Karlsruhe Institute of Technology

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Exercise Sheet 13 – Random Graphs

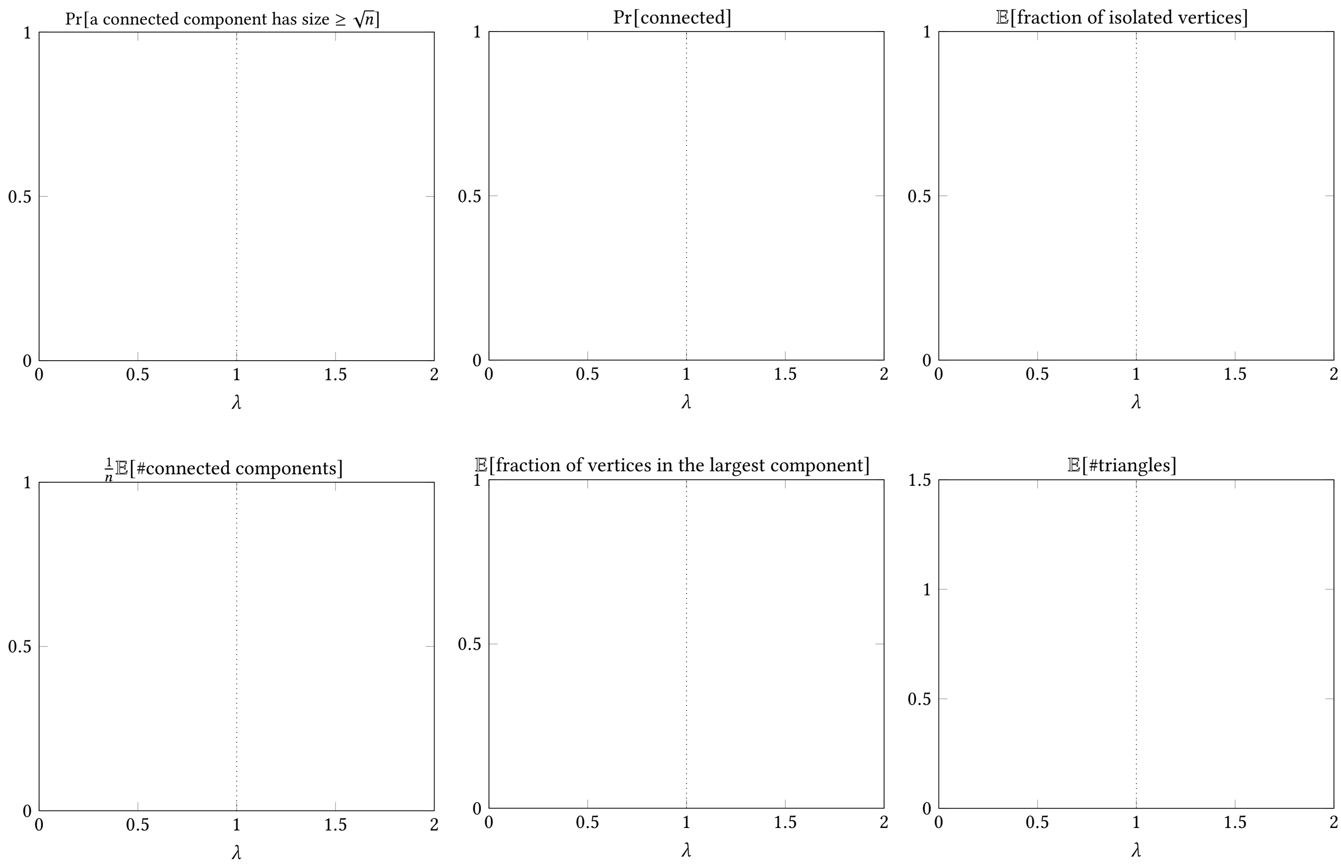
Probability and Computing

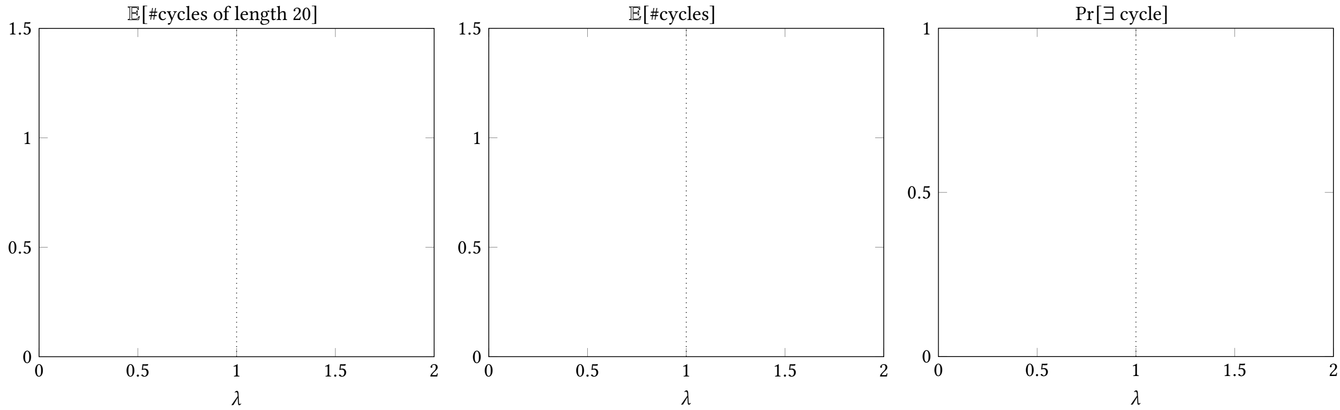
Exercise 1 – Intuition for Erdős–Rényi Graphs

