

Exercise Sheet 4 – Probability

Amplification

Probability and Computing

Exercise 1 – Probability Amplification with Two-Sided Error

Suppose a Monte Carlo algorithm A solves a decision problem correctly with probability $1/2 + \varepsilon$ and incorrectly otherwise (for some $\varepsilon > 0$). Let A' be the algorithm that executes A independently t times and decides according to the majority outcome. Show that the error probability of A' is at most $e^{-2t\varepsilon^2}$.

Hint: The following may be useful: $\sum_{i=0}^t \binom{t}{i} = 2^t$.

Exercise 2 – Pathfinder

Let G be an undirected graph with n vertices and m edges. We seek a path of length k that does not visit any vertex more than once. The naive brute-force approach runs in time $O(n^k)$. We now consider a simple randomized approach that works as follows. In the first step, each vertex v is assigned a label $L(v)$ uniformly at random from $\{1, \dots, k\}$. In the second step, for every vertex v with $L(v) = 1$, a modified breadth-first search is started, where a vertex w can be discovered from a vertex u only if $L(u) = L(w) + 1$ holds. If a vertex with label k is reached, a path of length k is constructed and output.

- (a) Show that the algorithm finds a path of length k with probability $1/k^k$, if such a path exists.
- (b) Improve the success probability to $1 - 1/n$ by probability amplification. What is the resulting total running time?

Remark: This idea is known as “Color Coding”. There exist more refined variants.

Exercise 3 – Bonus: Random-Walk Solver for 3-SAT

Let $\varphi(x_1, \dots, x_n)$ be a 3-SAT formula with n variables and m clauses. We seek a satisfying assignment using the following algorithm:

Algorithm randomWalkSolver(φ):

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    sample  $x_1, \dots, x_n \sim \text{Ber}(1/2)$ 
    for  $k = 1$  to  $n/2$  do
        if  $\varphi(x_1, \dots, x_n) = 1$  then
            break
        let  $C$  be an arbitrary unsatisfied clause of  $\varphi$ 
        sample  $j \sim \mathcal{U}(\{1, 2, 3\})$ 
        let  $x_i$  be the  $j$ th variable in  $C$ 
         $x_i \leftarrow 1 - x_i$ 
    if  $\varphi(x_1, \dots, x_n) = 1$  then
        return  $(x_1, \dots, x_n)$ 
    return  $\perp$ 

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If φ is unsatisfiable, clearly $\text{randomWalkSolver}(\varphi) = \perp$. Otherwise, let x^* be a satisfying assignment. Show that:

- (a) With probability at least $1/2$, the initial random assignment $(x_1, \dots, x_n)$ agrees with x^* on at least $n/2$ variables.
- (b) If $\varphi(x_1, \dots, x_n) = 0$, then one iteration of the loop will, with probability $1/3$, flip a variable so that it matches x^* on one more position.
- (c) Use probability amplification to obtain an algorithm with success probability $1 - 1/n$. What is the total running time?