Probability and Computing – Probabilistic Method

Stefan Walzer | WS 2024/2025



The Name of the Game

The Probabilistic Method (pioneered by Paul Erdős)

Show that something exists by proving that it has a positive probability of arising from a random process.

- Used to proved statements that don't involve randomness at all.
- Probabilistic arguments replace combinatorial arguments.

First Example: Ramsey Numbers

Definition: Ramsey Number

$R(k, k) := \min\{n \in \mathbb{N} \mid$ any red-blue colouring of the edges of K_n contains a monochromatic k-clique$\}$. [a]

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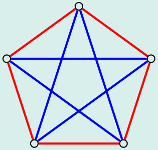
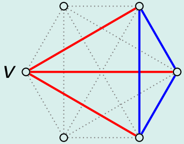
$R(3,3) > 5$



$R(3,3) \leq 6$



- v has 3 incident edges of the same colour
- wlog that colour is red
- if there is no red triangle then w_1, w_2, w_3 form a blue triangle.

Hence: $R(3,3) = 6$.

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- The number M of monochromatic k-cliques satisfies:

$$\mathbb{E}[M] = \binom{n}{k} \cdot 2^{-\binom{k}{2}+1} \leq \frac{n^k}{k!} \cdot 2^{-k^2/2+k/2+1} \leq \frac{(2^{k/2})^k}{(k/2)^{k/2}} \cdot 2^{-k^2/2} 2^{k/2} 2 = 2\left(\frac{4}{k}\right)^{k/2} < 1.$$

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Since $\mathbb{E}[M] < 1$ it is possible that $M = 0$. In particular a colouring with no monochromatic k-cliques exists. □

Expectation Argument

We have implicitly used:

$\Pr[X \leq \mathbb{E}[X]] > 0$ and $\Pr[X \geq \mathbb{E}[X]] > 0$.

Probabilistic Method with Expectation Argument

Show that an object x with $f(x) \overset{\leq}{\geq} b$ exists by proving that a random object X satisfies $\mathbb{E}[f(X)] \overset{\leq}{\geq} b$.

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Proof.

- Assign each $v \in V$ to V_1 or V_2 uniformly at random.
- Each edge crosses the cut (V_1, V_2) with probability $1/2$.
- $\mathbb{E}[\text{weight of } (V_1, V_2)] = \mathbb{E}\Big[\sum_{e \in E}[e \text{ crosses } (V_1, V_2)]\Big] = \sum_{e \in E} \Pr[e \text{ crosses } (V_1, V_2)] = |E| \cdot \frac{1}{2}$. □

Example: Independent Sets

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Theorem

Let $G = (V, E)$ with $n = |V|$, $m = |E|$ and $m \geq \frac{n}{2}$.
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Algorithm sampleAndReject:

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V⁺ ← ∅
for v ∈ V do
    with probability n/(2m) do
        V⁺ ← V⁺ ∪ {v}
V⁻ ← ∅
for {u, v} ∈ E do
    if u ∈ V⁺ and v ∈ V⁺ then
        with probability 1/2 do
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        otherwise
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return V⁺ \ V⁻
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Proof.

- sampleAndReject computes an independent set $V^+ \setminus V^-$.
- $\mathbb{E}[|V^+|] = n \cdot \frac{n}{2m} = \frac{n^2}{2m}$.
- $\mathbb{E}[|V^-|] \leq \sum_{\{u,v\} \in E} \Pr[u \in V^+, v \in V^+] = \sum_{\{u,v\} \in E} \left(\frac{n}{2m}\right)^2 = \frac{n^2}{4m}$.
- $\mathbb{E}[|V^+ \setminus V^-|] = \mathbb{E}[|V^+|] - \mathbb{E}[|V^-|] \geq \frac{n^2}{2m} - \frac{n^2}{4m} = \frac{n^2}{4m}$. □

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Remark: sampleAndReject seems suitable for a parallel / distributed setting.

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Context

Given: Family $\mathcal{E} = \{E_1, \ldots, E_n\}$ of “bad” events with $\Pr[E_i] \leq p < 1$.

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If $\mathcal{E}$ is an independent family then $\Pr[\text{none of } \mathcal{E}] = \prod_{i=1}^{n} \Pr[\bar{E}_i] \geq (1-p)^{|\mathcal{E}|} > 0$.

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Lovász Local Lemma (László Lovász and Paul Erdős, 1973)

If each $E \in \mathcal{E}$ has $\Pr[E] < p$ and depends on at most d events[a] from $\mathcal{E}$ and $4pd \le 1$ then $\Pr[\text{none of } \mathcal{E}] > 0$.

[a] Little challenge: State what this means formally.

