Stefan Walzer, Stefan Hermann
ITI Prof. Sanders
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Exercise Sheet X – Mixed

Probability and Computing

Exercise 1 – It's all connected

We consider random graphs in the Gilbert model. For $n \in \mathbb{N}$ and $p \in [0, 1]$ let

$$z(n, p) := \Pr[G(n, p) \text{ is connected}].$$

Show: For all $n \in \mathbb{N}$, $z(n, p)$ is monotonically increasing in p.

Exercise 2 – Sampling Gilbert Graphs

Let $p = \frac{\lambda}{n-1}$ for $\lambda = \Theta(1)$. Describe how one can sample $G(n, p)$ in expected time $O(n)$.

Hint: Let $X_1, X_2, \ldots \sim \text{Ber}(p)$ and $Y = \min\{i \in \mathbb{N} \mid X_i = 1\}$. Sample Y in time $O(1)$.

Exercise 3 – Concentration around an unknown expectation

For two strings S and T let $\text{lcs}(S, T)$ denote the maximum possible length of a string L that can be obtained from both S and T by deleting characters. For example, for $S = \text{ELEFANT}$ and $T = \text{ELLENLANG}$ we have $\text{lcs}(S, T) = 5$ because $L = \text{ELEAN}$. One calls L a *Longest Common Subsequence*.

In the following let $n \in \mathbb{N}$ and let $S, T \sim \mathcal{U}(\{0, 1\}^n)$ be random bit strings.

(a) Show that $\mathbb{E}[\text{lcs}(S, T)] \geq n/2$.

Remark: The number $\gamma_2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\text{lcs}(S, T)]$ is a so-called Chvátal–Sankoff number. It is known that it exists and that $\gamma_2 \in [0.788071, 0.826280]$. The exact value is unknown (simulations suggest $\gamma_2 \approx 0.811$).

(b) Let $\varepsilon > 0$. be an arbitrary constant. Show $\Pr[|\text{lcs}(S, T) - \mathbb{E}[\text{lcs}(S, T)]| \geq \varepsilon n] \leq \exp(-\varepsilon^2 n)$.

Exercise 4 – A delicacy to conclude

A circular cake stands on a turntable. A *random cut* means that we rotate the cake randomly and then cut from the center towards the north.

(a) We cut twice at random. What is the expected size of the smaller piece (as a fraction of the whole cake)?

(b) There is a cherry in the cake (point-like, not exactly in the center). What is the probability that the cherry is in the smaller piece after the two random cuts?