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Exercise Sheet 12 – Probabilistic Method

Probability and Computing

Exercise 1 – A children's game

Alice and Bob play an asymmetric game on a sequence of fields $0, 1, 2, \ldots, n$. Initially, k tokens are placed on field 0. Each round proceeds as follows.

1. Alice chooses two disjoint sets T_1 and T_2 of tokens.
2. Bob then chooses $i \in \{1, 2\}$.
3. The tokens from T_i are removed.
4. The tokens from T_{2-i} each move one field to the right.

Alice wins as soon as a token reaches field n. She loses as soon as she chooses two empty sets. Solve the following tasks.

(i) Give a strategy for Alice with which she wins for $k \geq 2^n$.

(ii) Use the probabilistic method to show that there is a winning strategy for Bob if $k < 2^n$.

(iii) Bonus: Construct a winning strategy for Bob (without the probabilistic method).

Solution 1

(i) Without loss of generality let $k = 2^n$, since Alice can ignore additional tokens. Alice's strategy is to always split the remaining tokens evenly between T_1 and T_2. Then exactly half of the tokens will always move on, and the other half will be removed. A simple induction shows that after $0 \leq i \leq n$ rounds exactly 2^{n-i} tokens lie on field i. Thus Alice has won after n rounds.

(ii) Bob plays uniformly at random. For Alice we consider an arbitrary strategy. We can now view the game from the perspective of a single token t. Each time t is selected by Alice, it moves one step to the right with probability $1/2$ and is removed with probability $1/2$. We may assume without loss of generality that Alice never loses (i.e. chooses two empty sets) as long as tokens still exist, and continues playing even if she has already won. Then the probability that token t ever reaches position $i \in \mathbb{N}$ is exactly 2^{-i}. The probability that t ever reaches position n is therefore 2^{-n}. The expected number of tokens that reach position n is thus $2^{-n} \cdot k$, which by assumption is < 1. Hence it is possible that no token reaches position n. Therefore the probability that Bob wins is positive. Thus Alice's strategy is not a winning strategy. Since Alice's strategy was arbitrary, there is no winning strategy for Alice. Hence there exists a winning strategy for Bob.

(iii) We imagine that a token lying on field i has a value of 2^i, that is, the value of a token doubles when it moves one field to the right. If Alice now chooses two sets T_1 and T_2 with values w_1 and w_2, then Bob should always remove the set of tokens with the larger value. Without loss of generality let this be T_1. He then allows the value of the tokens in T_2 to double. Since for $w_1 \geq w_2$ the inequality $2w_2 \leq w_1 + w_2$ holds, the total value cannot have increased. Since initially a value of k is present and a win for Alice requires a value of at least 2^n, Alice cannot win if $k < 2^n$ holds.

Exercise 2 – Larger[1] independent sets

Let $G = (V, E)$ be a graph with n vertices and m edges. Show using the probabilistic method that G contains an independent set of size $\sum_{v \in V} \frac{1}{\deg(v)+1}$.

Hint: Random permutation of the vertices.

Solution 2

We randomly permute the vertices. Let

$$I = \{v \in V \mid v \text{ appears in the permutation before all of its neighbors}\}.$$

It should be clear that I is an independent set. It is also clear that v is included in I with probability $\frac{1}{\deg(v)+1}$, since for this to happen v must be the first among $\deg(v) + 1$ vertices in the random permutation. Thus

$$\mathbb{E}[|I|] = \mathbb{E}\Big[\sum_{v \in V} [v \in I]\Big] = \sum_{v \in V} \Pr[v \in I] = \sum_{v \in V} \frac{1}{\deg(v) + 1}.$$

By the expectation argument, in particular there exists an independent set of the required size.

[1]**Remark:** Let $d = \frac{2m}{n}$ be the average degree of the vertices. In the lecture we constructed an independent set of size $\frac{n}{2d}$. For the size U of the independent set guaranteed by this exercise, the following holds using an inequality between arithmetic and harmonic means:

$$U = \sum_{v \in V} \frac{1}{\deg(v) + 1} = n \cdot \left(\frac{1}{n} \sum_{v \in V} \frac{1}{\deg(v) + 1}\right) \geq n\left(\frac{1}{n} \sum_{v \in V} \deg(v)+1\right)^{-1} = \frac{n}{d+1}.$$

This is larger than $\frac{n}{2d}$ for $d > 1$.[2]

[2]"*But what if $d < 1$ holds?*" Then the theorem from the lecture is not applicable at all.

Exercise 3 – Independent rainbow sets again

Let $G = (V, E)$ be a graph with $|V| = kc$ vertices that are colored with c colors, where each color appears exactly k times. The maximum degree is Δ. Show: If $k \geq 8\Delta$, then there exists an independent rainbow set.

Solution 3

This problem is a simple generalization of the necklace analysis from the lecture.

We again choose the rainbow set R by selecting one vertex uniformly at random from each color class. The goal is to show that R is an independent set with positive probability.

For this we define a bad event $B_{\{u,v\}}$ for each edge $\{u, v\}$ of the graph, which states that both u and v are contained in R. Again we have $\Pr[B_{\{u,v\}}] \leq \frac{1}{k^2} =: p$.

Another event $B_{\{u',v'\}}$ can only interact with $B_{\{u,v\}}$ if u' or v' has a color that is also the color of u or v. Hence there are at most $d = 2k\Delta - 2$ such events (2 relevant colors, k relevant vertices each, Δ incident edges each, where $\{u, v\}$ itself is not counted from either u or v).

Thus we have $4pd = 4\frac{1}{k^2}(2k\Delta - 2) < \frac{8\Delta}{k} \leq 1$. Therefore we can apply the Lovász Local Lemma.