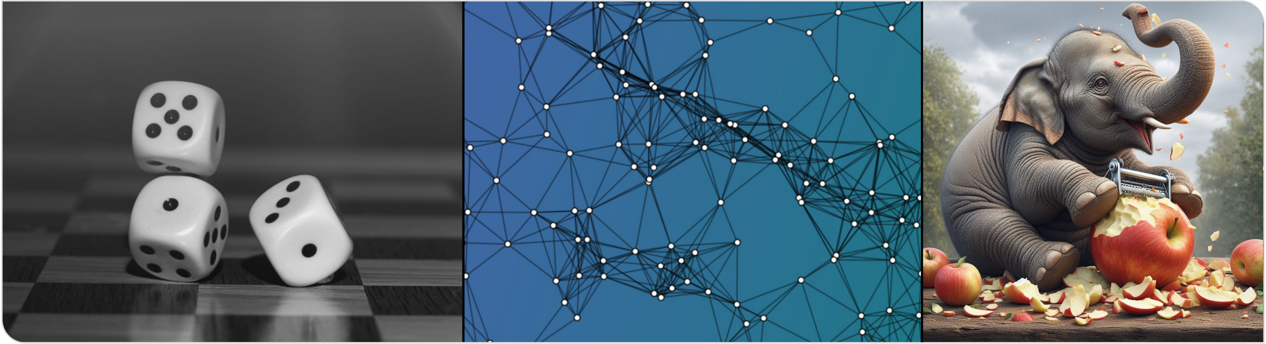
Probability and Computing – The Peeling Algorithm

Stefan Walzer | WS 2024/2025



Content

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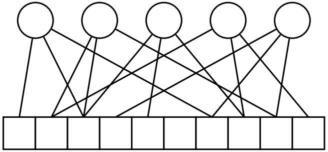
Cuckoo Hashing with one table and *k* hash functions



$n \in \mathbb{N}$	keys
$m \in \mathbb{N}$	table size
$\alpha = \frac{n}{m}$	load factor
$h_1, \ldots, h_k \sim \mathcal{U}([m]^D)$	hash functions

$\hookrightarrow$ Could also use a separate table per hash function.

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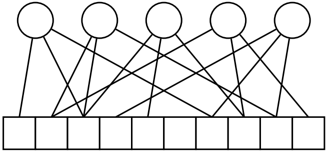

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randomWalkInsert(x)

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while $x \neq \bot$ do // TODO: limit
    sample $i \sim \mathcal{U}([k])$
    swap($x, T[h_i(x)]$)
```

(some improvements possible)

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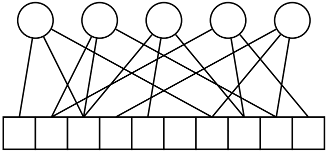

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**while** $x \neq \perp$ **do** // TODO: limit
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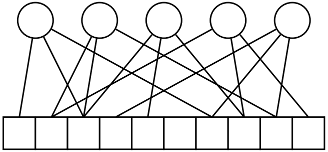
Theorem (without proof)

For each $k \in \mathbb{N}$ there is a **threshold** c_k^* such that:

- if $\alpha < c_k^*$ all keys can be placed with probability $1 - \mathcal{O}(\frac{1}{m})$.
- if $\alpha > c_k^*$ **not** all keys can be placed with probability $1 - \mathcal{O}(\frac{1}{m})$.

$c_2^* = \frac{1}{2}, \quad c_3^* \approx 0.92, \quad c_4^* \approx 0.98, \dots$

Cuckoo Hashing with one table and k hash functions



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Theorem (Bell, Frieze, 2024)

If $k \geq 4$ and $\alpha < c_k^*$ then, conditioned on a high probability event[a], the expected insertion time is $\mathcal{O}(1)$.

---

[a] Without this conditioning, randomWalkInsert might be trapped in an infinite loop.

Static Hash Tables

Static Hash Table

`construct`(S): builds table T with key set S

`lookup`(x): checks if x is in T or not

↪ no insertions or deletions after construction!

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Constructing cuckoo hash tables:

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- matching algorithm resembling preflow push
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Greedily constructing cuckoo hash tables

- Peeling: simple algorithm but sophisticated analysis
- interesting applications beyond hash tables (see “retrieval” in next lecture)

Content

The Peeling Algorithm

constructByPeeling($S \subseteq D,\ h_1, h_2, h_3 \in [m]^D$)

```
$T \leftarrow [\bot, \dots, \bot]$ // empty table of size $m$
while $\exists i \in [m] : \exists$ exactly one $x \in S : i \in \{h_1(x), h_2(x), h_3(x)\}$ do
    // $x$ is only unplaced key that may be placed in $i$
    $T[i] \leftarrow x$
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if $S = \varnothing$ then
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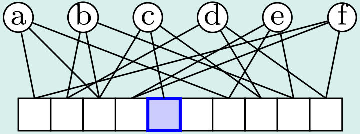

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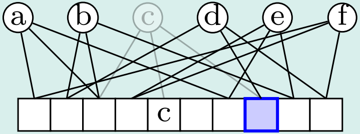

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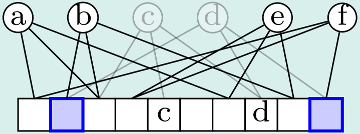

