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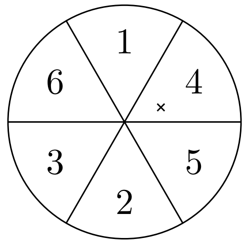
Exercise Sheet 1 – Basic Concepts

Probability and Computing

Exercise 1 – I probably still know it ...

A darts player throws a dart at a dartboard. The board is divided into 6 equally sized segments, each assigned a different score from $\{1, \ldots, 6\}$. Since the player is still in training, the dart lands uniformly at random on the board (but never off target). In the example on the right, the throw scored 4 points.



(a) Model the random experiment of the throw (not the resulting score) by describing the sample space and a probability measure of a continous probability space.

(b) We are now particularly interested in the following properties of a throw:
- the distance of the dart tip from the center of the board
- the resulting score
- the resulting score is even
- the resulting score modulo 2

Define the corresponding random variables and events.

(c) Determine the cumulative distribution function of the distance from part (b).

In the following, we are no longer interested in the position of the dart tip, but only in the resulting score.

(d) Model this random experiment by specifying a suitable *discrete* probability space.

(e) Let X be the random variable representing the score. Determine the following values:
- The expectation of X
- The variance of X
- The expectation of $X \cdot \mathbb{1}_{X \text{ is odd}}$. This is the expectation of the score in a game variant where only odd numbers count.
- The expectation of a throw that produced an odd score.

Solution 1

(a) The sample space is the unit disk:

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

The probability distribution is the uniform distribution over Ω. In other words, the probability measure assigns to each subset of Ω with area A the probability A/π. What a probability measure formally is, is not part of this lecture.

(b) We are asked about the following things.

- The *random variable* D that describes the distance. It holds that $D((x, y)) = \sqrt{x^2 + y^2}$ for $(x, y) \in \Omega$.
- The *random variable* P that describes the score. If we number the segments of the dartboard in mathematically positive order as $A_1, A_2, \ldots, A_6 \subseteq \Omega$, then:

$$P(\omega) = \begin{cases} 4 & \text{if } \omega \in A_1 \\ 1 & \text{if } \omega \in A_2 \\ 6 & \text{if } \omega \in A_3 \\ 3 & \text{if } \omega \in A_4 \\ 2 & \text{if } \omega \in A_5 \\ 5 & \text{if } \omega \in A_6. \end{cases}$$

- The event G, that the score is even, can now be written in various equivalent ways. Here are three suggestions:

$$G = \{\omega \in \Omega \mid P(w) \in \{2, 4, 6\}\} = \{P \in \{2, 4, 6\}\} = A_1 \cup A_3 \cup A_5.$$

  Note: In the middle case, the curly braces do not have their usual set notation meaning, but indicate that an event is being defined.

- The random variable $G(\omega) = P(\omega) \bmod 2$ is special in that it can only take the values 0 and 1. Such random variables are called indicator random variables. It holds that $\mathbb{E}[G] = \Pr[G = 1]$.

(c) For $d \leq 0$, $\Pr[D \leq d] = 0$. For $d \geq 1$, $\Pr[D \leq d] = 1$. For $0 \leq d \leq 1$, consider the event $E_d = \{(x, y) \in \Omega \mid x^2 + y^2 \leq d^2\}$. This has an area of $d^2\pi$. Thus:

$$\Pr[D \leq d] = \Pr[E_d] = d^2\pi/\pi = d^2.$$

(d) $\Omega = \{1, 2, 3, 4, 5, 6\}$ with uniform distribution.

(e) The following quantities were asked for:

- $\mathbb{E}[X] = (1 + 2 + 3 + 4 + 5 + 6)/6 = 3.5$.
- $\text{Var}(X) = ((1-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(6-3.5)^2)/6 \approx 2.92$.
- $\mathbb{E}[X \cdot \mathbb{1}_{X \text{ is odd}}] = (1 + 0 + 3 + 0 + 5 + 0)/6 = 1.5$.
- $\mathbb{E}[X \mid X \text{ is odd}] = (1 + 3 + 5)/3 = 3$.

Exercise 2 – Analogies to the calculation rules

Let Ω be the set of inhabitants of the distant land Omegon. Consider the following four statements and identify which of the five calculation rules from the slides each one is analogous to. For the remaining rule, come up with your own analogy. Argue (formally or intuitively, as you wish) why the calculation rules hold.

