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Exercise Sheet 11
Game Theory & Yao's Principle

Probability and Computing

Exercise 1 – Lower bounds for randomized sorting

Let $n \in \mathbb{N}$ and **Inputs** be the set of all permutations of $\{1, \dots, n\}$. Let **Algos** be the set of all comparison-based *deterministic* sorting algorithms.

(i) Give an $A \in \textbf{Algos}$ (without proof) with $\max_{I \in \textbf{Inputs}} C(A, I) = O(n \log n)$.

(ii) Warm-up: For $n = 3$ there exists a randomized algorithm $\mathcal{A}$ that beats every deterministic algorithm in the following sense:

$$\min_{A \in \textbf{Algos}} \max_{I \in \textbf{Inputs}} C(A, I) = 3 > 2 + \tfrac{2}{3} = \max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}}[C(A, I)].$$

Our plan in the following is to show that this advantage disappears in O-notation.

(iii) Show that there is no "hard input", that is, that the following holds:

$$\max_{I \in \textbf{Inputs}} \min_{A \in \textbf{Algos}} C(A, I) = n - 1.$$

In order to immediately focus on the best possible cost for an input *distribution*, we need some preparation:

(iv) Show that in a binary tree with k leaves, the average depth of a leaf is at least $\lfloor \log_2(k) \rfloor$.
Hint: To do so, show that the average leaf depth is minimal for balanced trees; more precisely, that any tree in which the leaf depths differ by at least 2 can be rearranged such that the average leaf depth decreases while the number of leaves remains the same.

(v) Now conclude using Yao's principle that:

$$\min_{\mathcal{A} \text{ dist. on } \textbf{Algos}} \max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}}[C(A, I)] = \Omega(n \log n).$$

Exercise 2 – Yao's principle without game theory

Prove Yao's principle without resorting to game-theoretic theorems (no Nash theorem, Loomis theorem, etc.). That is, prove that in the setting of the lecture, for an arbitrary distribution $\mathcal{A}_0$ on **Algos** and an arbitrary distribution $\mathcal{I}_0$ on **Inputs**, the following holds:

$$\max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}_0}[C(A, I)] \geq \min_{A \in \textbf{Algos}} \mathbb{E}_{I \sim \mathcal{I}_0}[C(A, I)].$$

Hint: The game-theoretic framework of the lecture is not required here, because we do not wish to show that "=" is achievable.

Exercise 3 – Recommendation: Simulating the Evolution of Teamwork

The YouTube channel *Primer* deals with evolutionary game theory. In the following video, among other things, all possible 2-player games with two pure strategies are classified according to how many and which types of Nash equilibria they possess. The video is entertaining and invites active thinking, but is only marginally relevant to the lecture.

https://www.youtube.com/watch?v=TZfh8hpJIxo