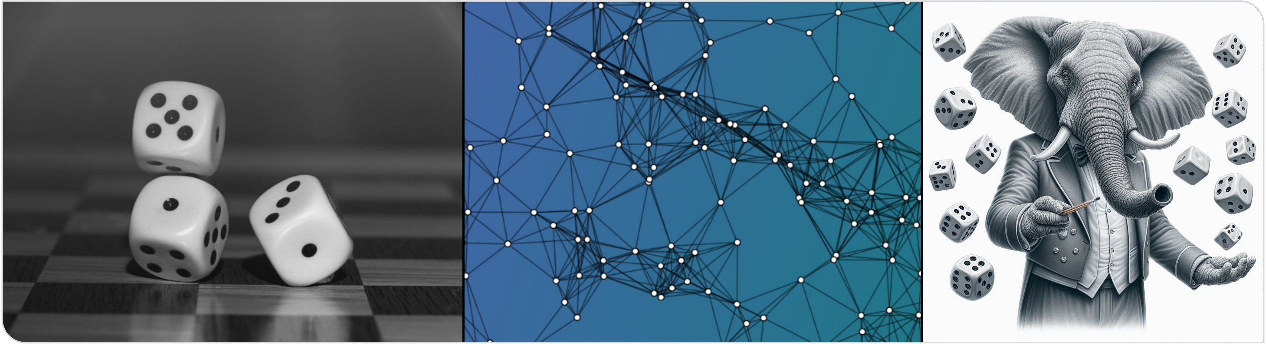
Karlsruhe Institute of Technology

Probability and Computing –
Important Random Variables and How to Sample Them

Stefan Walzer | WS 2024/2025



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2. Bernoulli Distribution
3. Uniform Distribution
4. Rejection Sampling
5. Inverse Transform Sampling
6. Geometric Distribution
7. Sampling Without Replacement
8. Reservoir Sampling

What is a Probability?

Physical Accounts

Probabilities are persistent rates of outcomes when observing the same (random) process over and over again.

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See `https://en.wikipedia.org/wiki/Probability_interpretations`.

In this lecture, we use a naive notion.

Bernoulli Distribution

Definition: $Ber(p)$ for $p \in [0, 1]$

$B \sim Ber(p)$ is a random variable with

$$\Pr[B = 1] = p \text{ and } \Pr[B = 0] = 1 - p.$$

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Standard Assumption: Access to Coin Flips

Algorithms have access to a sequence $B_1, B_2, \ldots \sim Ber(1/2)$ in independent uniformly random bits.

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Exercise: $Ber(1/3)$ from $Ber(1/2)$

Design an algorithm that outputs B such that $B \sim Ber(1/3)$.

Uniform Distribution

Definition: $\mathcal{U}(D)$ on finite D

If $|D| < \infty$, then $X \sim \mathcal{U}(D)$ is a random variable with

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Definition: $\mathcal{U}(D)$ on infinite D

If D is infinite but has finite measure[a] then $X \sim \mathcal{U}(D)$ is a random variable with uniform density function on D.
Important example:

$$X \sim \mathcal{U}([0,1]) \Leftrightarrow \forall x \in [0,1] : \Pr[X < x] = x.$$

[a] Formal details: Not in this lecture.

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Algorithms have access to $X_1, X_2, \ldots \sim \mathcal{U}([0,1])$. In practice: Initialise the significand[a] of floating point number with random bits.

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Exercise: $\mathcal{U}(\{1,\ldots,n\})$ from $\mathcal{U}([0,1])$

Design an algorithm that outputs X such that $X \sim \mathcal{U}(\{1,\ldots,n\})$.

Uniform Distribution on a Disc

Task

Sample $P \sim \mathcal{U}(D)$ for $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

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Flawed Attempt

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sample $\Phi \sim \mathcal{U}([0, 2\pi])$
sample $R \sim \mathcal{U}([0, 1])$
**return** $(R \cdot \cos\Phi, R \cdot \sin\Phi)$
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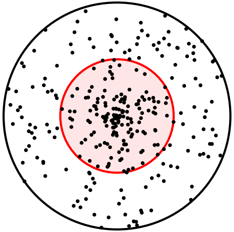

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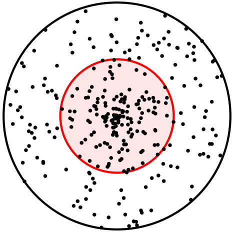

Issue

Disc of half the radius is hit 50% of the time but makes up only $1/4$ of the area!