We consider the Erdős–Rényi graph $G(n, \lambda n/2)$ or the Gilbert graph $G(n, \lambda/n)$ (both lead to the same result). The expected vertex degree is therefore $\lambda \pm O(1/n)$.

(i) Sketch the behavior of the following probabilities and expectations for $n \longrightarrow \infty$ as a function of $\lambda \in [0, 2]$.

Hint: This is not about numerical exactness but about the qualitative behavior. Where does the curve take the value 0, 1, or ∞? Does anything special happen at $\lambda = 1$?





(ii) Let $\lambda = \Theta(1)$. Guess: What holds with high probability for (the order of magnitude of) the minimum and maximum degree as a function of n?

$$\min_{v\in[n]} \deg(v) = \dots\dots\dots\dots \qquad \max_{v\in[n]} \deg(v) = \Theta\Big(\dots\dots\dots\dots\Big)$$

Exercise 2 – Vertex Degrees in Erdős–Rényi Graphs

On slide 15 of the section on balls into bins and Poissonization we showed that $\mathrm{Bin}(n, \frac{\lambda}{n})$ *converges to* $\mathrm{Pois}(\lambda)$ *as* $n \longrightarrow \infty$ *(or, respectively, that the CDFs converge). In the following exercise you may use without proof:*

> **Lemma.** Let $t_1, t_2, \dots \in \mathbb{N}$ and $p_1, p_2, \dots \in (0,1)$ as well as $\lambda \in \mathbb{R}_+$. Furthermore let $X_n \sim \mathrm{Bin}(t_n, p_n)$ for all $n \in \mathbb{N}$ and $X \sim \mathrm{Pois}(\lambda)$. If $t_n \longrightarrow \infty$ and $t_n \cdot p_n \longrightarrow \lambda$ as $n \longrightarrow \infty$, then $X_n \xrightarrow{\mathrm{d}} X$ as $n \longrightarrow \infty$.

Let $\lambda > 0$ and $X \sim \mathrm{Pois}(\lambda)$. We consider variants of Erdős–Rényi graphs from the lecture. We set the expected degree to approximately λ and want to show (using the above lemma) that the distribution of a single vertex converges asymptotically to $\mathrm{Pois}(\lambda)$ as $n \longrightarrow \infty$ (while λ remains constant).

(i) Let X_n be the degree of vertex 1 in $G(n, p)$ with $p = \lambda/n$. Show $X_n \xrightarrow{\mathrm{d}} X$.

(ii) Let X_n be the degree of vertex 1 in $G^{\mathrm{UE}}(n, m)$ with $m = \lfloor \lambda n/2 \rfloor$. Show $X_n \xrightarrow{\mathrm{d}} X$.

(iii) Let X_n be the degree of vertex 1 in $G(n, m)$ with $m = \lfloor \lambda n/2 \rfloor$. Show $X_n \xrightarrow{\mathrm{d}} X$.

Hint: The last part is by far the most difficult. It is helpful to "trap" $G(n, m)$ using a coupling between two Gilbert graphs $G^- = (n, p^-)$ and $G^+ = (n, p^+)$.

Exercise 3 – Extinction Probability in Galton–Watson Trees

Let $\mathrm{GWT}(\lambda)$ be the Galton–Watson tree with offspring distribution $\mathrm{Pois}(\lambda)$.

(i) Let n_i be the number of vertices in level i of $\mathrm{GWT}(\lambda)$. The root level is level 0. What is $\mathbb{E}[n_i]$?

(ii) For $i \in \mathbb{N}_0$ let p_i be the probability that $\mathrm{GWT}(\lambda)$ has at least one vertex at level i. Express p_{i+1} in terms of p_i.
Hint: Use Exercise 3 (iv) from Sheet 9.

(iii) Determine, for $\lambda \in \{0, 0.5, 1, 1.1, 1.5\}$ (or for arbitrary λ), approximations for the probability $s(\lambda)$ that $\mathrm{GWT}(\lambda)$ is infinite. A computer algebra system may be useful (e.g. Wolfram Alpha).

Additional consideration: What is the expected number of vertices at level i?