,\alpha m}^{x,R}$ :




$T_\alpha^R$ :



# Distribution of $T_\alpha^R$

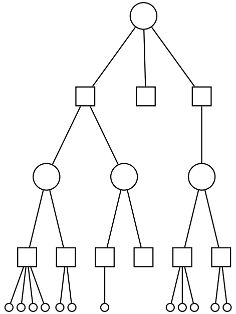
e.g. for $y = (2, 0, 1, 4, 2, 1, 0, 3, 2)$:

**Lemma**

*Let $T_y$ be a possible outcome of $T_\alpha^R$ given by a finite sequence $y = (y_1, \ldots, y_k) \in \mathbb{N}_0^k$ specifying the number of children of $\square$-nodes in level order. Then*

$$\Pr[T_\alpha^R = T_y] = \prod_{i=1}^{k} \Pr_{Y \sim \mathrm{Pois}(3\alpha)}[Y = y_i].$$

$T_y =$



# iv No cycles in $G^{x,R}_{m,\alpha m}$

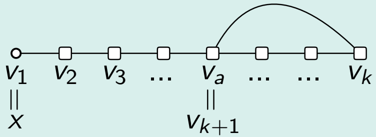
## Lemma

*Assume $R = \mathcal{O}(1)$. The probability that $G^{x,R}_{m,\alpha m}$ contains a cycle is $\mathcal{O}(1/m)$.*

## Proof.

If $G^{x,R}_{m,\alpha m}$ contains a cycle then we have

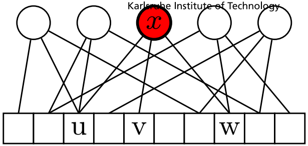
- a sequence $(v_1 = x, v_2, \dots, v_k, v_{k+1} = v_a)$ of nodes with $a \in [k]$
- of length $k \leq 4R$ (consider BFS tree for $x$ and additional edge in it)
- for each $i \in \{1, \dots, k\}$ an index $j_i \in \{1, 2, 3\}$ of the hash function connecting $v_i$ and $v_{i+1}$. (If $a = k - 1$ then $j_k \neq j_{k-1}$.)




$\Pr[\exists \text{cycle in } G^{x,R}_{m,\alpha m}] \leq \Pr[\exists 2 \leq k \leq 4R : \exists v_2, \dots, v_k : \exists a \in [k] : \exists j_1, \dots, j_k \in [3] : \forall i \in [k] : h_{j_i} \text{ connects } v_i \text{ to } v_{i+1}]$

$$\leq \sum_{k=2}^{4R} \sum_{v_2,\dots,v_k} \sum_{a=1}^{k} \sum_{j_1,\dots,j_k} \prod_{i=1}^{k} \Pr[h_{j_i} \text{ connects } v_i \text{ to } v_{i+1}] \leq \sum_{k=2}^{4R} (\max\{m, n\})^{k-1} \cdot k \cdot 3^k \left(\tfrac{1}{m}\right)^k = \tfrac{1}{m} \sum_{k=2}^{4R} k \cdot 3^k = \mathcal{O}(1/m). \qquad \square$$

# iv Distribution of $G^{x,R}_{m,\alpha m}$




## Lemma

*Let $T_y$ be a possible outcome of $T^R_\alpha$ as before. Then*

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

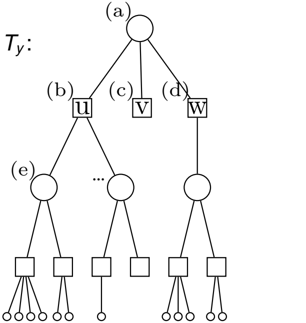
## “Proof by example”, using $T_y$ shown on the right.

The following things have to “go right” for $G^{x,R}_{m,\alpha m} = T_y$.

- a $h_1(x), h_2(x), h_3(x)$ pairwise distinct: probability $\xrightarrow{m\to\infty} 1$
  $\hookrightarrow$ non-distinct would give cycle of length 2. Unlikely by lemma.

Note: $3\lfloor\alpha m\rfloor - 3$ remaining hash values $\sim \mathcal{U}([m])$.

$T_y$:



# $\boxed{\text{iv}}$ Distribution of $G^{x,R}_{m,\alpha m}$

## Lemma

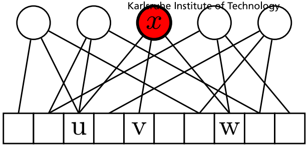
*Let $T_y$ be a possible outcome of $T^R_\alpha$ as before. Then*

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

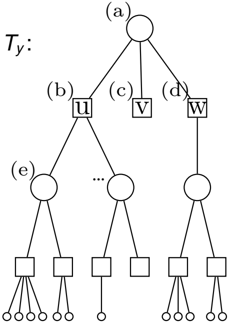
## “Proof by example”, using $T_y$ shown on the right.