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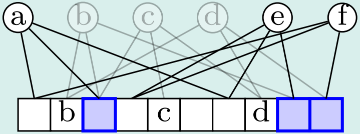

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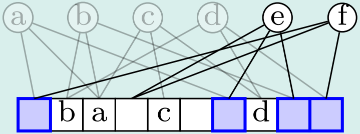

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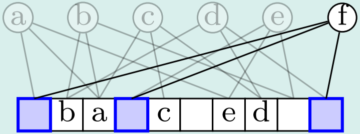

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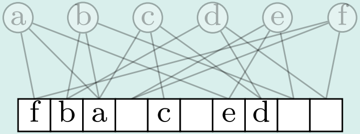

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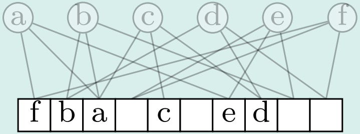


Exercise

- Success of constructByPeeling does not depend on choices for i made by while.
- constructByPeeling can be implemented in linear time.

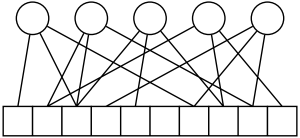
Peelability and the Cuckoo Graph

Cuckoo Graph and Peelability

- The **Cuckoo Graph** is the bipartite graph

$$G_{S,h_1,h_2,h_3} = (S, [m], \{(x, h_i(x)) \mid x \in S, i \in [3]\})$$

- Call G_{S,h_1,h_2,h_3} **peelable** if constructByPeeling(S, h_1, h_2, h_3) succeeds.
- If $h_1, h_2, h_3 \sim \mathcal{U}([m]^D)$ then the distribution of G_{S,h_1,h_2,h_3} does not depend on S. We then simply write $G_{m,\alpha m}$.
  - m □-nodes and $\lfloor \alpha m \rfloor$-○-nodes
  - think: α is constant and $m \to \infty$.



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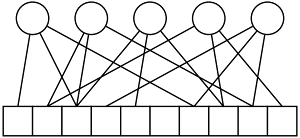

Peeling simplified (not computing placement)

```
while ∃ □-node of degree 1 do
    remove it and its incident ○
```

G is peelable if and only if
this algorithm removes all ○-nodes.

Content

Peeling Theorem

Peeling Threshold

Let $c_3^\Delta = \min_{y\in[0,1]} \frac{y}{3(1-e^{-y})^2} \approx 0.81$.

Theorem (today's goal)

Let $\alpha < c_3^\Delta$. Then $\Pr[G_{m,\alpha m}$ is peelable$] = 1 - o(1)$.

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Remark: More is known.

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Exercise: What about $k = 2$?

Peeling does not reliably work for $k = 2$ for any $\alpha > 0$.

Peeling Theorem: Proof outline

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Let $\alpha < c_3^\Delta$. Then $\Pr[G_{m,\alpha m} \text{ is peelable}] = 1 - o(1)$.

Proof Idea

The random (possibly) infinite tree T_α can be peeled for $\alpha < c_3^\Delta$ and T_α is locally like $G_{m,\alpha m}$.

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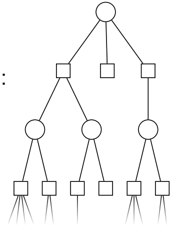
Steps

- I What is T_α?
- II What does peeling mean in this setting?
- III What role does c_3^Δ play?
- IV What does it mean for T_α to be locally like $G_{m,\alpha m}$?
- V What is the probability that a fixed key of $G_{m,\alpha m}$ is peeled?
- VI What is the probability that *all* keys of $G_{m,\alpha m}$ are peeled?

T_α :



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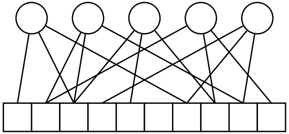

Observations for the finite Graph $G_{m,\alpha m}$

- each ○ has 3 □ as neighbours (rare exception: $h_1(x)$, $h_2(x)$, $h_3(x)$ not distinct)
- each □ has random number X of ○ as neighbours with $X \sim \text{Bin}(3n, \frac{1}{m}) = \text{Bin}(3\lfloor \alpha m \rfloor, \frac{1}{m})$. In an exercise you'll show

$$\Pr[X = i] \xrightarrow{m \to \infty} \Pr_{Y \sim \text{Pois}(3\alpha)}[Y = i].$$

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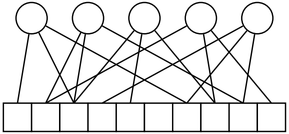

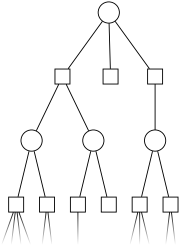

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Definition of the (possibly) infinite random tree T_α

- root is ○ and has three □ as children
- each □ has random number of ○ children, sampled $\text{Pois}(3\alpha)$ (independently for each □).
- each non-root ○ has two □ as children.

Remark: T_α is finite with positive probability > 0, e.g. when the first three $\text{Pois}(3\alpha)$ random variables come out as 0. But T_α is also infinite with positive probability.

ii What does peeling mean in this setting?

