Karlsruhe Institute of Technology

Probability & Computing

Probability Amplification

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✗	true neg	false neg
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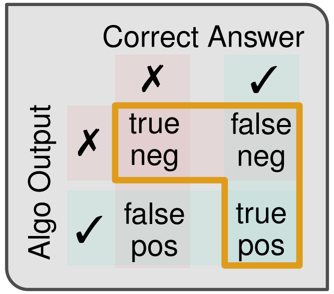

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Probability Amplification for true-biased algorithms

- Execute independently t times.
  - If ✓ at least once: Return ✓. (surely correct)
  - Otherwise: Return ✗. $\Pr[\text{“correct”}] \geq 1 - (1-p)^t \geq 1 - e^{-pt}$

$1 + x \leq e^x$ for $x \in \mathbb{R}$

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Probability Amplification for optimization algorithms

- Execute independently t times.
  - output best result

$$\Pr[\text{“optimal”}] \geq 1 - (1-p)^t \geq 1 - e^{-pt}$$

The Segmentation Problem

Input

- Set P of points in a feature space (e.g., $\mathbb{R}^d$)
- Similarity measure $\sigma\colon P \times P \mapsto \mathbb{R}_+$

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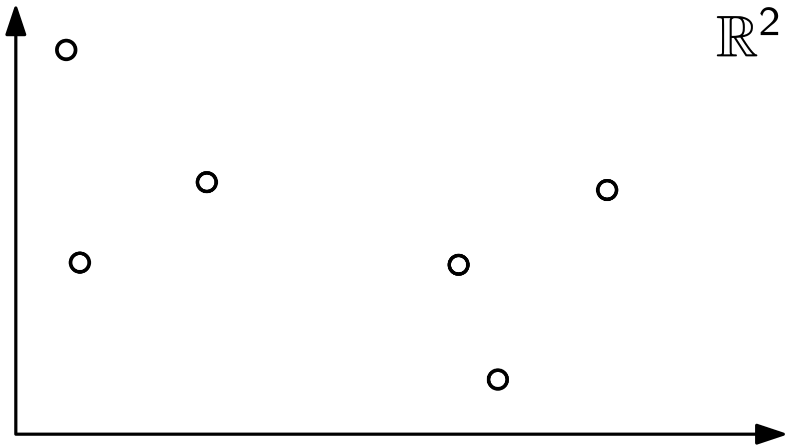

Example

- six points in $\mathbb{R}^2$
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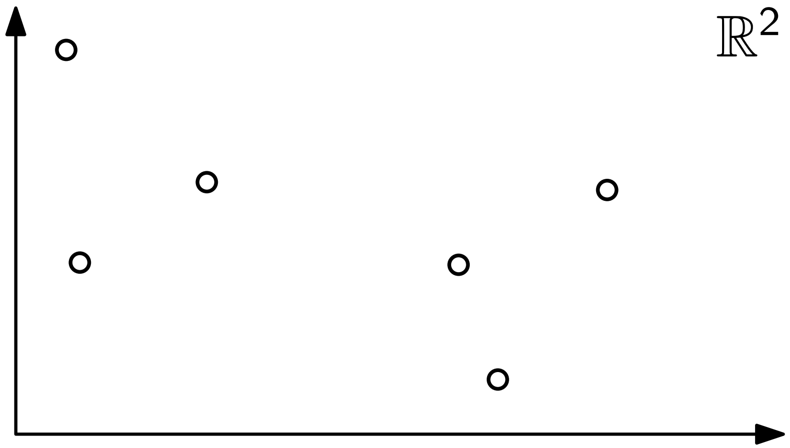

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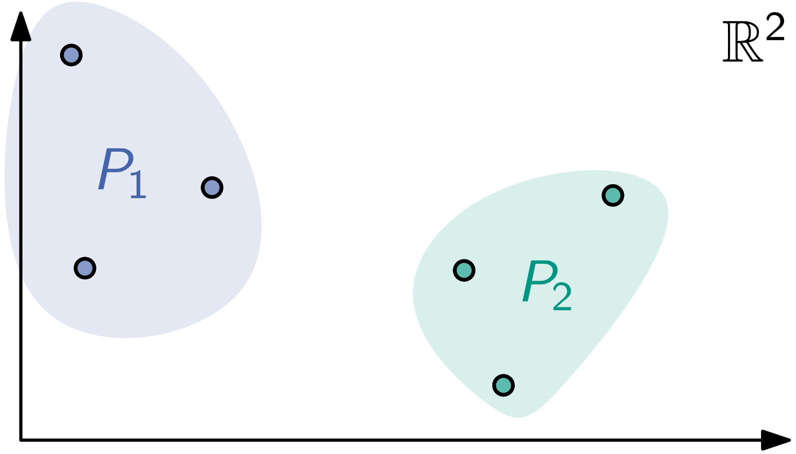

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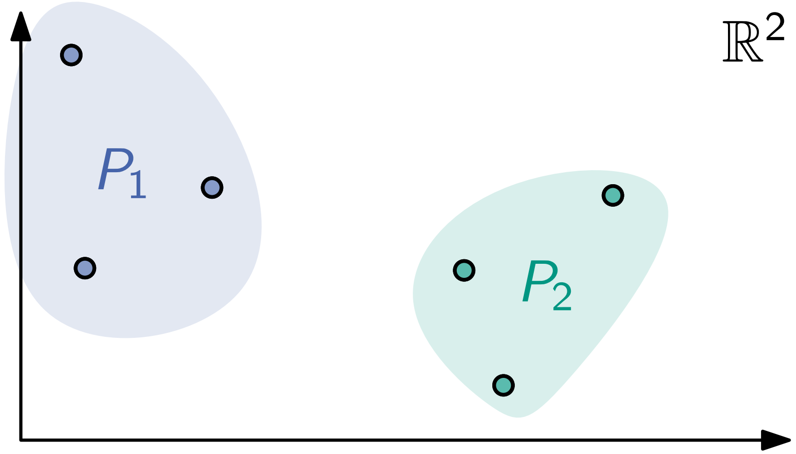

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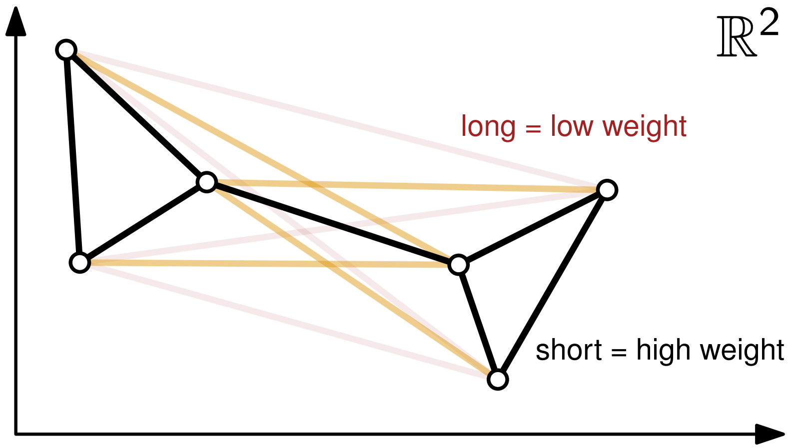
- Edges between all node pairs, with the weight given by the similarity of the two nodes