Uniform Distribution on a Disc with Rejection Sampling

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Solution with Rejection Sampling

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repeat
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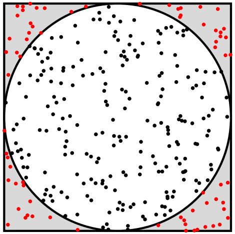

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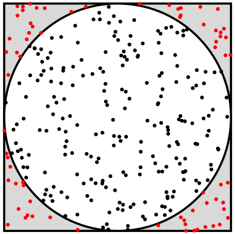

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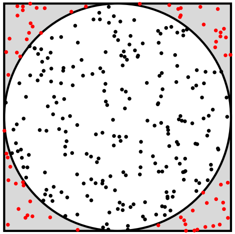

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- Expected number of rounds is $1/(\pi/4) = \mathcal{O}(1)$.

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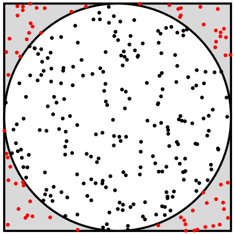

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Spoiler alert: We'll get worst-case constant time with inverse transform sampling later.

Probability? ○ Bernoulli ○ Uniform ○○ Rejection ●○ Inverse Transform ○○ Geometric ○ No Replacement ○ Reservoir ○ Appendix ○○

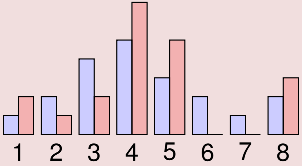
Rejection Sampling in General Discrete Distributions

Exercise

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be distributions on a finite[a] set D. Assume

1. We can sample in constant time from $\mathcal{D}_1$.
2. There exists $C > 0$ such that for any $x \in D$ we have

$$\Pr_{X \sim \mathcal{D}_2}[X = x] \leq C \cdot \Pr_{X \sim \mathcal{D}_1}[X = x].$$



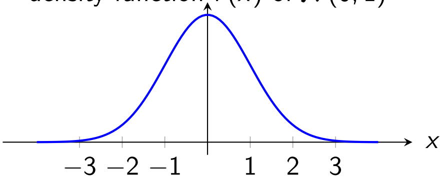
Design an algorithm that generates a sample from $\mathcal{D}_2$ in expected time $\mathcal{O}(C)$.

[a] This can be generalised.

Inverse Transform Sampling

- Let $\mathcal{D}$ be a distribution on $\mathbb{R}$.
  $\hookrightarrow$ e.g. $\mathcal{D} = \mathcal{N}(0,1)$

density function $f(x)$ of $\mathcal{N}(0,1)$



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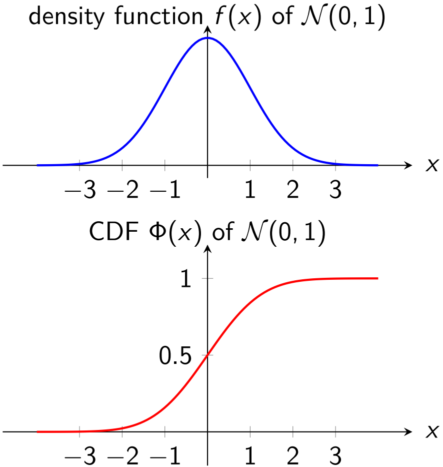
- Let $X \sim \mathcal{D}$ and $F_X(x) = \Pr[X \leq x]$.
  $\hookrightarrow$ F_X is the *cumulative distribution function* of X
  $\hookrightarrow$ the CDF of the normal distribution is called Φ



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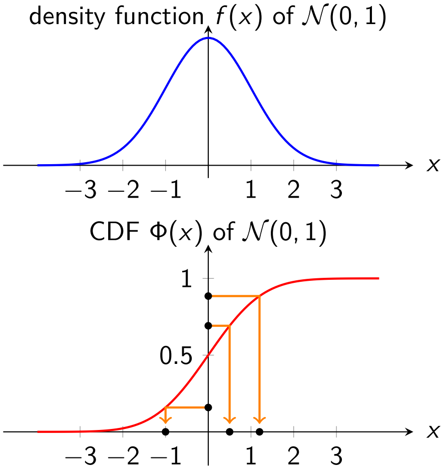
- Let $F_X^{-1}(u) := \inf\{x \in \mathbb{R} \mid F_X(x) \geq u\}$.
  $\hookrightarrow$ ordinary inverse for strictly monotone F_X