- $\boxed{\text{b}}$ Exactly $y_1 = 2$ of the remaining hash values are $u$.
  $\hookrightarrow \Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor-3,\frac{1}{m})}[Y = 2] \xrightarrow{m\to\infty} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = 2]$. $\to$ exercise
  Moreover: The two hash values must belong to 2 distinct keys. Probability $\xrightarrow{m\to\infty}$ 1.
  $\hookrightarrow$ non-distinct would give cycle of length 2.

Note: The $3\lfloor\alpha m\rfloor - 5$ remaining hash values are $\sim\mathcal{U}([m]\setminus\{u\})$. $\to$ exercise




$T_y$:



# iv Distribution of $G^{x,R}_{m,\alpha m}$

## Lemma

*Let $T_y$ be a possible outcome of $T^R_\alpha$ as before. Then*

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \overset{m\to\infty}{\longrightarrow} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

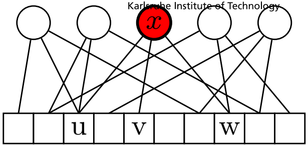
## "Proof by example", using $T_y$ shown on the right.

- c None of the remaining hash values are $v$.
  $\hookrightarrow \Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor-5,\frac{1}{m-1})}[Y=0] \overset{m\to\infty}{\longrightarrow} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y=0].$

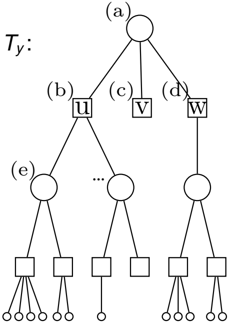
Note: The $3\lfloor\alpha m\rfloor - 5$ remaining hash values are $\sim\mathcal{U}([m]\setminus\{u,v\})$.

- d One of the remaining hash values is $w$.
  $\hookrightarrow \Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor-5,\frac{1}{m-2})}[Y=1] \overset{m\to\infty}{\longrightarrow} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y=1].$

...




$T_y$:

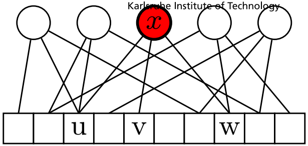


# iv Distribution of $G^{x,R}_{m,\alpha m}$

## Lemma

*Let $T_y$ be a possible outcome of $T^R_\alpha$ as before. Then*

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$




## Proof sketch in general (some details ommitted)

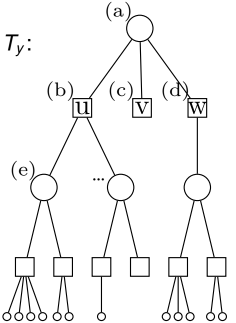
- General case at $i$-th $\square$-node. Want: probability that $G^{x,R}_{m,\alpha m}$ continues to match $T_y$. Note: $T_y$ is fixed, so $i$ and the number $c_i$ of previously revealed hash values is bounded.

$$\Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor - c_i, \frac{1}{m-i+1})}[Y = y_i] \xrightarrow{m\to\infty} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

  Moreover, those $y_i$ hash values must belong to distinct fresh keys. Probability $\xrightarrow{m\to\infty}$ 1
  $\hookrightarrow$ otherwise we'd have a cycle.

- General case for $\bigcirc$-node. The two children must be fresh: probability $\xrightarrow{m\to\infty}$ 1
  $\hookrightarrow$ otherwise there would be a cycle.

$T_y$:



# Probability that a specific key survives peeling

**Lemma**

*Let $\alpha < c_3^\Delta$. Let $x$ be any ◯-node in $G_{m,\alpha m}$ as before (chosen before sampling the hash functions). Let*

$$\mu_m := \Pr_{h_1,h_2,h_3 \sim \mathcal{U}([m]^D)}[x \text{ is removed when peeling } G_{m,\alpha m}].$$

*Then* $\lim_{m\to\infty} \mu_m = 1$.

# $\mu_m := \Pr[x \text{ is removed when peeling } G_{m,\alpha m}] \xrightarrow{m\to\infty} 1$

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Let $\delta > 0$ be arbitrary. We will show $\lim_{m\to\infty} \mu_m \geq 1 - 2\delta$.

Let $R \in \mathbb{N}$ be such that $q_R < \delta$. — possible because $\lim_{R\to\infty} q_R = 0$

$\mathcal{Y}^R := \{\text{all possibilities for } T_\alpha^R\}$

$\mathcal{Y}^R_{\text{peel}} := \{T \in \mathcal{Y}^R \mid \text{ peeling } T \text{ removes the root}\}$ — note: $\Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{peel}}] = q_R \leq \delta$.