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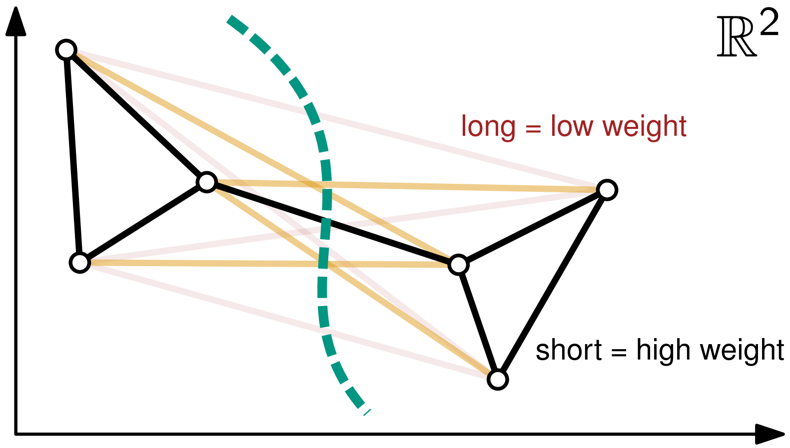

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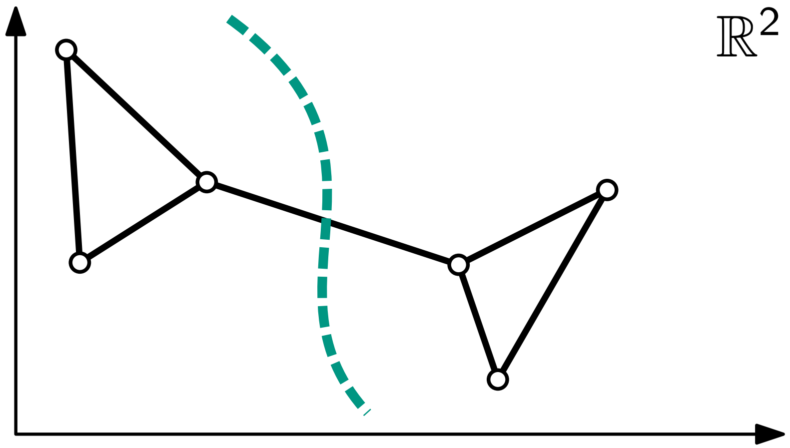

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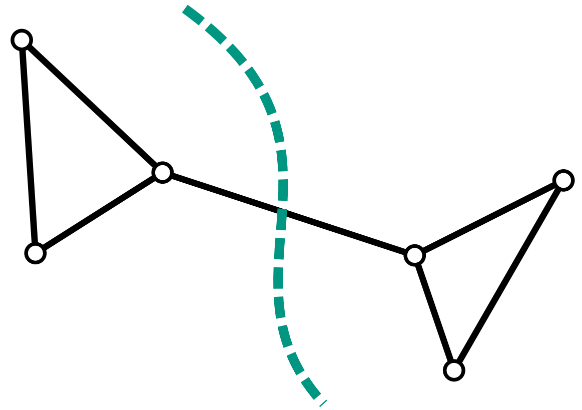
Today

$k = 2$ and $\sigma\colon P \times P \mapsto \{0, 1\}$

Computing Min Cuts

Cuts

- $G = (V, E)$ an unweighted, undirected, connected graph
- *Cut*: partition of V into non-empty parts V_1, V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$.



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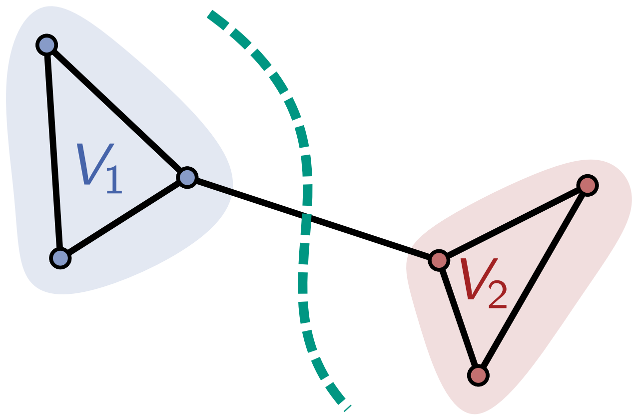

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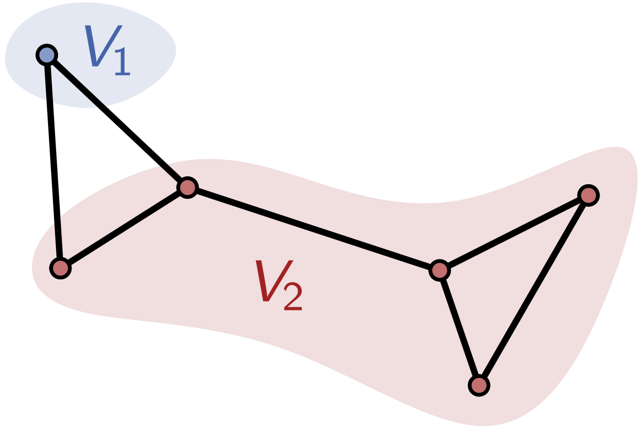

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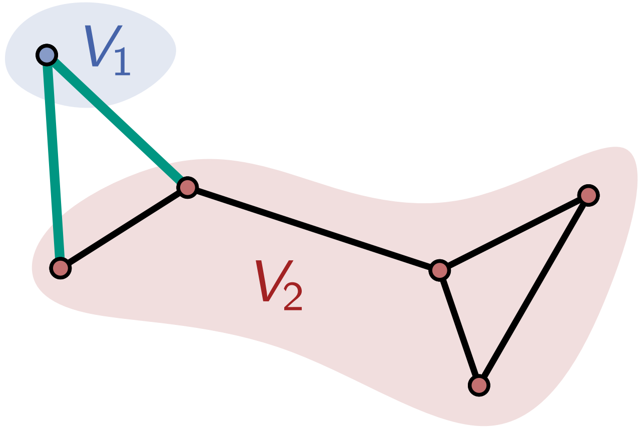

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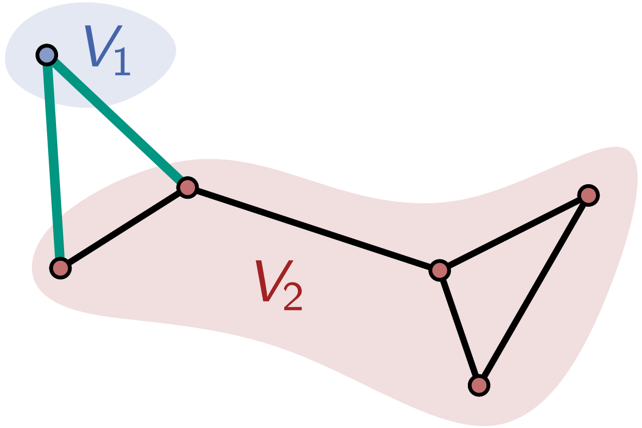

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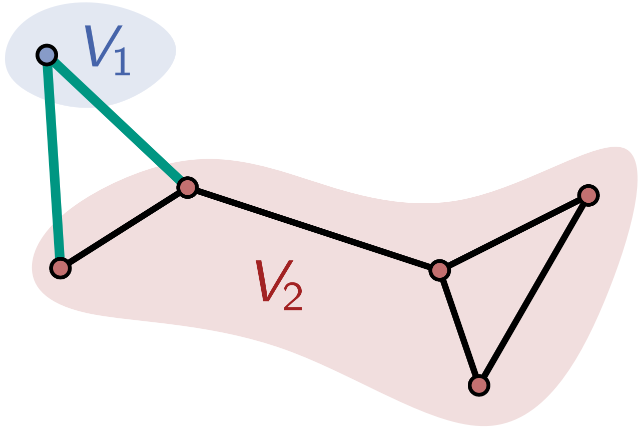

Today Goal: Compute a Min-Cut

i.e. a cut of minimum weight or cut-set of minimum size
the weight of the min-cut is known as the edge-connectivity of G

- Known deterministic strategies have worst case running time $\Omega(n^3)$.
- We'll see randomised algorithm with running time $O(n^2 \cdot \log^3(n))$.