Peeling Algorithm

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while ∃ □-node of degree 1 do
    remove it and its incident ○
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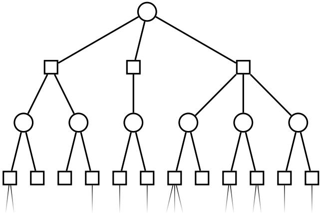
↪ not well defined outcome on T_α!



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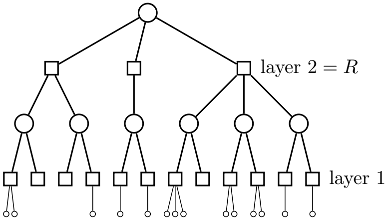

Peel only the first $R \in \mathbb{N}$ layers

- Let T_α^R be the first $2R + 1$ levels of T_α.
- *R layers* of □-nodes, labeled bottom to top.
- Run peeling on T_α^R (later $R \to \infty$).

↪ Why not consider the first $2R$ levels? (without $+1$)

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while $\exists$ $\square$-node of degree 1 do
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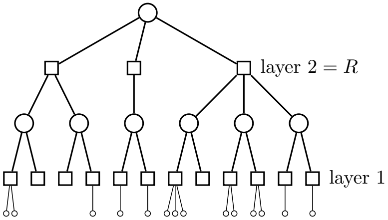

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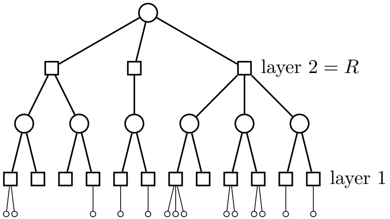

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Only care whether root is removed (root represents arbitrary node in $G_{m,\alpha m}$)

We may then simplify the peeling algorithm.

- replace "□-node of degree 1" condition with stronger "childless □-node".
  - prevents peeling of □-nodes with one child and no parent
  - no matter: such nodes are disconnected from the root anyway
- whether node is peeled only depends on subtree
  ↪ one bottom up pass suffices for peeling

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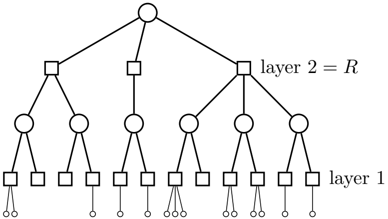

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- replace "$\square$-node of degree 1" condition with stronger "childless $\square$-node".
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  ↪ one bottom up pass suffices for peeling

ii What does peeling mean in this setting?

Peeling Algorithm

```
while $\exists$ childless $\square$-node do
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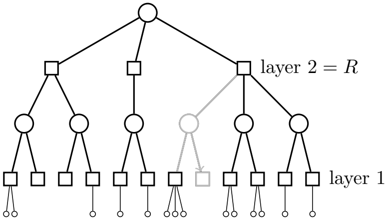
```

↪ not well defined outcome on T_α!

↪ but well defined on T_α^R!



Peel only the first $R \in \mathbb{N}$ layers

- Let T_α^R be the first $2R + 1$ levels of T_α.
- *R layers* of □-nodes, labeled bottom to top.
- Run peeling on T_α^R (later $R \to \infty$).

↪ Why not consider the first $2R$ levels? (without $+1$)

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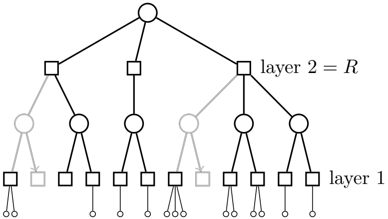

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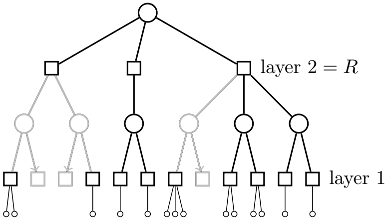

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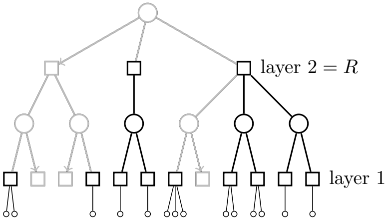

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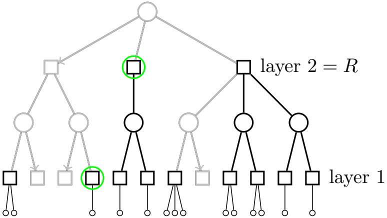

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ⅱ What does peeling mean in this setting? (2)

Observation

Let $q_R = \Pr[\text{root survives when peeling } T_\alpha^R]$.
The values q_R are decreasing in R.

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Let $q_R = \Pr[\text{root survives when peeling } T_\alpha^R]$.
The values q_R are decreasing in R.