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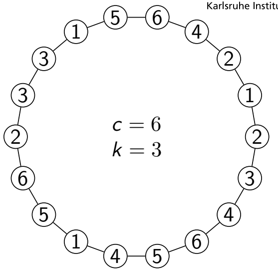

Setting

Consider a necklace of ck beads with k beads of each of c colours.
An *independent rainbow* is a set of beads

- containing one bead of each colour // rainbow
- and not containing a pair of adjacent beads. // independent

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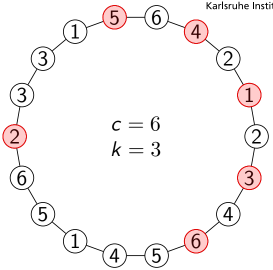

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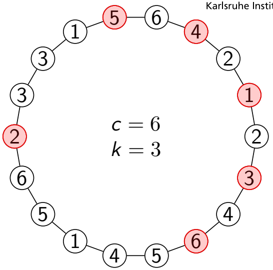

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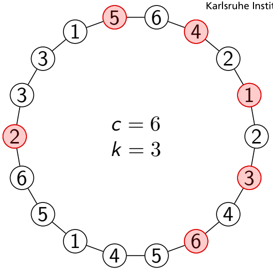

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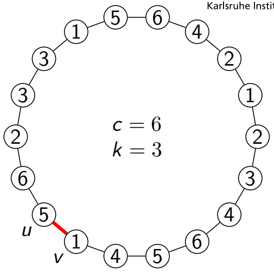

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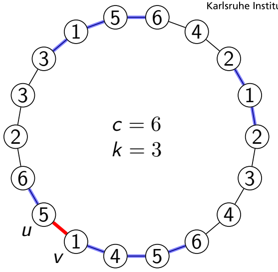

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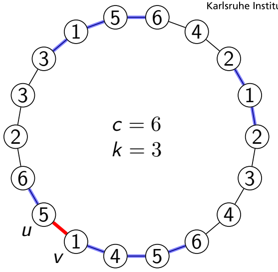

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$2k$ relevant beads, hence $4k-2$ relevant pairs.

$$\Rightarrow d = 4k-2, \qquad 4pd \leq 4\frac{1}{k^2}(4k-2) < \frac{16}{k} \leq 1.$$

$$\Pr[R \text{ independent}] = \Pr[\text{none of } (E_{\{u,v\}})_{u,v}] \overset{\text{LLL}}{>} 0.$$

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Proof of LLL using the Claim.

$\Pr[\text{none of } \mathcal{E}] = \prod_{i=1}^{n} \Pr[\bar{E}_i \mid \text{none of } \{E_1, \ldots, E_{i-1}\}] \geq (1-2p)^n \overset{4pd \leq 1}{>} 2^{-n} > 0.$ □

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- Base case: If $|S| = 0$ then $\Pr[E^* \mid \text{none of } \varnothing] = \Pr[E^*] \leq p \leq 2p$. ✓ Let now $|S| > 0$.

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$$\Pr[E^* \mid \text{none of } S] = \frac{\Pr[E^* \wedge \text{none of } S]}{\Pr[\text{none of } S]} \leq \frac{\Pr[E^* \wedge \text{none of } I]}{\Pr[\text{none of } D \mid \text{none of } I]\Pr[\text{none of } I]}$$

$$= \frac{\Pr[E^*]\Pr[\text{none of } I]}{\Pr[\text{none of } D \mid \text{none of } I]\Pr[\text{none of } I]} = \frac{p}{\Pr[\text{none of } D \mid \text{none of } I]} \qquad \square$$

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Claim: $\forall S \subseteq \mathcal{E} : \forall E^* \in \mathcal{E} \setminus S : \Pr[E^* \mid \text{none of } S] \leq 2p$.

Proof of the Claim by Induction on $|S|$.

- Base case: If $|S| = 0$ then $\Pr[E^* \mid \text{none of } \varnothing] = \Pr[E^*] \leq p \leq 2p$. ✓ Let now $|S| > 0$.
- Partition $S = I \mathbin{\dot\cup} D$ such that E^* is independent of I and $1 \leq |D| \leq d$. // $\leq d$ possible by assumption, > 0 is our choice.
- $\Pr[\text{none of } D \mid \text{none of } I] = 1 - \Pr[\bigcup_{E \in D} E \mid \text{none of } I] \overset{\text{UB}}{\geq} 1 - \sum_{E \in D} \underbrace{\Pr[E \mid \text{none of } I]}_{\leq 2p \text{ (Induction, using } |I| < |S|)} \geq 1 - 2dp \overset{4pd \leq 1}{\geq} \frac{1}{2}$. (☆).

$$\Pr[E^* \mid \text{none of } S] = \frac{\Pr[E^* \wedge \text{none of } S]}{\Pr[\text{none of } S]} \leq \frac{\Pr[E^* \wedge \text{none of } I]}{\Pr[\text{none of } D \mid \text{none of } I]\Pr[\text{none of } I]}$$

$$= \frac{\Pr[E^*]\Pr[\text{none of } I]}{\Pr[\text{none of } D \mid \text{none of } I]\Pr[\text{none of } I]} = \frac{p}{\Pr[\text{none of } D \mid \text{none of } I]}$$

□

Proof of Lovász Local Lemma

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Lovász Local Lemma (László Lovász and Paul Erdős, 1973)

If each $E \in \mathcal{E}$ has $\Pr[E] < p$ and depends on at most d events from $\mathcal{E}$ and $4pd \le 1$ then $\Pr[\text{none of } \mathcal{E}] > 0$.