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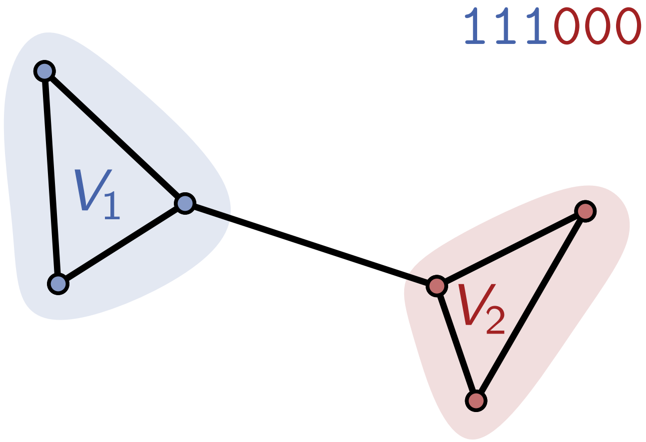

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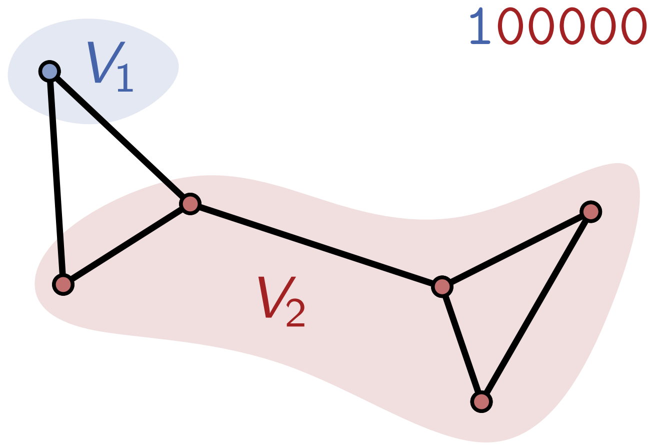

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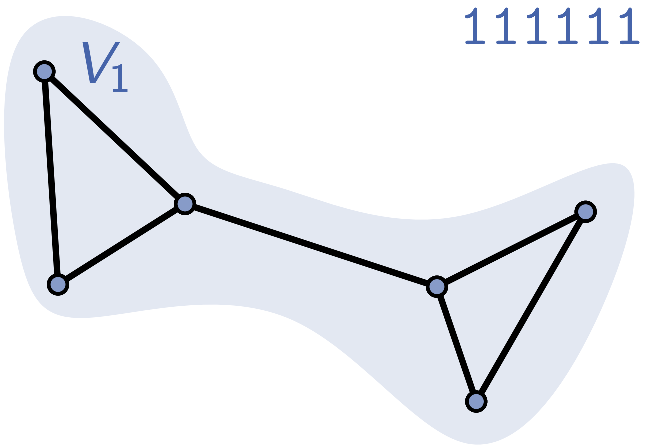

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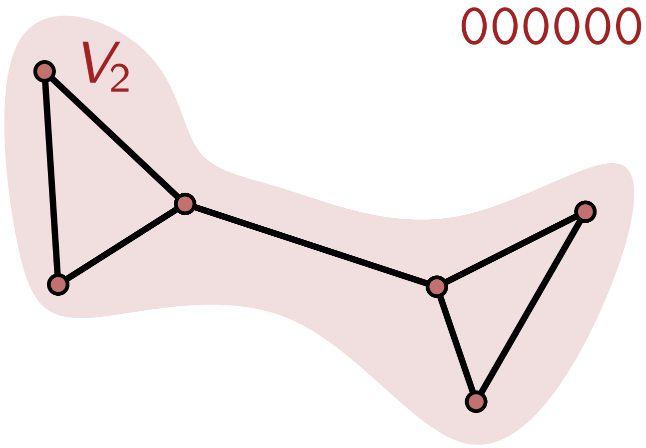

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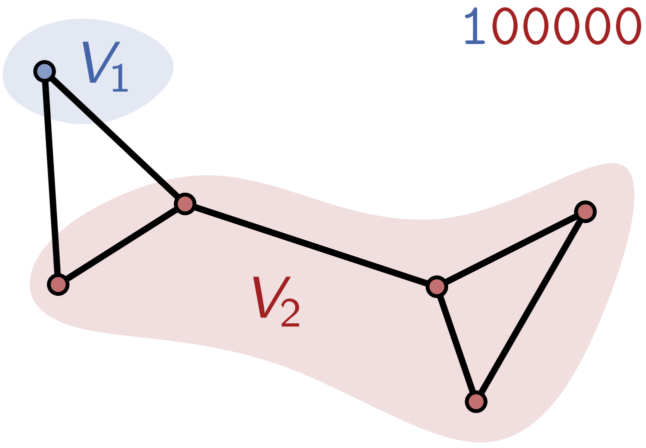

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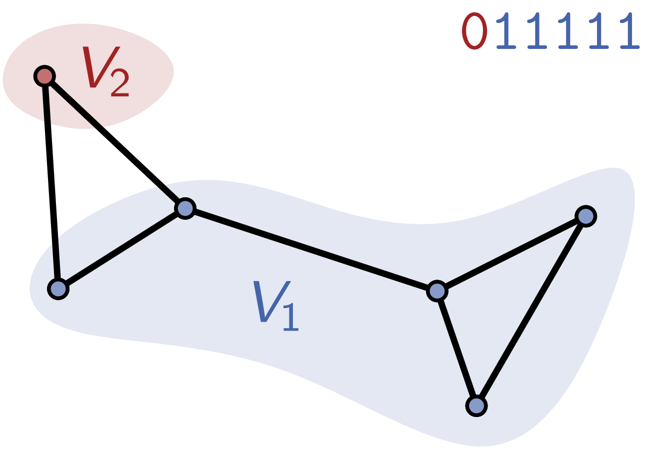

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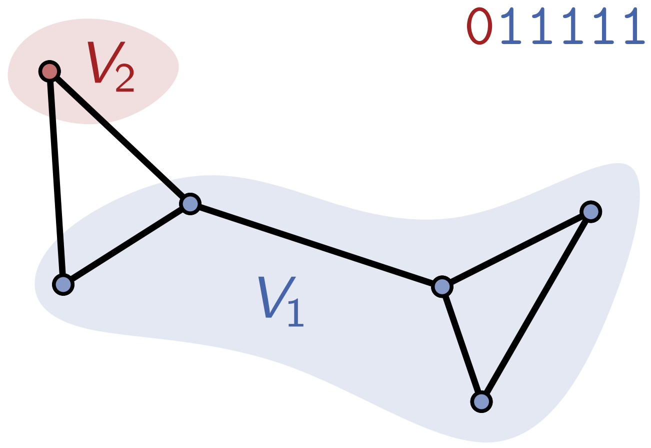

Algorithm: Random Cut

- Return a uniformly random cut.

