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Exercise Sheet 10
Approximation Algorithms
Probability and Computing

Exercise 1 – Jensen's Inequality

Let $D \subseteq \mathbb{R}$ be a connected domain and $f : D \to \mathbb{R}$ be a function. The function f is called convex if it is "curved to the left" and concave if it is "curved to the right".[1] A function is convex if and only if its negation is concave. For a formal definition see: Wikipedia

(a) Decide (without proof) for the following functions whether they are convex on their respective domains, concave, both, or neither.

$$f_1(x) = x, \qquad f_2(x) = x^2, \qquad f_3(x) = x^3, \qquad f_4(x) = \log(x), \qquad f_5(x) = \log^2(x).$$

(b) Let f be a convex function with domain D. Argue geometrically that for every $x_0 \in D$ there exists a linear function g such that:

  (i) $f(x) \geq g(x)$ for all $x \in D$

  (ii) $f(x_0) = g(x_0)$.

(c) Conclude that for every convex function f and for every random variable X with values in the domain D of f the following holds:

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]).$$

Hint: Consider $x_0 = \mathbb{E}[X]$ and the corresponding g from the previous subproblem.

(d) Show that analogously, for every concave function f with domain D and for every random variable X with values in D the following holds:

$$\mathbb{E}[f(X)] \leq f(\mathbb{E}[X]).$$

The inequality from (c) as well as variants as in (d) are called Jensen's inequality.

[1] The "curved to the left" in quotation marks allows, besides left curvatures (of a twice continuously differentiable function), also left kinks and linear behavior.

Exercise 2 – Analysis of Lossy Counting

Reminder: Lossy Counting is a simple streaming algorithm that approximately counts the length m of a stream. It involves a parameter $p \in (0, 1]$. The algorithm itself as well as the way it is used are shown on the right. Prove:

(a) $\mathbb{E}[\text{result}] = m$

(b) $\Pr[|\text{result} - m| \leq \varepsilon m] \geq 1 - 2\exp(-\varepsilon^2 pm/3)$.

(c) $\mathbb{E}[\text{space}] \leq \log(1 + mp) + 1$.
Hint: By space we denote the maximum memory usage required for the state Z of LossyCounting. A number $i \in \mathbb{N}$ can be encoded with $\lceil \log_2(i+1) \rceil$ bits. Use Jensen's inequality from Exercise 1.

```
Algorithm init:
    Z ← 0
    return Z
Algorithm update(Z, a):
    with probability p do
        Z ← Z + 1
    return Z
Algorithm result(Z):
    return Z/p
```

Usage:

```
Z ← init()
for i = 1 to m do
    Z ← update(Z, a_i)
return result(Z)
```