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Exercise Sheet 12 – Probabilistic Method

Probability and Computing

Exercise 1 – A children's game

Alice and Bob play an asymmetric game on a sequence of fields $0, 1, 2, \ldots, n$. Initially, k tokens are placed on field 0. Each round proceeds as follows.

1. Alice chooses two disjoint sets T_1 and T_2 of tokens.
2. Bob then chooses $i \in \{1, 2\}$.
3. The tokens from T_i are removed.
4. The tokens from T_{2-i} each move one field to the right.

Alice wins as soon as a token reaches field n. She loses as soon as she chooses two empty sets. Solve the following tasks.

(i) Give a strategy for Alice with which she wins for $k \geq 2^n$.

(ii) Use the probabilistic method to show that there is a winning strategy for Bob if $k < 2^n$.

(iii) Bonus: Construct a winning strategy for Bob (without the probabilistic method).

Exercise 2 – Larger[1] independent sets

Let $G = (V, E)$ be a graph with n vertices and m edges. Show using the probabilistic method that G contains an independent set of size $\sum_{v \in V} \frac{1}{\deg(v)+1}$.

Hint: Random permutation of the vertices.

[1]**Remark:** Let $d = \frac{2m}{n}$ be the average degree of the vertices. In the lecture we constructed an independent set of size $\frac{n}{2d}$. For the size U of the independent set guaranteed by this exercise, the following holds using an inequality between arithmetic and harmonic means:

$$U = \sum_{v \in V} \frac{1}{\deg(v) + 1} = n \cdot \left(\frac{1}{n} \sum_{v \in V} \frac{1}{\deg(v) + 1} \right) \geq n \left(\frac{1}{n} \sum_{v \in V} \deg(v) + 1 \right)^{-1} = \frac{n}{d+1}.$$

This is larger than $\frac{n}{2d}$ for $d > 1$.[2]

[2]"*But what if $d < 1$ holds?*" Then the theorem from the lecture is not applicable at all.

Exercise 3 – Independent rainbow sets again

Let $G = (V, E)$ be a graph with $|V| = kc$ vertices that are colored with c colors, where each color appears exactly k times. The maximum degree is Δ. Show: If $k \geq 8\Delta$, then there exists an independent rainbow set.