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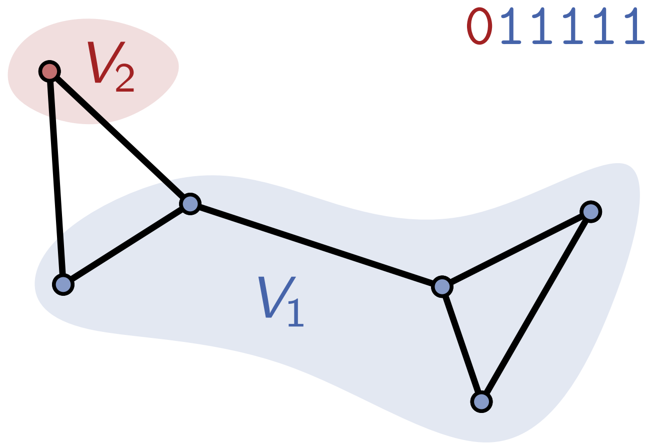

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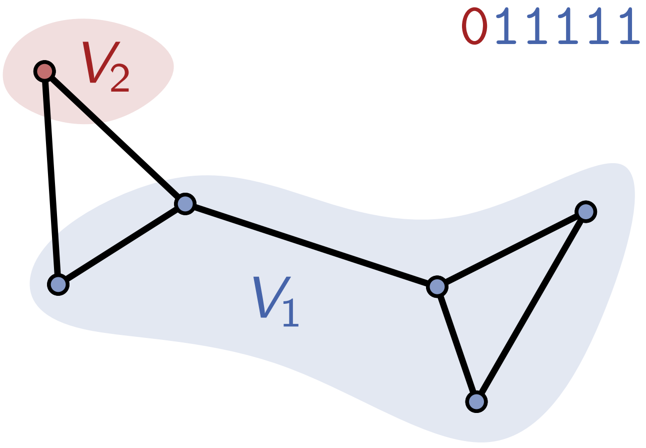

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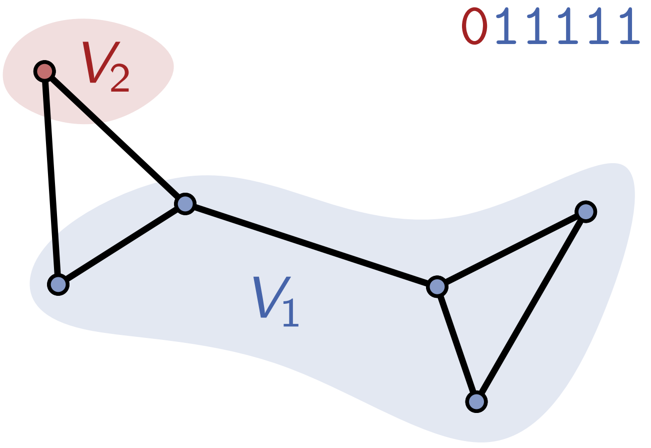

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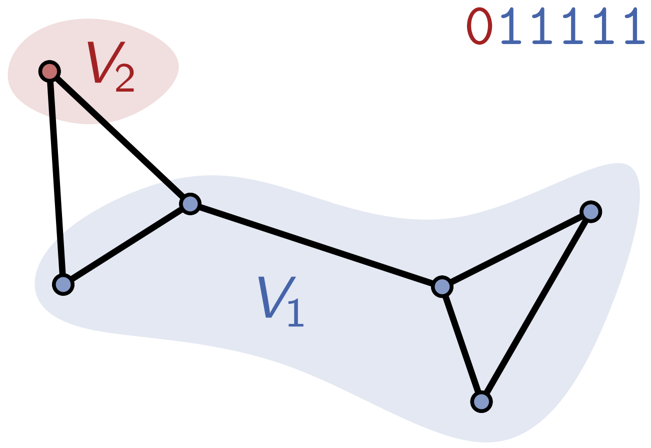

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- Number of possible assignments of n nodes to 2 parts (2^n)
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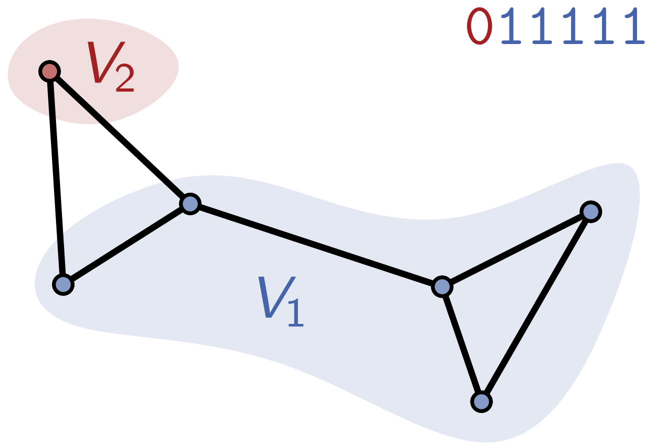
- Swapping parts does not yield a new partition ($/2$)



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    - actually for bounded running time: declare failure rather than sampling again
    - samples each cut with probability $1/2^{n-1}$

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- Repeat the algorithm to obtain t independent random cuts, return the smallest

$$\Pr[\text{"min cut found"}] \geq 1 - \left(1 - 1/2^{n-1}\right)^t$$

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$$\Pr[\text{“min cut found”}] \geq 1 - \left(1 - \underbrace{1/2^{n-1}}_{\text{minimum}}\right)^t$$

Random Cut: Analysis

Running time: $O(n)$ much better than the $\Omega(n^3)$ in the deterministic setting , but...

Success probability: $\geq 1/2^{n-1}$ "$=$" if there is only one min-cut.

$\rightarrow$ exponentially small!

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- Repeat the algorithm to obtain t independent random cuts, return the smallest

$$\Pr[\text{"min cut found"}] \geq \underbrace{1 -}_{\text{not}} \left(1 - \underbrace{1/2^{n-1}}_{\text{minimum}}\right)^t$$

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Amplification

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$$\Pr[\text{“min cut found”}] \geq \underbrace{1 -}_{\text{not}} \left(1 - \underbrace{1/2^{n-1}}_{\text{minimum}}\right)^{\overset{t}{\downarrow\, t \text{ times}}}$$

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$$\Pr[\text{"min cut found"}] \geq 1 - \left(1 - 1/2^{n-1}\right)^t \geq 1 - e^{-t/2^{n-1}}$$

$1 + x \leq e^x$ for $x \in \mathbb{R}$

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- For $t = 2^{n-1}$ min cut found with constant probability $1 - 1/e \approx 0.63$

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- For $t = 2^{n-1}$ min cut found with constant probability $1 - 1/e \approx 0.63$
- For $t = 2^{n-1} \cdot \ln(n)$ min cut found with high probability $1 - 1/n$

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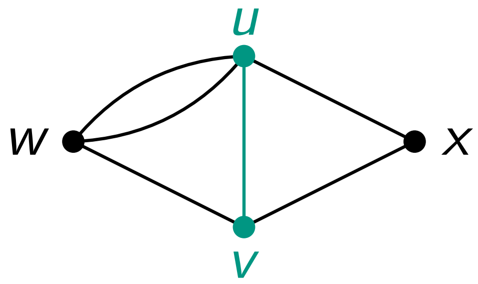
this is terrible so far...