Peeling Algorithm

```
while ∃ childless □-node do
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```

Proof.

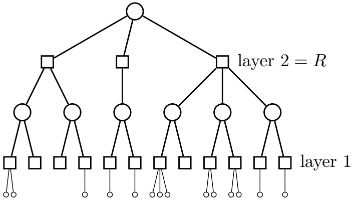
Assume when peeling T_α^R the sequence $\vec{x} = (x_1, \ldots, x_k)$ is a valid sequence of □-node choices. Then $\vec{x}$ is also valid when peeling T_α^{R+1}.

$$\text{peeling } T_\alpha^R \text{ removes the root} \Rightarrow \text{peeling } T_\alpha^{R+1} \text{ removes the root}$$

$$\text{root survives when peeling } T_\alpha^{R+1} \Rightarrow \text{peeling } T_\alpha^R \text{ removes the root}$$

$$q_{R+1} \leq q_R$$

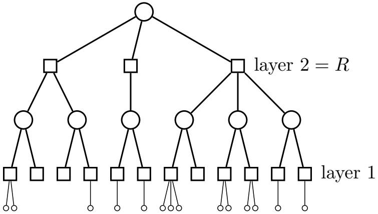
□



ii What does peeling mean in this setting? (3)

Peeling T_α^R bottom up

```
for $i = 1$ to $R$ do // □-layers bottom to top
    for each □-node $v$ in layer $i$ do
        if $v$ has no children then
            remove $v$ and its parent ○
```



Survival probabilities $p_i :=$ Pr[■-node in layer i is *not* peeled]

$$
\begin{aligned}
p_1 &= \Pr[\square\text{-node has} \geq 1 \text{ child}] \\
&= \Pr_{Y \sim \mathrm{Pois}(3\alpha)}[Y > 0] = 1 - e^{-3\alpha}.
\end{aligned}
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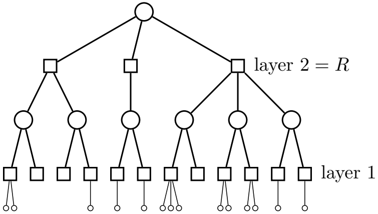

Survival probabilities $p_i := \Pr[\blacksquare\text{-node in layer } i \text{ is } \textit{not} \text{ peeled}]$

$$p_1 = \Pr[\square\text{-node has} \geq 1 \text{ child}]$$

$$= \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y > 0] = 1 - e^{-3\alpha}.$$

$$p_i = \Pr[\text{layer } i\ \square\text{-node } v \text{ has} \geq 1 \textit{ surviving} \text{ child}]$$

$$= \Pr_{X\sim\mathrm{Pois}(3\alpha p_{i-1}^2)}[X > 0] = 1 - e^{-3\alpha p_{i-1}^2}.$$

$Y :=$ number of (initial) children of v

$X :=$ number of surviving children of v

each child ○-node survives if both its □-children from layer $i-1$ survive ⇝ probability p_{i-1}^2.

$\Rightarrow Y \sim \mathrm{Pois}(3\alpha)$ and $X \sim \mathrm{Bin}(Y, p_{i-1}^2)$.

$\Rightarrow X \sim \mathrm{Pois}(3\alpha p_{i-1}^2)$. ⇝ exercise!

ii What does peeling mean in this setting? (3)

Peeling T_α^R bottom up

```
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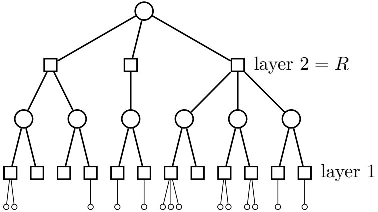

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$$p_i = \Pr[\text{layer } i\ \square\text{-node } v \text{ has} \geq 1 \textit{ surviving} \text{ child}]$$

$$= \Pr_{X \sim \text{Pois}(3\alpha p_{i-1}^2)}[X > 0] = 1 - e^{-3\alpha p_{i-1}^2}.$$

□-survival probabilities. With $p_0 := 1$ we have

$$p_i = \begin{cases} 1 & \text{if } i = 0 \\ 1 - e^{-3\alpha p_{i-1}^2} & \text{if } i = 1, 2, \ldots \end{cases}$$

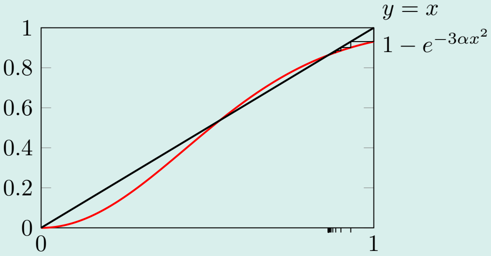
Moreover: $q_R :=$ Pr[root survives] $= p_R^3$.

What role does $c_3^\Delta \approx 0.81$ play?

$$p_i = \begin{cases} 1 & \text{if } i = 0 \\ 1 - e^{-3\alpha p_{i-1}^2} & \text{if } i = 1, 2, \dots \end{cases}$$

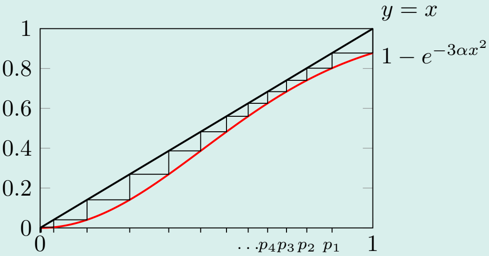
$\hookrightarrow$ consider $f(x) = 1 - e^{-3\alpha x^2}$

Case 1: $\exists x > 0 : f(x) = x$.



$$\Rightarrow \lim_{i \to \infty} p_i = x^* = \max\{x \in [0,1] \mid f(x) = x\}.$$