Let $\mathcal{Y}^R_{\text{fin}} \subseteq \mathcal{Y}^R$ be a *finite* set such that $\Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{fin}}] \leq \delta$ — uses that $\mathcal{Y}^R$ is countable and $\sum\limits_{T\in\mathcal{Y}^R} \Pr[T_\alpha^R = T] = 1$.

$$\lim_{m\to\infty} \mu_m \geq \lim_{m\to\infty} \Pr[G^{x,R}_{m,\alpha m} \in \mathcal{Y}^R_{\text{peel}}]$$
peeling only in $R$-neighbourhood of $x$ is "weaker"

$$\geq \lim_{m\to\infty} \Pr[G^{x,R}_{m,\alpha m} \in \mathcal{Y}^R_{\text{peel}} \cap \mathcal{Y}^R_{\text{fin}}]$$

$$= \lim_{m\to\infty} \sum\nolimits_{T\in\mathcal{Y}^R_{\text{peel}}\cap\mathcal{Y}^R_{\text{fin}}} \Pr[G^{x,R}_{m,\alpha m} = T]$$

$$= \sum\nolimits_{T\in\mathcal{Y}^R_{\text{peel}}\cap\mathcal{Y}^R_{\text{fin}}} \lim_{m\to\infty} \Pr[G^{x,R}_{m,\alpha m} = T]$$
*finite* sums commute with limit

$$= \sum\nolimits_{T\in\mathcal{Y}^R_{\text{peel}}\cap\mathcal{Y}^R_{\text{fin}}} \Pr[T_\alpha^R = T]$$
previous lemmas

$$= \Pr[T_\alpha^R \in \mathcal{Y}^R_{\text{peel}} \cap \mathcal{Y}^R_{\text{fin}}] = 1 - \Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{peel}} \cap \mathcal{Y}^R_{\text{fin}}]$$

$$= 1 - \Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{peel}} \vee T_\alpha^R \notin \mathcal{Y}^R_{\text{fin}}]$$
De Morgan's laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$$\geq 1 - \Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{peel}}] - \Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{fin}}] \geq 1 - 2\delta.$$
union bound: $\Pr[E_1 \vee E_2] \leq \Pr[E_1] + \Pr[E_2]$ $\square$

# Proof of the Peeling Theorem

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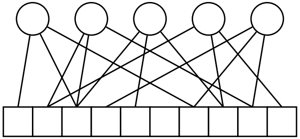

**Theorem**

Let $\alpha < c_3^\Delta$. Then

$$\Pr[G_{m,\alpha m} \text{ is peelable}] = 1 - o(1).$$

**Proof**

Let $n = \lfloor \alpha m \rfloor$ and $0 \leq s \leq n$ the number of ◯ nodes surviving peeling.

last lemma: each ◯ survives with probability $o(1)$.

linearity of expectation $\mathbb{E}[s] = n \cdot o(1) = o(n)$.

Exercise: $\Pr[s \in \{1, \dots, \delta n\}] = \mathcal{O}(1/m)$ if $\delta > 0$ is a small enough constant.

Markov: $\Pr[s > \delta n] \leq \frac{\mathbb{E}[s]}{\delta n} = \frac{o(n)}{\delta n} = o(1)$.

finally: $\Pr[s > 0] = \Pr[s \in \{1, \dots, \delta n\}] + \Pr[s > \delta n] = \mathcal{O}(1/m) + o(1) = o(1)$. $\square$

# Conclusion

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## Peling Process

- greedy algorithm for placing keys in cuckoo table
- works up to a load factor of $c_3^\Delta \approx 0.81$

## We saw glimpses of important techniques

- *Local interactions in large graphs.* Also used in statistical physics.
- *Galton-Watson Processes / Trees.* Random processes related to $T_\alpha$.
- *Local weak convergence.* How the finite graph $G_{m,\alpha m}$ is locally like $T_\alpha$.

## But wait, there's more!

- Further applications of peeling
  - retrieval data structures (next lecture)
  - perfect hash functions (next lecture)
  - set sketches
  - linear error correcting codes

# Anhang: Mögliche Prüfungsfragen I

- Cuckoo Hashing und der Schälalgorithmus
  - (Wie) kann man Cuckoo Hashing mit mehr als 2 Hashfunktionen aufziehen?
  - Welcher Vorteil ergibt sich im Vergleich zu 2 Hashfunktionen?
  - Wie funktioniert der Schälalgorithmus zur Platzierung von Schlüsseln in einer Cuckoo Hashtabelle?
  - Schälen lässt sich als einfacher Prozess auf Graphen auffassen. Wie?
  - Was besagt das Hauptresultat, das wir zum Schälprozess bewiesen haben?
- Beweis des Schälsatzes. *Mir ist klar, dass der Beweis äußerst kompliziert ist.*
  - Im Beweis haben zwei Graphen eine Rolle gespielt ein endlicher und ein (potentiell) unendlicher. Wie waren diese Graphen definiert?
  - Welcher Zusammenhang besteht zwischen der Verteilung der Knotengrade in $T_\alpha$ und $G_{m,\alpha m}$?