Karger's Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops



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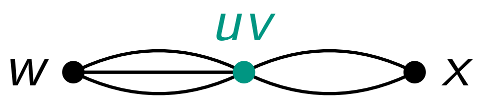

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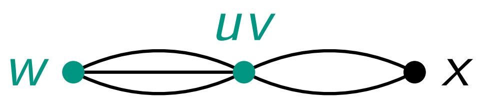

Karger’s Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops
- A (multi) graph with two nodes has a unique cut-set

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- Motivation: distinguish *non-essential* (not part of a min-cut) as well as *essential* edges (part of a min-cut) & hope there are few essential ones

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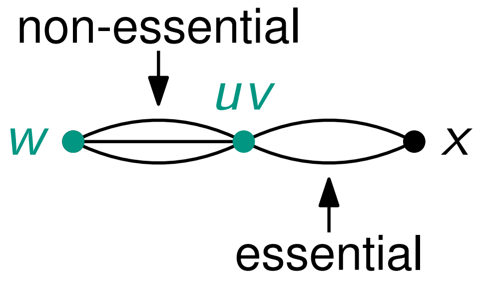

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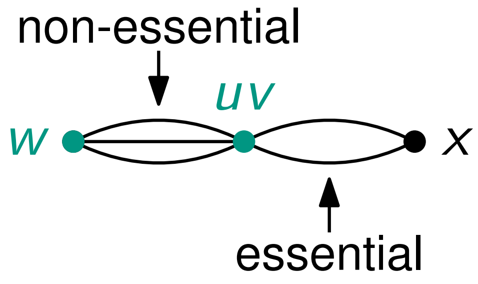

Contraction Algorithm

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Karger$(G_0 = (V_0, E_0))$

for $i = 1$ to $n - 2$ **do**

  $e := \mathcal{U}(E_{i-1})$

  $G_i = G_{i-1}$.**contract**(e)

return unique cut-set in G_{n-2}

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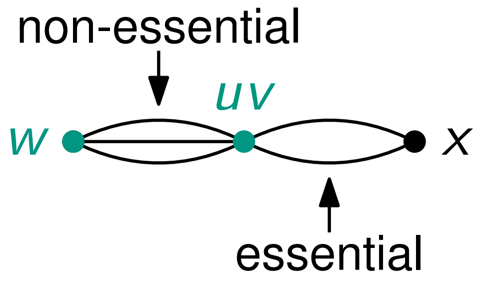

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  for $i = 1$ to $n - 2$ **do** // $O(n)$

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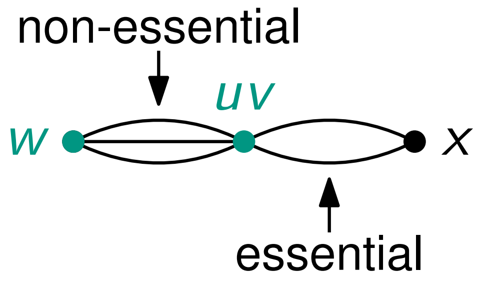

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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

(the converse does not hold)

- Let C be a cut-set in G_i.
  - $G_i \setminus C$ is disconnected
- Assume C is not a cut-set in G_0.
  - $G_0 \setminus C$ is connected.
- $G_i \setminus C$ arises from $G_0 \setminus C$ by i edge contractions.
- $\lightning$ contractions cannot disconnect a graph

Karger's Algorithm

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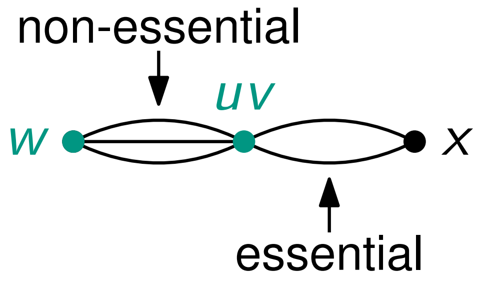

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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

- Consider min-cut in G_0 with cut-set C and $|C| = k$

Karger's Algorithm

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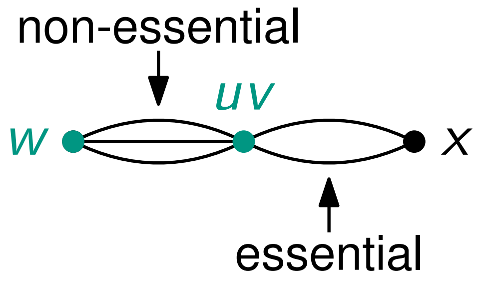

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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i =$ "C in G_i"

$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m}$

Karger's Algorithm

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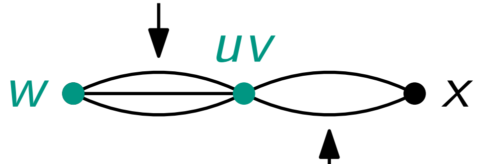

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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0. ⚡

- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i$ = "C in G_i"

Observation: min-degree $\geq k$

$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m}$

Karger's Algorithm

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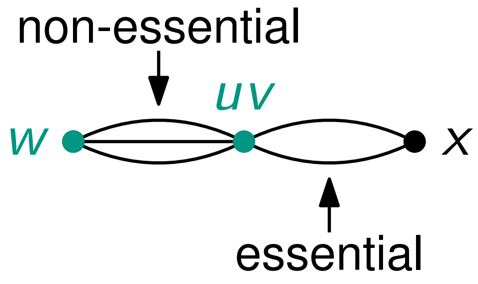

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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0. ↯

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- $\mathcal{E}_i =$ "C in G_i"

Observation: min-degree $\geq k$

(holds for all G_i due to 1st observation)

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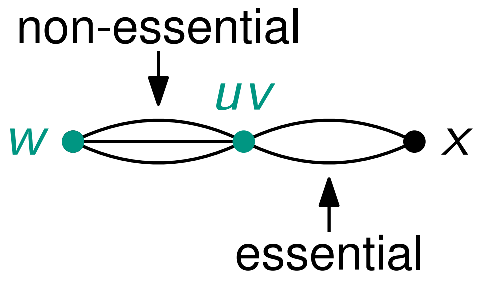

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- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i =$ "C in G_i"

Observation: min-degree $\geq k$

(holds for all G_i due to 1st observation)

$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m}$

$$m = \frac{1}{2}\sum_{v \in V} \deg(v) \geq \frac{1}{2}\sum_{v \in V} k \geq \frac{1}{2}nk$$

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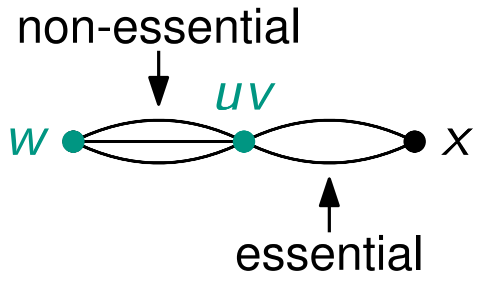