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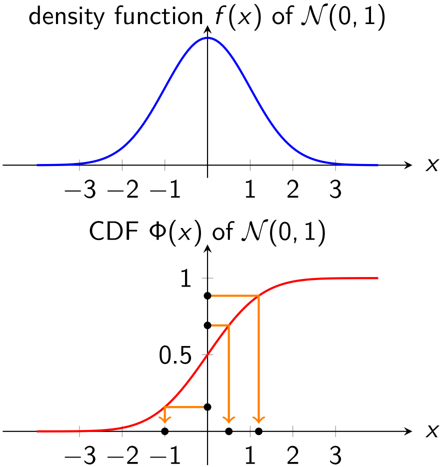
Theorem (Inverse Transform Sampling)

If $U \sim \mathcal{U}([0,1])$ then $F_X^{-1}(U) \overset{\text{d}}{=} X$, i.e. $F_X^{-1}(U) \sim \mathcal{D}$.

("$\overset{\text{d}}{=}$" means: "has the same distribution as")



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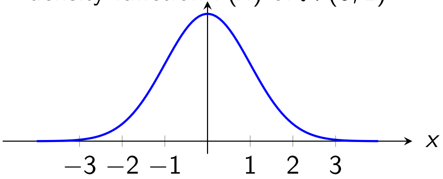
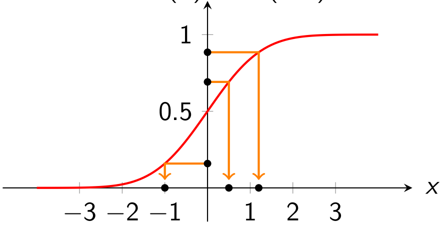
Reason: $\Pr[F_X^{-1}(U) \leq x] = \Pr[U \leq F_X(x)] = F_X(x)$.

density function $f(x)$ of $\mathcal{N}(0, 1)$



CDF $\Phi(x)$ of $\mathcal{N}(0, 1)$



Uniform Distribution on a Disc with Inverse Transform Sampling

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Preparation

If $(x, y) \sim \mathcal{U}(D)$ then $R = \sqrt{x^2 + y^2}$ satisfies

$$F_R(r) = \Pr[R \leq r] = r^2\pi/\pi = r^2 \text{ hence } F_R^{-1}(u) = \sqrt{u}.$$

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Solution with Inverse Transform Sampling

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$R \leftarrow \sqrt{U}$
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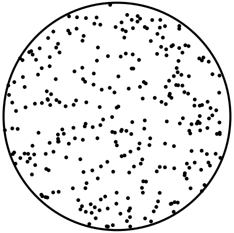

Geometric Distribution

Definition: $G \sim \mathit{Geom}_1(p)$ and $G' \sim \mathit{Geom}_0(p)$

Let $p \in (0,1]$ and $B_1, B_2, \ldots \sim \mathit{Ber}(p)$.
Then we define the geometric random variables

$$G := \min\{i \in \mathbb{N} \mid B_i = 1\}$$

↪ number of $\mathit{Ber}(p)$ trials until (and including) the first success

$$G' := G - 1$$

↪ number of $\mathit{Ber}(p)$ failures before the first success

We write $G \sim \mathit{Geom}_1(p)$ and $G' \sim \mathit{Geom}_0(p)$.[a]

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Sampling $G \sim Geom_1(p)$ in time $\mathcal{O}(G)$

```
$i \leftarrow 0$
repeat
    $i \leftarrow i + 1$
    sample $X \sim Ber(p)$
until $X = 1$
return $i$
```

Quite bad: $\mathbb{E}[G] = 1/p$ might be large.

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Exercise

Use inverse transform sampling to sample $G \sim Geom_1(p)$ in time $\mathcal{O}(1)$.

Sampling Without Replacement

Exercise

Design an algorithm that, given $k, n \in \mathbb{N}$ with $0 \leq k \leq n$ outputs a set $S \subseteq [n]$ of size $|S| = k$ uniformly at random.