Claim: $\forall S \subseteq \mathcal{E} : \forall E^* \in \mathcal{E} \setminus S : \Pr[E^* \mid \text{none of } S] \le 2p$.

Proof of the Claim by Induction on $|S|$.

- Base case: If $|S| = 0$ then $\Pr[E^* \mid \text{none of } \varnothing] = \Pr[E^*] \le p \le 2p$. ✓ Let now $|S| > 0$.
- Partition $S = I \,\dot\cup\, D$ such that E^* is independent of I and $1 \le |D| \le d$. // $\le d$ possible by assumption, > 0 is our choice.
- $\Pr[\text{none of } D \mid \text{none of } I] = 1 - \Pr[\bigcup_{E \in D} E \mid \text{none of } I] \overset{\text{UB}}{\ge} 1 - \sum_{E \in D} \underbrace{\Pr[E \mid \text{none of } I]}_{\le 2p \text{ (Induction, using } |I| < |S|)} \ge 1 - 2dp \overset{4pd \le 1}{\ge} \frac{1}{2}.$ (☆).

$$\Pr[E^* \mid \text{none of } S] = \frac{\Pr[E^* \wedge \text{none of } S]}{\Pr[\text{none of } S]} \le \frac{\Pr[E^* \wedge \text{none of } I]}{\Pr[\text{none of } D \mid \text{none of } I]\Pr[\text{none of } I]}$$

$$= \frac{\Pr[E^*]\Pr[\text{none of } I]}{\Pr[\text{none of } D \mid \text{none of } I]\Pr[\text{none of } I]} = \frac{p}{\Pr[\text{none of } D \mid \text{none of } I]} \overset{(☆)}{\le} \frac{p}{1/2} = 2p. \qquad \square$$

Summary

What the Probabilistic Method is all About

- Goal: Prove the existence of objects with certain properties.
- Use probabilistic language as a tool.

Vanilla Variant:

Goal: Show that $P \subseteq \Omega$ is not empty.

1. Define a random object $X \in \Omega$.
2. Show: $\Pr[X \in P] > 0$.
3. Conclude: $\exists x \in \Omega : x \in P$.

Variant with Expectation Argument

Goal: Show that $f : \Omega \to \mathbb{R}$ has maximum at least q.

1. Define a random object $X \in \Omega$.
2. Show: $\mathbb{E}[f(X)] \geq q$.
3. Conclude: $\exists x \in \Omega : f(x) \geq q$.

Variant with Lovász Local Lemma

Goal: Show that $P \subseteq \Omega$ is not empty.

1. Define random object X.
2. Define family $\mathcal{E}$ of bad events such that $\bigcap_{E \in \mathcal{E}} \bar{E} \Rightarrow X \in P$.
3. Show that $E \in \mathcal{E}$ satisfies $\Pr[E] \leq p$.
4. Show $E \in \mathcal{E}$ depends on at most d other events from $\mathcal{E}$.
5. Show $4dp \leq 1$.
6. Conclude with LLL: $\exists x : x \in P$.

Anhang: Mögliche Prüfungsfragen I

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- Was ist das Ziel der probabilistischen Methode?
- Bezüglich der grundlegenden Methode:
  - Welche “Kreativleistung” muss man erbringen und was muss man dann ausrechnen?
  - Verdeutliche die Methode an einem Beispiel.
- Bezüglich der Variante mit Erwartungswertargument:
  - Welche “Kreativleistung” muss man erbringen und was muss man dann ausrechnen?
  - Verdeutliche die Methode an einem Beispiel.
  - Wir haben gezeigt, dass jeder Graph einen Schnitt von Gewicht $|E|/2$ besitzt. Wie?
  - Wir haben gezeigt, dass jeder Graph eine unabhängige Menge der Größe $\frac{n^2}{4m}$ besitzt. Wie?
- Bezüglich Lovász Local Lemma:
  - Formuliere die Aussage des Lemmas.
  - Was ist der Bezug zur probabilistischen Methode?
  - Wir haben gezeigt, dass gefärbte Graphen unabhängige Regenbogenmengen gewisser Größe besitzen. Wie sind wir vorgegangen?