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- **for** $i = 1$ to $n-2$ **do** // $O(n)$
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- Running time in $O(n^2)$
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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i =$ "C in G_i"

Observation: min-degree $\geq k$

(holds for all G_i due to 1st observation)

$$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m} \geq 1 - \frac{k}{nk/2} = 1 - \frac{2}{n}$$

$$m = \frac{1}{2}\sum_{v \in V} \deg(v) \geq \frac{1}{2}\sum_{v \in V} k \geq \frac{1}{2} nk$$

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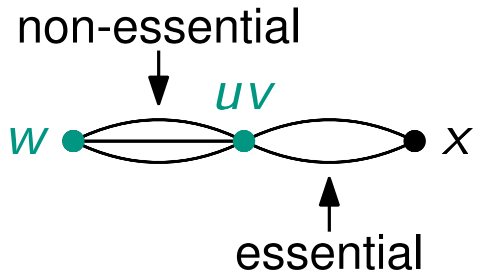

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- **for** $i = 1$ to $n-2$ **do** // $O(n)$
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Observation: min-degree $\geq k$

(holds for all G_i due to 1st observation)

$\Pr[\mathcal{E}_1] \geq 1 - \frac{2}{n}$

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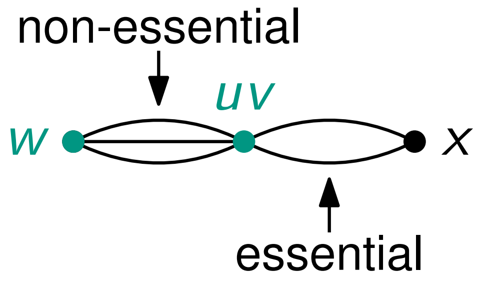