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

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Algorithm init($k$):
    allocate reservoir$[1..k]$
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir$[n] \leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir$[I] \leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

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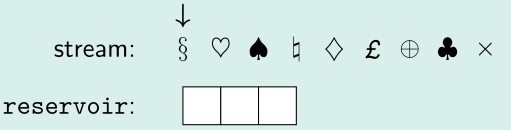

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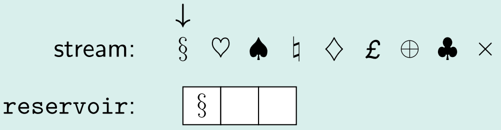

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    **if** $n \leq k$ **then**
        reservoir[$n$] $\leftarrow x$
    **else**
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        **if** $I \leq k$ **then**
            reservoir[$I$] $\leftarrow x$
```
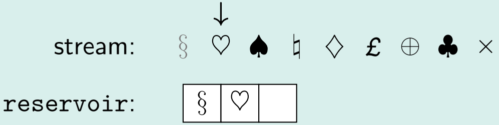

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

reservoir: | § | ♡ | |

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

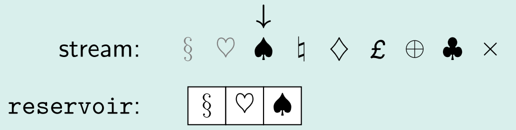
```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

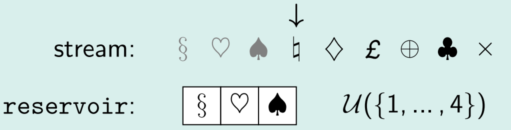
```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$

**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    **if** $n \leq k$ **then**
        reservoir[$n$] $\leftarrow x$
    **else**
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        **if** $I \leq k$ **then**
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
**Algorithm** init($k$):
    allocate reservoir[$1..k$]
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**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    **if** $n \leq k$ **then**
        reservoir[$n$] $\leftarrow x$
    **else**
        sample $I \sim \mathcal{U}(\{1,...,n\})$
        **if** $I \leq k$ **then**
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | § | ♡ | ♠ | $\mathcal{U}(\{1, \dots, 4\}) \leadsto I = 2$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
    $n \leftarrow 0$
Algorithm observeItem(x):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[n] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[I] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: § ♡ ♠ $\overset{\downarrow}{\natural}$ ◇ £ ⊕ ♣ ×

`reservoir`: | § | ♮ | ♠ | $\mathcal{U}(\{1, \dots, 4\}) \rightsquigarrow I = 2$ ✓

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

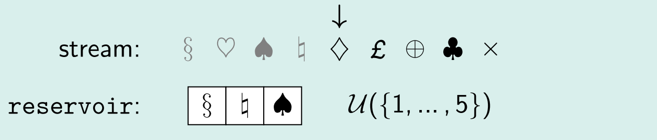
```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $l \sim \mathcal{U}(\{1, \dots, n\})$
        if $l \leq k$ then
            reservoir[$l$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
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Algorithm observeItem(x):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[n] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[I] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: § ♡ ♠ ♮ $\overset{\downarrow}{\diamondsuit}$ £ ⊕ ♣ ×

`reservoir`: | § | ♮ | ♠ | $\quad \mathcal{U}(\{1, \dots, 5\}) \rightsquigarrow I = 5$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: $\S \quad \heartsuit \quad \spadesuit \quad \natural \quad \overset{\downarrow}{\diamondsuit} \quad \pounds \quad \oplus \quad \clubsuit \quad \times$

`reservoir`: $\boxed{\S}\boxed{\natural}\boxed{\spadesuit}$ $\quad \mathcal{U}(\{1, \dots, 5\}) \rightsquigarrow I = 5$ ✗

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[$1..k$]
    $n \leftarrow 0$

Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
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```
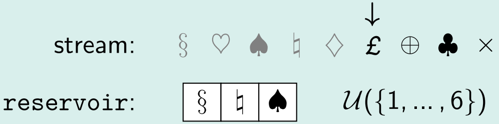

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

reservoir: | § | ♮ | ♠ | $\mathcal{U}(\{1, \dots, 6\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
**Algorithm** observeItem($x$):
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    **if** $n \leq k$ **then**
        reservoir[$n$] $\leftarrow x$
    **else**
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        **if** $I \leq k$ **then**
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```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

reservoir: | § | ♮ | ♠ | $\mathcal{U}(\{1, \dots, 6\}) \rightsquigarrow I = 1$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$

Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓ (over £)

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

reservoir: | £ | ♮ | ♠ | $\mathcal{U}(\{1, \dots, 6\}) \rightsquigarrow I = 1$ ✓

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$

Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: $\S \quad \heartsuit \quad \spadesuit \quad \natural \quad \diamondsuit \quad \pounds \quad \overset{\downarrow}{\oplus} \quad \clubsuit \quad \times$

`reservoir`: $\boxed{\pounds}\boxed{\natural}\boxed{\spadesuit}$ $\quad \mathcal{U}(\{1, \dots, 7\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

