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Exercise Sheet 6 – Concentration Bounds

Probability and Computing

Exercise 1 – Algebraic Rule for Expectation

Let X, Y be independent random variables whose expectation exists. Show that

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

Hint: Use the definition of independence for discrete random variables, which guarantees

$$\Pr[X = i \wedge Y = j] = \Pr[X = i] \cdot \Pr[Y = j] \quad \text{for all } i, j.$$

Exercise 2 – Algebraic Rules for Variance

Let X, Y be independent random variables with existing variance. Let $s, t > 0$. Show:

(a) $\mathsf{Var}(sX) = s^2 \mathsf{Var}(X)$

(b) $\mathsf{Var}(X + Y) = \mathsf{Var}(X) + \mathsf{Var}(Y)$

(c) $\mathsf{Var}(sX + tY) = s^2 \mathsf{Var}(X) + t^2 \mathsf{Var}(Y)$

Hint: Use linearity of expectation and the result of the previous exercise, i.e., $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ for independent X and Y.

Exercise 3 – Chernoff in Even Simpler Form for Large Deviations

Let $X = X_1 + \cdots + X_n$ be a sum of independent Bernoulli random variables with $\mu = \mathbb{E}[X]$ and let $b \geq 6\mu$. Show

$$\Pr[X \geq b] \leq 2^{-b}.$$

Hint: Use the Chernoff bound $\Pr[X \geq (1 + \delta)\mu] \leq (\frac{e^\delta}{(1+\delta)^{1+\delta}})^\mu$.

Exercise 4 – Comparing Concentration Inequalities

For $n \in \mathbb{N}$ let X_n be the number of sixes when rolling a fair die n times. Let p_n be the probability that X_n exceeds its expectation by at least 10%. For each of the following, find an upper bound on p_n using...

(a) ... Markov's Inequality.

(b) ... Chebyshev's Inequality.

(c) ... the Chernoff bound (or a variant).

(d) Compare the asymptotic strength of the bounds.