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```
for i = 1 to n − 2 do            // O(n)
    e := U(E_{i−1})              // O(1)
    G_i = G_{i−1}.contract(e)    // O(n)
return unique cut-set in G_{n−2}
```

- Running time in $O(n^2)$
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- $\mathcal{E}_i =$ "C in G_i"

Observation: min-degree $\geq k$

(holds for all G_i due to 1st observation)

$\Pr[\mathcal{E}_1] \geq 1 - \frac{2}{n}$

$\Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \geq 1 - \frac{2}{n-1}$

- none of the k edges of C contracted
- do not contract k edges in an $n-1$-node graph

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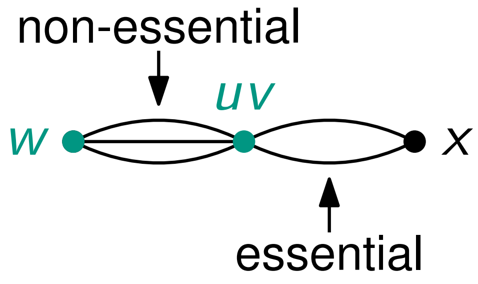

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$\Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \geq 1 - \frac{2}{n-1} \longrightarrow \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}] \geq 1 - \frac{2}{n-i+1}$

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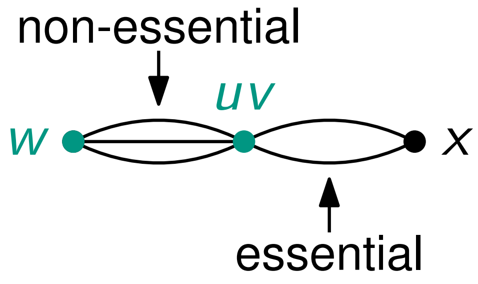

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$$\underbrace{\Pr[\mathcal{E}_{n-2}] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_{n-2} \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{n-3}]}_{\text{chain rule of probability}}$$

Karger's Algorithm

Karlsruhe Institute of Technology

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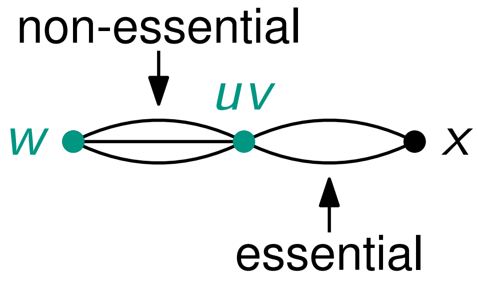

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Karger's Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops
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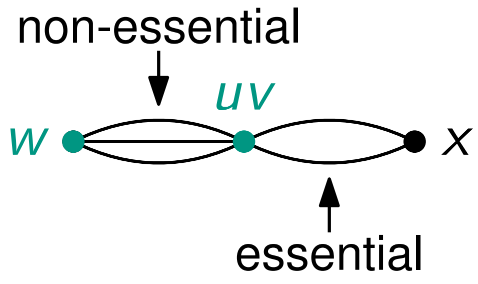

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$1 = \frac{n-i}{n-i}$

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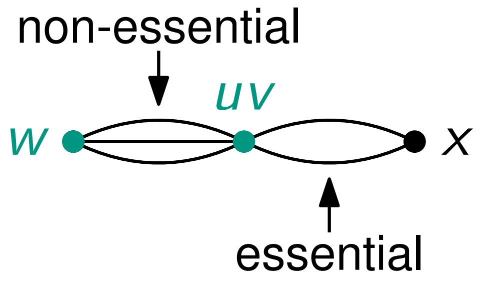

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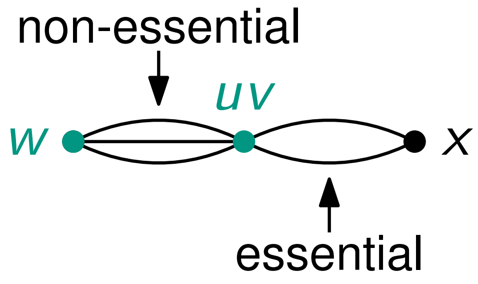

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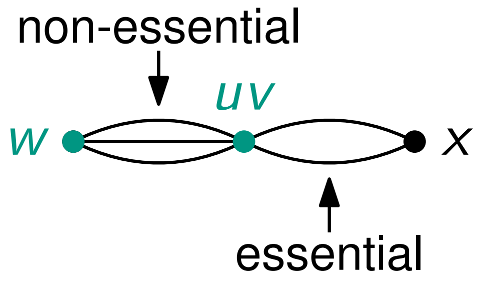

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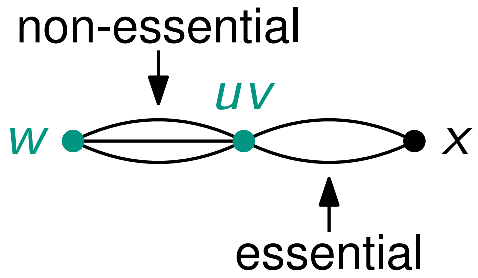

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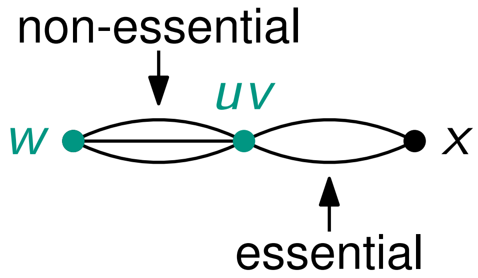

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Karlsruhe Institute of Technology

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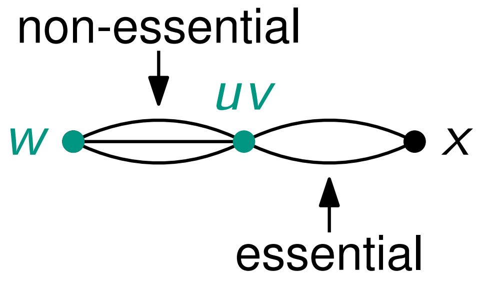

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$$= \frac{2}{n(n-1)} \geq \frac{2}{n^2}$$

Karger's Algorithm Amplified

Karlsruhe Institute of Technology

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Number of repetitions t

Amplified prob. $\geq 1 - e^{-pt}$

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More Amplification: Karger-Stein

Karlsruhe Institute of Technology

Motivation

- Probability that a min-cut survives i contractions

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- Probability that a min-cut survives i contractions

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Karlsruhe Institute of Technology

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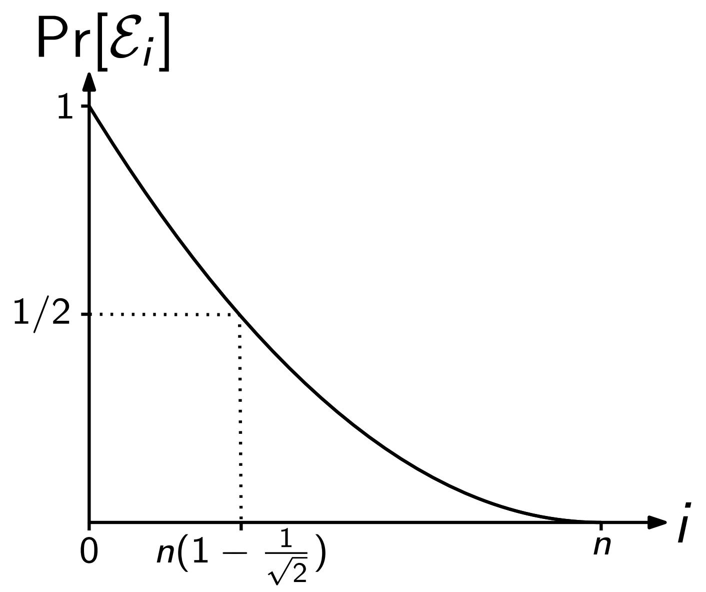

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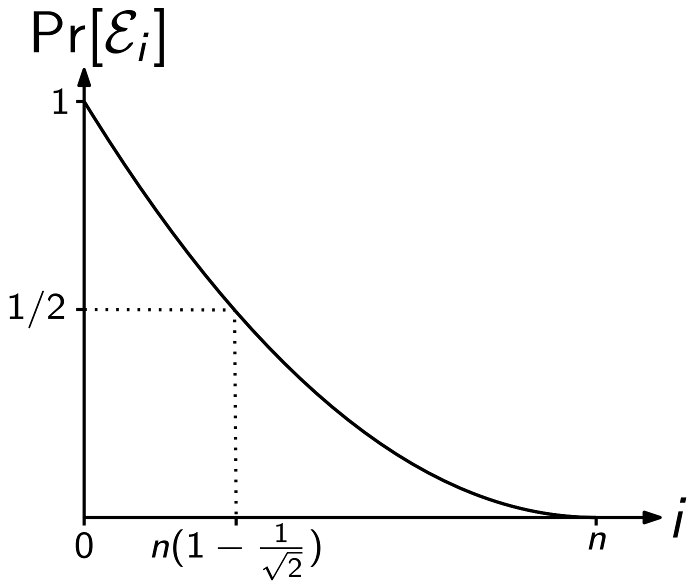

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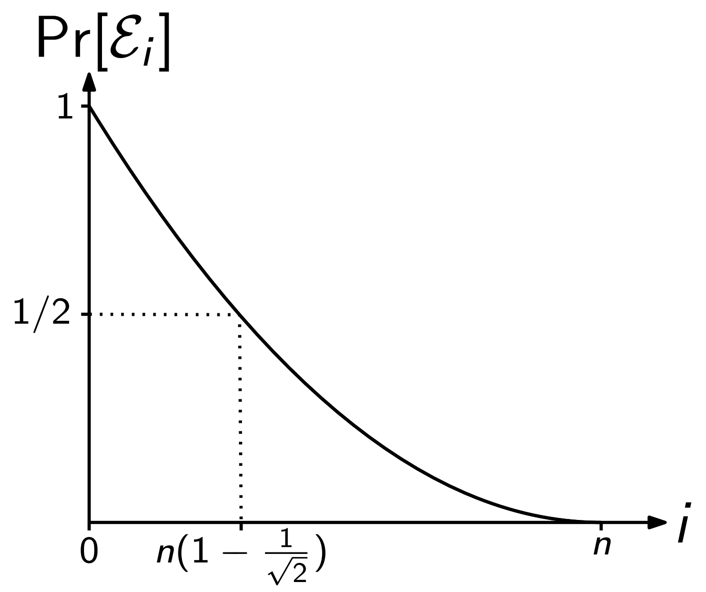

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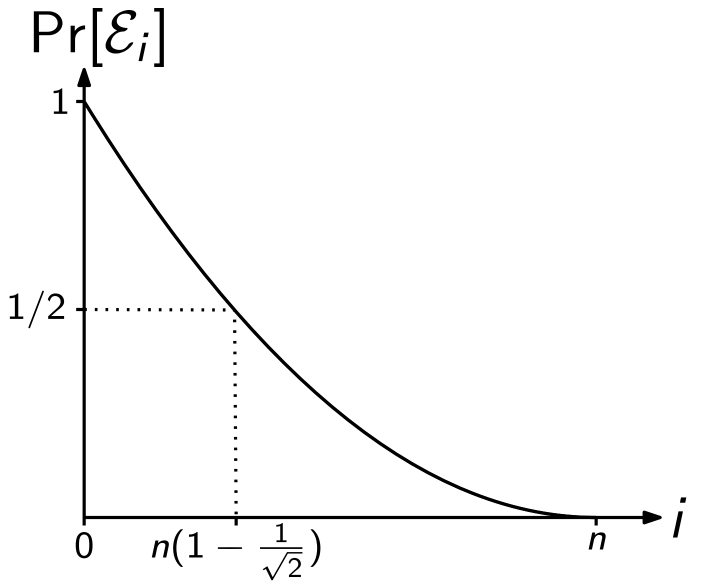

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KargerStein$(G_0 = (V_0, E_0))$
- **if** $|V_0| = 2$ **then** return unique cut-set
- **for** $i = 1$ to $s = |V_0| - \frac{|V_0|}{\sqrt{2}} - 1$ **do**
  - $e := \mathcal{U}(E_{i-1})$
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- $C_1 :=$ **KargerStein**(G_s) // inde-
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Karger-Stein: Running Time