Case 2: $\forall x \in (0, 1] : f(x) < x$



$$\Rightarrow \lim_{i \to \infty} p_i = 0.$$

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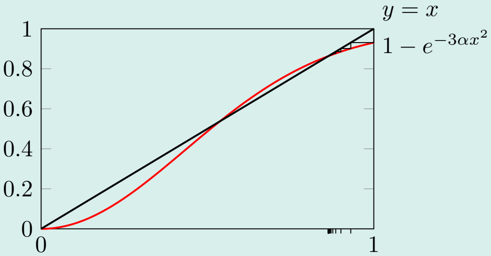
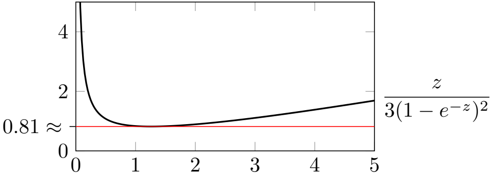


$$\Rightarrow \lim_{i\to\infty} p_i = x^* = \max\{x \in [0,1] \mid f(x) = x\}.$$

$$\begin{aligned} \text{Case 1} &\Leftrightarrow \exists x > 0 : x = 1 - e^{-3\alpha x^2} \\ &\Leftrightarrow \exists x > 0 : x^2 = (1 - e^{-3\alpha x^2})^2 \\ &\Leftrightarrow \exists z > 0 : \frac{z}{3\alpha} = (1 - e^{-z})^2 // z = 3\alpha x^2 \\ &\Leftrightarrow \exists z > 0 : \alpha = \frac{z}{3(1 - e^{-z})^2} \\ &\Leftrightarrow \alpha \geq \min_{z>0} \frac{z}{3(1 - e^{-z})^2} =: c_3^\Delta \approx 0.81 \end{aligned}$$



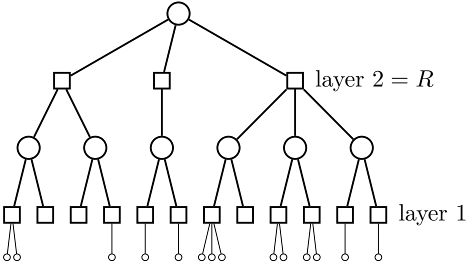
iii Interim Conclusion: What we learned about peeling T_α

Lemma

For $\alpha < c_3^\Delta \approx 0.81$ we have

- $\lim\limits_{i\to\infty} p_i = 0.$
- $\lim\limits_{R\to\infty} q_R = \lim\limits_{R\to\infty} p_R^3 = 0.$

"Root rarely survives for large R."

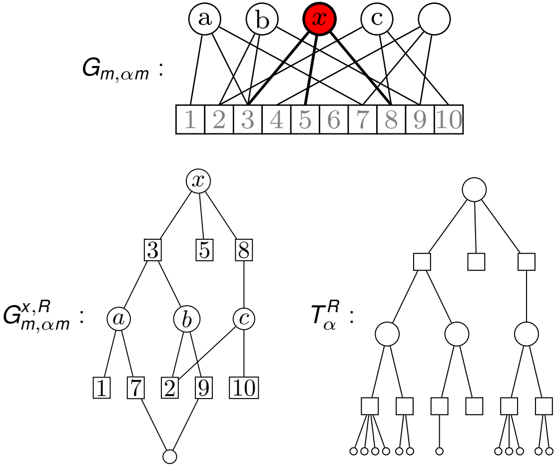


iv What does it mean for T_α to be locally like $G_{m,\alpha m}$?

Neighbourhoods in T_α and G

Let $R \in \mathbb{N}$. We consider

- T_α^R as before and
- for any fixed $x \in S$ the subgraph $G_{m,\alpha m}^{x,R}$ of $G_{m,\alpha m}$ induced by all nodes with distance at most $2R$ from x.



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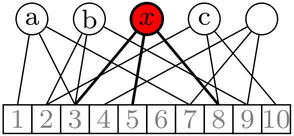
Lemma

For any $R \in \mathbb{N}$, the **distribution** of $G_{m,\alpha m}^{x,R}$ converges the distribution of T_α^R, i.e.

$$\forall T : \lim_{m\to\infty} \Pr[G_{m,\alpha m}^{x,R} = T] = \Pr[T_\alpha^R = T].$$

$G_{m,\alpha m}$:



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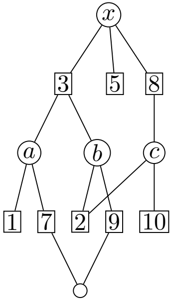
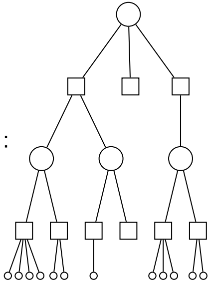


T_α^R :



iv Distribution of T_α^R

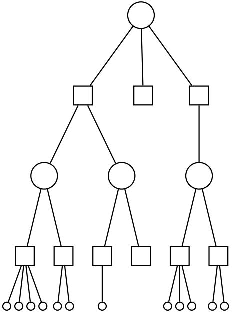
e.g. for $y = (2, 0, 1, 4, 2, 1, 0, 3, 2)$:

Lemma

Let T_y be a possible outcome of T_α^R given by a finite sequence $y = (y_1, \dots, y_k) \in \mathbb{N}_0^k$ specifying the number of children of $\square$-nodes in level order. Then

$$\Pr[T_\alpha^R = T_y] = \prod_{i=1}^{k} \Pr_{Y \sim \mathrm{Pois}(3\alpha)}[Y = y_i].$$

$T_y =$



iv No cycles in $G_{m,\alpha m}^{x,R}$

Lemma

Assume $R = \mathcal{O}(1)$. *The probability that* $G_{m,\alpha m}^{x,R}$ *contains a cycle is* $\mathcal{O}(1/m)$.

iv No cycles in $G^{x,R}_{m,\alpha m}$

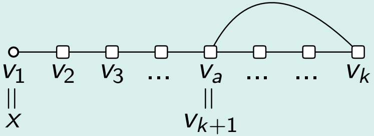
Lemma

Assume $R = \mathcal{O}(1)$. The probability that $G^{x,R}_{m,\alpha m}$ contains a cycle is $\mathcal{O}(1/m)$.

Proof.

If $G^{x,R}_{m,\alpha m}$ contains a cycle then we have

- a sequence $(v_1 = x, v_2, \dots, v_k, v_{k+1} = v_a)$ of nodes with $a \in [k]$
- of length $k \leq 4R$ (consider BFS tree for x and additional edge in it)
- for each $i \in \{1, \dots, k\}$ an index $j_i \in \{1, 2, 3\}$ of the hash function connecting v_i and v_{i+1}. (If $a = k - 1$ then $j_k \neq j_{k-1}$.)



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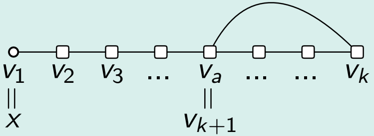


$\Pr[\exists\text{cycle in } G^{x,R}_{m,\alpha m}] \le \Pr[\exists 2 \le k \le 4R : \exists v_2, \dots, v_k : \exists a \in [k] : \exists j_1, \dots, j_k \in [3] : \forall i \in [k] : h_{j_i} \text{ connects } v_i \text{ to } v_{i+1}]$

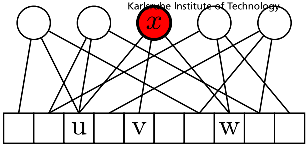
$$\le \sum_{k=2}^{4R} \sum_{v_2,\dots,v_k} \sum_{a=1}^{k} \sum_{j_1,\dots,j_k} \prod_{i=1}^{k} \Pr[h_{j_i} \text{ connects } v_i \text{ to } v_{i+1}] \le \sum_{k=2}^{4R} (\max\{m,n\})^{k-1} \cdot k \cdot 3^k \left(\frac{1}{m}\right)^k = \frac{1}{m} \sum_{k=2}^{4R} k \cdot 3^k = \mathcal{O}(1/m). \qquad \square$$

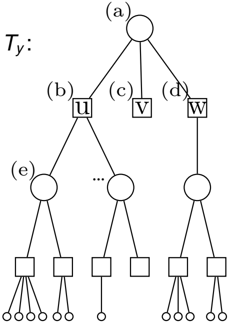
iv Distribution of $G^{x,R}_{m,\alpha m}$

Lemma

Let T_y be a possible outcome of T^R_α as before. Then

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$





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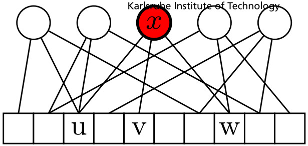
“Proof by example”, using T_y shown on the right.

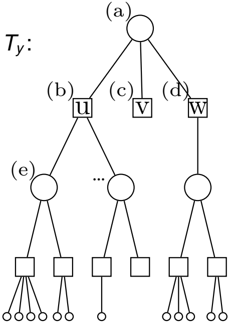
The following things have to “go right” for $G^{x,R}_{m,\alpha m} = T_y$.

a $h_1(x), h_2(x), h_3(x)$ pairwise distinct: probability $\xrightarrow{m\to\infty} 1$
$\hookrightarrow$ non-distinct would give cycle of length 2. Unlikely by lemma.

Note: $3\lfloor\alpha m\rfloor - 3$ remaining hash values $\sim\mathcal{U}([m])$.



T_y:



iv Distribution of $G^{x,R}_{m,\alpha m}$

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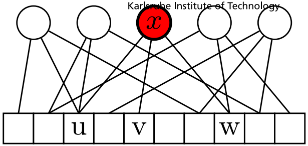
Let T_y be a possible outcome of T^R_α as before. Then

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

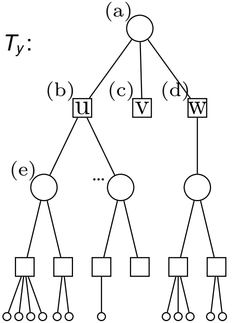
“Proof by example”, using T_y shown on the right.