1. Let h be the proportion of dog owners, k the proportion of cat owners, and t the proportion of inhabitants with a dog or a cat. Then: $t \leq h + k$.
2. Suppose 40% of the inhabitants live in the west, the rest in the east. If g_1 is the average height of westerners and g_2 the average height of easterners, then $g_1 \cdot 0.4 + g_2 \cdot 0.6$ is the average height in Omegon.
3. Suppose 40% of the inhabitants live in the west, the rest in the east. Let k_1 be the proportion of cat owners among westerners and k_2 the proportion of cat owners among easterners. Then the total proportion of cat owners is $k = k_1 \cdot 0.4 + k_2 \cdot 0.6$.
4. If an inhabitant eats on average w white and b brown chicken eggs per year, then on average they eat $w + b$ chicken eggs per year.

Solution 2

1. Union bound. To show it, we need that for disjoint events A and B, $\Pr[A \cup B] = \Pr[A] + \Pr[B]$ and that probabilities are non-negative. For two events E_1, E_2 it follows that:

$$\begin{aligned}\Pr[E_1 \cup E_2] &= \Pr[E_1 \cup (E_2 \setminus E_1)] = \Pr[E_1] + \Pr[E_2 \setminus E_1] \\ &\leq \Pr[E_1] + \Pr[E_2 \setminus E_1] + \Pr[E_1 \cap E_2] \\ &= \Pr[E_1] + \Pr[(E_2 \setminus E_1) \cup (E_1 \cap E_2)] = \Pr[E_1] + \Pr[E_2].\end{aligned}$$

   For more than two events, one can use induction.

2. Law of total expectation. Here is an informal proof:

$$\begin{aligned}&\sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \cdot \Pr[E_i] \\ &= \sum_{i=1}^{n} \sum_{x} x \cdot \Pr[X = x \mid E_i] \cdot \Pr[E_i] && \text{(Definition of conditional expectation)} \\ &= \sum_{i=1}^{n} \sum_{x} x \cdot \Pr[\{X = x\} \cap E_i] && \text{(Definition of conditional probability)} \\ &= \sum_{x} x \cdot \sum_{i=1}^{n} \Pr[\{X = x\} \cap E_i] = \sum_{x} x \cdot \Pr[X = x] && \text{(Disjointness of } E_1, \ldots, E_n) \\ &= \mathbb{E}[X] && \text{(Definition of expectation)}\end{aligned}$$

3. Law of total probability. This is a special case of the law of total expectation for an indicator random variable $X = \mathbb{1}_F$, since then $\Pr[F] = \mathbb{E}[X]$ and $\Pr[F \mid E_i] = \mathbb{E}[X \mid E_i]$.

4. Linearity of expectation. The statement is hardly clearer with a proof, but here is one anyway. For discrete probability spaces, by expanding over all possible outcomes:

$$\begin{aligned}\mathbb{E}[X+Y] &= \sum_{\omega\in\Omega}(X(\omega)+Y(\omega))\cdot\Pr[\{\omega\}]\\ &= \sum_{\omega\in\Omega}X(\omega)\Pr[\{\omega\}] + \sum_{\omega\in\Omega}Y(\omega)\Pr[\{\omega\}] = \mathbb{E}[X]+\mathbb{E}[Y].\end{aligned}$$

5. Missing: Tail Sum Formula. Analogy:

   > Every inhabitant ω can do a certain number ℓ_ω of push-ups. Let z_j be the number of inhabitants who can do at least j push-ups. Then $\sum_{\omega\in\Omega}\ell_\omega = \sum_{j\geq 1} z_j$, and thus also $\frac{1}{|\Omega|}\sum_{\omega\in\Omega}\ell_\omega = \sum_{j\geq 1}\frac{z_j}{|\Omega|}$. The left sum is the average number of push-ups, and $\frac{z_j}{|\Omega|}$ is the proportion of inhabitants who can do j push-ups.

   The formula can be derived by swapping summations. For arbitrary non-negative real numbers $(x_{i,j})_{i,j\in\mathbb{N}}$ it holds that $\sum_{j\geq 1}\sum_{i=1}^{j} x_{i,j} = \sum_{i\geq 1}\sum_{j\geq i} x_{i,j}$, since in both cases $x_{i,j}$ is counted exactly when $j \geq i$. We apply this to $x_{i,j} = \Pr[X = i]$.

$$\begin{aligned}\mathbb{E}[X] &= \sum_{j\geq 1} j\cdot\Pr[X=j] = \sum_{j\geq 1}\sum_{i=1}^{j}\Pr[X=j]\\ &= \sum_{i\geq 1}\sum_{j\geq i}\Pr[X=j] = \sum_{i\geq 1}\Pr[X\geq i].\end{aligned}$$