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Karger-Stein: Running Time

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Solution (essentially by Master Theorem)

$$T(n) = O(n^2 \log n)$$

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Know: Each call to Karger-Stein breaks the min-cut with probability at most $\frac{1}{2}$.

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Given complete binary tree of height d where each node is randomly coloured red or green (with probability $\frac{1}{2}$ each).
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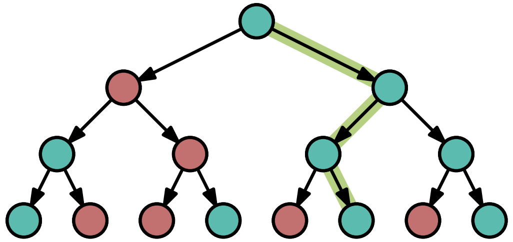

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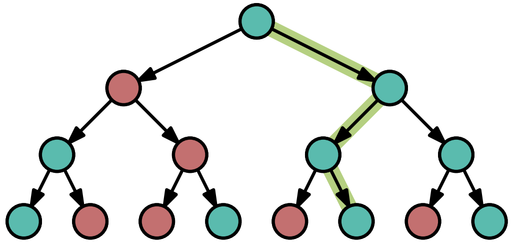

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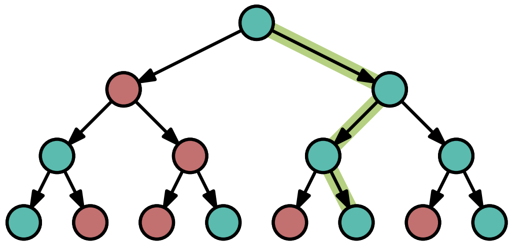

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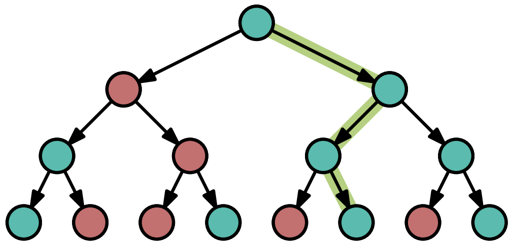

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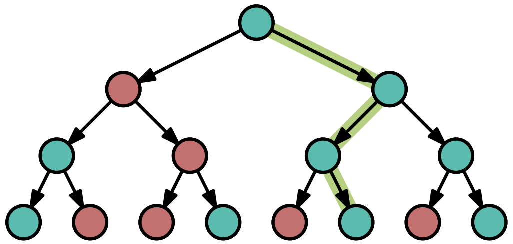

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Corollary: Karger-Stein succeeds with probability at least $p_{\log_{\sqrt{2}}(n)} = \frac{1}{O(\log n)}$.

Karger-Stein Amplified

Theorem: On a graph with n nodes, Karger-Stein runs in $O(n^2 \log(n))$ time and returns a minimum cut with probability at least $1/O(\log(n))$.

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Success probability $\geq p$

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$$\Pr[\text{“min-cut found”}] \geq 1 - \exp\left(-\frac{t}{O(\log(n))}\right) = 1 - O\left(\frac{1}{n}\right)$$

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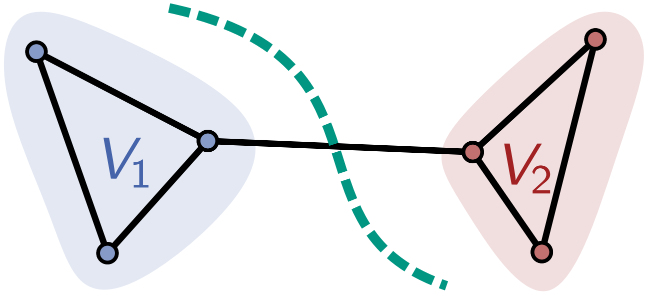
- Compared to $O(n^4 \log(n))$ for Karger
- Compared to $\Omega(n^3)$ for deterministic approaches

Conclusion

Minimum Cut

- Fundamental graph problem
- Many deterministic flow-based algorithms ...
- . . . with worst-case running times in $\Omega(n^3)$



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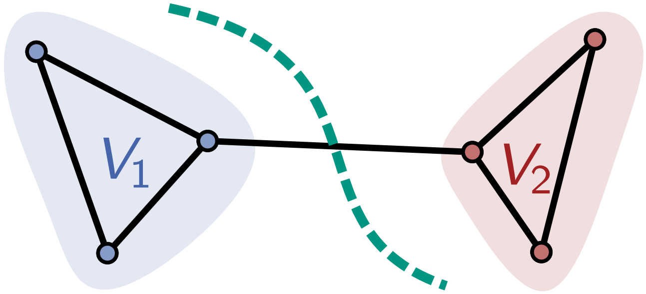
- Karger's edge-contraction algorithm



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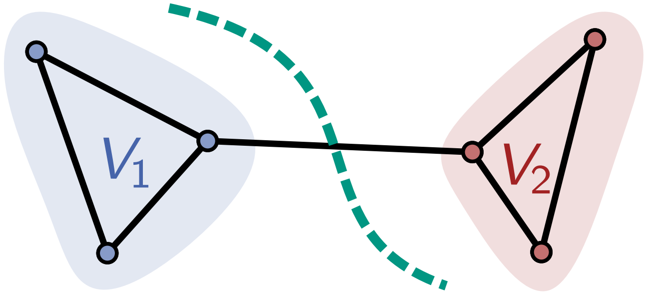

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- Monte Carlo algorithms with and without biases
- Repetitions amplify success probability
- Karger-Stein: Amplify before failure probability gets large

Algo Output \ Correct Answer	✗	✓
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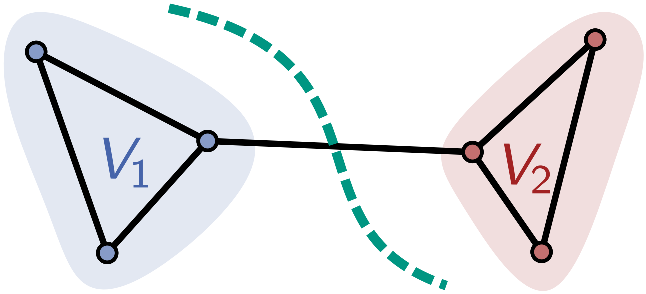

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Outlook

"Minimum cuts in near-linear time", Karger, J.Acm. '00

Success w.h.p. in time $O(m \log^3(n))$

"Faster algorithms for edge connectivity via random 2-out contractions", Ghaffari & Nowicki & Thorup, SODA'20

Success w.h.p. in time $O(m \log(n))$ and $O(m + n \log^3(n))$

Mögliche Prüfungsfragen

- Was ist ein Monte-Carlo-Algorithmus?
  - Welche Varianten gibt es?
- Was versteht man unter Probability Amplification?
- Wie funktioniert Probability Amplification...
  - ... bei einseitigem Fehler?
  - ... bei zweiseitigem Fehler?
  - ... bei Optimierungsproblemen?
  - Wie hängt die Fehlerwahrscheinlichkeit mit der Anzahl Wiederholungen zusammen?
- Was ist das Minimum Cut Problem?
  - Was leisten die besten bekannten deterministischen Algorithmen?
  - Was sind Erfolgswahrscheinlichkeit und Laufzeit des trivialen Random Cut Algorithmus?
  - Wie funktioniert der Algorithmus von Karger?
    - Was bedeutet $\Pr[\mathcal{E}_t]$ und wie haben wir diese Wahrscheinlichkeit abgeschätzt?
    - Was ergibt sich für die Laufzeit und die Erfolgswahrscheinlichkeit?
  - Wie ergibt sich der Algorithmus von Karger und Stein aus dem Algorithmus von Karger?
    - Wie haben wir die Erfolgswahrscheinlichkeit und Laufzeit abgeschätzt?
    - Wie erreiche ich eine Erfolgswahrscheinlichkeit von $1 - \frac{1}{n}$?