- b Exactly $y_1 = 2$ of the remaining hash values are u.
  $\hookrightarrow$ $\Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor-3,\frac{1}{m})}[Y=2] \xrightarrow{m\to\infty} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y=2]$. → exercise
  Moreover: The two hash values must belong to 2 distinct keys. Probability $\xrightarrow{m\to\infty}$ 1.
  $\hookrightarrow$ non-distinct would give cycle of length 2.

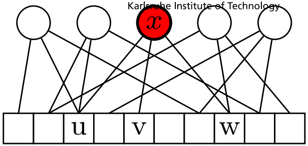
Note: The $3\lfloor\alpha m\rfloor - 5$ remaining hash values are $\sim\mathcal{U}([m]\setminus\{u\})$. → exercise



T_y:



iv Distribution of $G^{x,R}_{m,\alpha m}$



Lemma

Let T_y be a possible outcome of T^R_α as before. Then

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

"Proof by example", using T_y shown on the right.

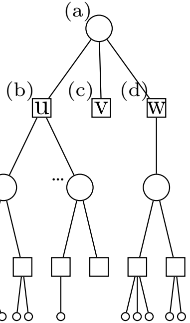
- c None of the remaining hash values are v.
  $\hookrightarrow \Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor-5,\frac{1}{m-1})}[Y=0] \xrightarrow{m\to\infty} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y=0].$

Note: The $3\lfloor\alpha m\rfloor - 5$ remaining hash values are $\sim\mathcal{U}([m]\setminus\{u,v\})$.

- d One of the remaining hash values is w.
  $\hookrightarrow \Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor-5,\frac{1}{m-2})}[Y=1] \xrightarrow{m\to\infty} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y=1].$

. . .

T_y:

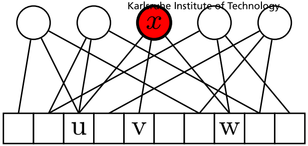


iv Distribution of $G^{x,R}_{m,\alpha m}$

Lemma

Let T_y be a possible outcome of T^R_α as before. Then

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m}=T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y=y_i].$$



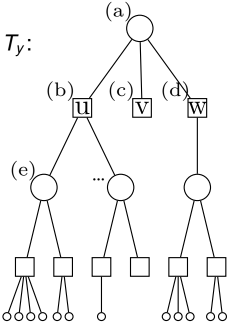
Proof sketch in general (some details ommitted)

- General case at i-th □-node. Want: probability that $G^{x,R}_{m,\alpha m}$ continues to match T_y. Note: T_y is fixed, so i and the number c_i of previously revealed hash values is bounded.

  $$\Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor - c_i, \frac{1}{m-i+1})}[Y=y_i] \xrightarrow{m\to\infty} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y=y_i].$$

  Moreover, those y_i hash values must belong to distinct fresh keys. Probability $\xrightarrow{m\to\infty}$ 1
  ↪ otherwise we'd have a cycle.

- General case for ○-node. The two children must be fresh: probability $\xrightarrow{m\to\infty}$ 1
  ↪ otherwise there would be a cycle.



v Probability that a specific key survives peeling

Lemma

Let $\alpha < c_3^\Delta$. *Let* x *be any* ○*-node in* $G_{m,\alpha m}$ *as before (chosen before sampling the hash functions). Let*

$$\mu_m := \Pr_{h_1,h_2,h_3 \sim \mathcal{U}([m]^D)}[x \textit{ is removed when peeling } G_{m,\alpha m}].$$

Then $\lim\limits_{m\to\infty} \mu_m = 1$.

v $\mu_m := \Pr[x \text{ is removed when peeling } G_{m,\alpha m}] \xrightarrow{m\to\infty} 1$

Let $\delta > 0$ be arbitrary. We will show $\lim_{m\to\infty} \mu_m \geq 1 - 2\delta$.

Let $R \in \mathbb{N}$ be such that $q_R < \delta$.

possible because $\lim_{R\to\infty} q_R = 0$

$\mathcal{Y}^R := \{\text{all possibilities for } T_\alpha^R\}$

$\mathcal{Y}^R_{\text{peel}} := \{T \in \mathcal{Y}^R \mid \text{ peeling } T \text{ removes the root}\}$

note: $\Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{peel}}] = q_R \leq \delta$.

Let $\mathcal{Y}^R_{\text{fin}} \subseteq \mathcal{Y}^R$ be a *finite* set such that $\Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{fin}}] \leq \delta$

uses that $\mathcal{Y}^R$ is countable and $\sum_{T\in\mathcal{Y}^R} \Pr[T_\alpha^R = T] = 1$.

v $\mu_m := \Pr[x \text{ is removed when peeling } G_{m,\alpha m}] \xrightarrow{m\to\infty} 1$

Let $\delta > 0$ be arbitrary. We will show $\lim_{m\to\infty} \mu_m \geq 1 - 2\delta$.

Let $R \in \mathbb{N}$ be such that $q_R < \delta$. — possible because $\lim_{R\to\infty} q_R = 0$

$\mathcal{Y}^R := \{\text{all possibilities for } T_\alpha^R\}$

$\mathcal{Y}^R_{\text{peel}} := \{T \in \mathcal{Y}^R \mid \text{ peeling } T \text{ removes the root}\}$ — note: $\Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{peel}}] = q_R \leq \delta$.

