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Exercise Sheet 4 – Probability Amplification

Probability and Computing

Exercise 1 – Probability Amplification with Two-Sided Error

Suppose a Monte Carlo algorithm A solves a decision problem correctly with probability $1/2+\varepsilon$ and incorrectly otherwise (for some $\varepsilon > 0$). Let A' be the algorithm that executes A independently t times and decides according to the majority outcome. Show that the error probability of A' is at most $e^{-2t\varepsilon^2}$.

Hint: The following may be useful: $\sum_{i=0}^{t}\binom{t}{i} = 2^t$.

Exercise 2 – Pathfinder

Let G be an undirected graph with n vertices and m edges. We seek a path of length k that does not visit any vertex more than once. The naive brute-force approach runs in time $O(n^k)$. We now consider a simple randomized approach that works as follows. In the first step, each vertex v is assigned a label $L(v)$ uniformly at random from $\{1,\ldots,k\}$. In the second step, for every vertex v with $L(v)=1$, a modified breadth-first search is started, where a vertex w can be discovered from a vertex u only if $L(u)=L(w)+1$ holds. If a vertex with label k is reached, a path of length k is constructed and output.

(a) Show that the algorithm finds a path of length k with probability $1/k^k$, if such a path exists.

(b) Improve the success probability to $1-1/n$ by probability amplification. What is the resulting total running time?

Remark: This idea is known as “Color Coding”. There exist more refined variants.

Exercise 3 – Bonus: Random-Walk Solver for 3-SAT

Let $\varphi(x_1,\ldots,x_n)$ be a 3-SAT formula with n variables and m clauses. We seek a satisfying assignment using the following algorithm:

```
Algorithm randomWalkSolver(φ):
    sample x_1, ..., x_n ~ Ber(1/2)
    for k = 1 to n/2 do
        if φ(x_1, ..., x_n) = 1) then
            break
        let C be an arbitrary unsatisfied clause of φ
        sample j ~ U({1, 2, 3})
        let x_i be the jth variable in C
        x_i ← 1 − x_i
    if φ(x_1, ..., x_n) = 1) then
        return (x_1, ..., x_n)
    return ⊥
```

If φ is unsatisfiable, clearly randomWalkSolver$(\varphi) = \bot$. Otherwise, let x^* be a satisfying assignment. Show that:

(a) With probability at least $1/2$, the initial random assignment $(x_1, \ldots, x_n)$ agrees with x^* on at least $n/2$ variables.

(b) If $\varphi(x_1, \ldots, x_n) = 0$, then one iteration of the loop will, with probability $1/3$, flip a variable so that it matches x^* on one more position.

(c) Use probability amplification to obtain an algorithm with success probability $1 - 1/n$. What is the total running time?