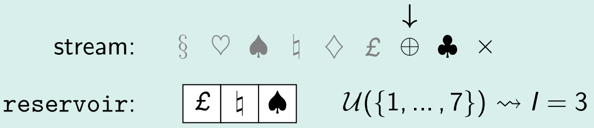
```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir:` | £ | ♮ | ♠ |

$\mathcal{U}(\{1, \dots, 7\}) \rightsquigarrow I = 3$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
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    if $n \leq k$ then
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```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: $\S \quad \heartsuit \quad \spadesuit \quad \natural \quad \diamondsuit \quad \pounds \quad \overset{\downarrow}{\oplus} \quad \clubsuit \quad \times$

`reservoir:` $\boxed{\pounds}\boxed{\natural}\boxed{\oplus}$ $\quad \mathcal{U}(\{1, \dots, 7\}) \rightsquigarrow I = 3$ ✓

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
    $n \leftarrow 0$

Algorithm observeItem(x):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[n] ← x
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[I] ← x
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | £ | ♮ | ⊕ | $\mathcal{U}(\{1, \dots, 8\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
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Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: $\S \quad \heartsuit \quad \spadesuit \quad \natural \quad \diamondsuit \quad \pounds \quad \oplus \quad \overset{\downarrow}{\clubsuit} \quad \times$

`reservoir:` $\boxed{\pounds}\boxed{\natural}\boxed{\oplus}$ $\quad \mathcal{U}(\{1, \dots, 8\}) \rightsquigarrow I = 3$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
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        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
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```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | £ | ♮ | ♣ | $\mathcal{U}(\{1, \dots, 8\}) \rightsquigarrow I = 3$ ✓

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
    $n \leftarrow 0$

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    if $n \leq k$ then
        reservoir[n] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[I] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: $\S$ $\heartsuit$ $\spadesuit$ $\natural$ $\diamondsuit$ £ $\oplus$ $\clubsuit$ $\overset{\downarrow}{\times}$

reservoir: $\boxed{£}\boxed{\natural}\boxed{\clubsuit}$ $\mathcal{U}(\{1, \ldots, 9\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$

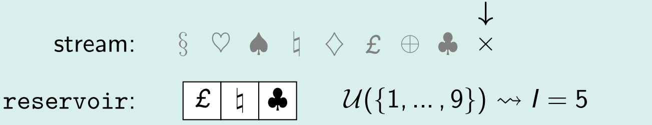
**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    **if** $n \leq k$ **then**
        reservoir[$n$] $\leftarrow x$
    **else**
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        **if** $I \leq k$ **then**
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

reservoir: | £ | ♮ | ♣ |

$\mathcal{U}(\{1, \dots, 9\}) \rightsquigarrow I = 5$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
    $n \leftarrow 0$

Algorithm observeItem(x):
    $n \leftarrow n + 1$
    if $n \le k$ then
        reservoir[n] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \le k$ then
            reservoir[I] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \ge k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ $\overset{\downarrow}{\times}$

reservoir: $\boxed{£}\boxed{\natural}\boxed{\clubsuit}$ $\mathcal{U}(\{1, \dots, 9\}) \rightsquigarrow I = 5$ ✗

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
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Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | £ | ♮ | ♣ |

Conclusion

General Techniques

- rejection sampling
- inverse transform sampling

Distributions

- Bernoulli distribution
- uniform distribution
- geometric distribution

Other Stuff

- sampling from a set without replacement
- reservoir sampling

Anhang: Mögliche Prüfungsfragen I

- Wie kann man $B \sim Ber(p)$ sampeln? Wie $X \sim \mathcal{U}(\{1, \dots, n\})$? Unter welchen Annahmen?
- Wie funktioniert Rejection Sampling allgemein? Unter welchen Voraussetzungen führt Rejection Sampling zu einem effizienten Algorithmus?
- Wie funktioniert Inverse Transform Sampling allgemein? Unter welchen Voraussetzungen führt Inverse Transform Sampling zu einem effizienten Algorithmus?
- Wie kann man einen zufälligen Punkt einer Kreisscheibe sampeln? Nenne zwei Techniken und nenne Vor- bzw. Nachteile.
- Gegeben eine Menge der Größe n. Wie kann ich eine zufällige Teilmenge der Größe $k \leq n$ bestimmen und wie lange dauert das?
- Erkläre Reservoir Sampling. Ist das nicht einfach ein langsamerer Algorithmus für „Sampling without Replacement“?