Let $\mathcal{Y}^R_{\text{fin}} \subseteq \mathcal{Y}^R$ be a *finite* set such that $\Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{fin}}] \leq \delta$ — uses that $\mathcal{Y}^R$ is countable and $\sum\limits_{T\in\mathcal{Y}^R} \Pr[T_\alpha^R = T] = 1$.

$$\begin{aligned}
\lim_{m\to\infty} \mu_m &\geq \lim_{m\to\infty} \Pr[G^{x,R}_{m,\alpha m} \in \mathcal{Y}^R_{\text{peel}}] && \text{peeling only in } R\text{-neighbourhood of } x \text{ is "weaker"}\\
&\geq \lim_{m\to\infty} \Pr[G^{x,R}_{m,\alpha m} \in \mathcal{Y}^R_{\text{peel}} \cap \mathcal{Y}^R_{\text{fin}}]\\
&= \lim_{m\to\infty} \textstyle\sum_{T\in\mathcal{Y}^R_{\text{peel}}\cap\mathcal{Y}^R_{\text{fin}}} \Pr[G^{x,R}_{m,\alpha m} = T]\\
&= \textstyle\sum_{T\in\mathcal{Y}^R_{\text{peel}}\cap\mathcal{Y}^R_{\text{fin}}} \lim_{m\to\infty} \Pr[G^{x,R}_{m,\alpha m} = T] && \textit{finite} \text{ sums commute with limit}\\
&= \textstyle\sum_{T\in\mathcal{Y}^R_{\text{peel}}\cap\mathcal{Y}^R_{\text{fin}}} \Pr[T_\alpha^R = T] && \text{previous lemmas}\\
&= \Pr[T_\alpha^R \in \mathcal{Y}^R_{\text{peel}} \cap \mathcal{Y}^R_{\text{fin}}] = 1 - \Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{peel}} \cap \mathcal{Y}^R_{\text{fin}}]\\
&= 1 - \Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{peel}} \vee T_\alpha^R \notin \mathcal{Y}^R_{\text{fin}}] && \text{De Morgan's laws: } \overline{A \cap B} = \overline{A} \cup \overline{B}\\
&\geq 1 - \Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{peel}}] - \Pr[T_\alpha^R \notin \mathcal{Y}^R_{\text{fin}}] \geq 1 - 2\delta. && \text{union bound: } \Pr[E_1 \vee E_2] \leq \Pr[E_1] + \Pr[E_2] \quad \square
\end{aligned}$$

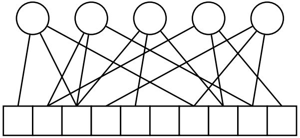
vi Proof of the Peeling Theorem

Karlsruhe Institute of Technology

Theorem

Let $\alpha < c_3^\Delta$. Then

$$\Pr[G_{m,\alpha m} \text{ is peelable}] = 1 - o(1).$$



Proof

Let $n = \lfloor \alpha m \rfloor$ and $0 \leq s \leq n$ the number of ○ nodes surviving peeling.

last lemma: each ○ survives with probability $o(1)$.

linearity of expectation $\mathbb{E}[s] = n \cdot o(1) = o(n)$.

Exercise: $\Pr[s \in \{1, \ldots, \delta n\}] = \mathcal{O}(1/m)$ if $\delta > 0$ is a small enough constant.

Markov: $\Pr[s > \delta n] \leq \frac{\mathbb{E}[s]}{\delta n} = \frac{o(n)}{\delta n} = o(1)$.

finally: $\Pr[s > 0] = \Pr[s \in \{1, \ldots, \delta n\}] + \Pr[s > \delta n] = \mathcal{O}(1/m) + o(1) = o(1)$. ☐

Conclusion

Peeling Process

- greedy algorithm for placing keys in cuckoo table
- works up to a load factor of $c_3^\Delta \approx 0.81$

We saw glimpses of important techniques

- *Local interactions in large graphs.* Also used in statistical physics.
- *Galton-Watson Processes / Trees*. Random processes related to T_α.
- *Local weak convergence*. How the finite graph $G_{m,\alpha m}$ is locally like T_α.

But wait, there's more!

- Further applications of peeling
  - retrieval data structures (next lecture)
  - perfect hash functions (next lecture)
  - set sketches
  - linear error correcting codes

Anhang: Mögliche Prüfungsfragen I

- Cuckoo Hashing und der Schälalgorithmus
  - (Wie) kann man Cuckoo Hashing mit mehr als 2 Hashfunktionen aufziehen?
  - Welcher Vorteil ergibt sich im Vergleich zu 2 Hashfunktionen?
  - Wie funktioniert der Schälalgorithmus zur Platzierung von Schlüsseln in einer Cuckoo Hashtabelle?
  - Schälen lässt sich als einfacher Prozess auf Graphen auffassen. Wie?
  - Was besagt das Hauptresultat, das wir zum Schälprozess bewiesen haben?
- Beweis des Schälsatzes. *Mir ist klar, dass der Beweis äußerst kompliziert ist.*
  - Im Beweis haben zwei Graphen eine Rolle gespielt ein endlicher und ein (potentiell) unendlicher. Wie waren diese Graphen definiert?
  - Welcher Zusammenhang besteht zwischen der Verteilung der Knotengrade in T_α und $G_{m,\alpha m}$?