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winter term 2025/2026

Exercise Sheet 6 – Concentration Bounds

Probability and Computing

Exercise 1 – Algebraic Rule for Expectation

Let X, Y be independent random variables whose expectation exists. Show that

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

Hint: Use the definition of independence for discrete random variables, which guarantees

$$\Pr[X = i \wedge Y = j] = \Pr[X = i] \cdot \Pr[Y = j] \quad \text{for all } i, j.$$

Solution 1

Let $R_X, R_Y \subseteq \mathbb{R}$ be countable sets containing all possible values of X and Y. Then:

$$\begin{aligned}
\mathbb{E}[X \cdot Y] &= \sum_{(x,y) \in R_X \times R_Y} x \cdot y \cdot \Pr[X = x \wedge Y = y] \\
&\overset{\text{indep.}}{=} \sum_{x \in R_X} \sum_{y \in R_Y} x \cdot y \cdot \Pr[X = x] \Pr[Y = y] \\
&= \sum_{x \in R_X} \Big(x \cdot \Pr[X = x] \sum_{y \in R_Y} y \cdot \Pr[Y = y] \Big) \\
&= \Big(\sum_{x \in R_X} x \cdot \Pr[X = x] \Big) \Big(\sum_{y \in R_Y} y \cdot \Pr[Y = y] \Big) \\
&= \mathbb{E}[X] \cdot \mathbb{E}[Y].
\end{aligned}$$

Exercise 2 – Algebraic Rules for Variance

Let X, Y be independent random variables with existing variance. Let $s, t > 0$. Show:

(a) $\mathsf{Var}(sX) = s^2 \mathsf{Var}(X)$

(b) $\mathsf{Var}(X + Y) = \mathsf{Var}(X) + \mathsf{Var}(Y)$

(c) $\mathsf{Var}(sX + tY) = s^2 \mathsf{Var}(X) + t^2 \mathsf{Var}(Y)$

Hint: Use linearity of expectation and the result of the previous exercise, i.e., $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ for independent X and Y.

Solution 2

We now prove the three variance rules (very explicitly).

(a) Here, besides the definition of variance, we only need the insight that $\mathbb{E}[sZ] = s\mathbb{E}[Z]$ (linearity of expectation). The latter holds for every random variable Z whose expectation exists, and for every $s \in \mathbb{R}$. Thus:

$$\begin{aligned}\mathsf{Var}(sX) &= \mathbb{E}[(sX - \mathbb{E}[sX])^2] = \mathbb{E}[(sX - s\mathbb{E}[X])^2] \\ &= \mathbb{E}[s^2(X - \mathbb{E}[X])^2] = s^2\mathbb{E}[(X - \mathbb{E}[X])^2] = s^2 \mathsf{Var}(X).\end{aligned}$$

(b) First, centered random variables have expectation 0:

$$\mathbb{E}[X - \mathbb{E}[X]] = \mathbb{E}[X] - \mathbb{E}[\mathbb{E}[X]] = \mathbb{E}[X] - \mathbb{E}[X] = 0. \tag{1}$$

If X, Y are independent, then $X - c$ and $Y - d$ are also independent for constants $c, d \in \mathbb{R}$. Setting $c = \mathbb{E}[X]$, $d = \mathbb{E}[Y]$, we obtain independence of $X - \mathbb{E}[X]$ and $Y - \mathbb{E}[Y]$. Thus:

$$\mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])] = \mathbb{E}[X - \mathbb{E}[X]] \cdot \mathbb{E}[Y - \mathbb{E}[Y]] \overset{(1)}{=} 0 \cdot 0 = 0. \tag{2}$$

Now consider $(X + Y - \mathbb{E}[X + Y])^2$, the expectation of which is the variance we are looking for:

$$\begin{aligned}(X + Y - \mathbb{E}[X + Y])^2 &= (X + Y - \mathbb{E}[X] - \mathbb{E}[Y])^2 \\ &= ((X - \mathbb{E}[X]) + (Y - \mathbb{E}[Y]))^2 \\ &= (X - \mathbb{E}[X])^2 + (Y - \mathbb{E}[Y])^2 + 2(X - \mathbb{E}[X])(Y - \mathbb{E}[Y]).\end{aligned}$$

Taking expectations:

$$\begin{aligned}\mathsf{Var}(X + Y) &= \mathbb{E}[(X + Y - \mathbb{E}[X + Y])^2] \\ &= \mathbb{E}[(X - \mathbb{E}[X])^2] + \mathbb{E}[(Y - \mathbb{E}[Y])^2] + 2\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \mathsf{Var}(X) + \mathsf{Var}(Y) + 0\end{aligned}$$

using the definition of variance and (2).

(c) Since sX and tY are independent when X and Y are, we obtain directly from (a) and (b):

$$\mathsf{Var}(sX + tY) \overset{(b)}{=} \mathsf{Var}(sX) + \mathsf{Var}(tY) \overset{(a)}{=} s^2 \mathsf{Var}(X) + t^2 \mathsf{Var}(Y).$$

Exercise 3 – Chernoff in Even Simpler Form for Large Deviations

Let $X = X_1 + \cdots + X_n$ be a sum of independent Bernoulli random variables with $\mu = \mathbb{E}[X]$ and let $b \geq 6\mu$. Show

$$\Pr[X \geq b] \leq 2^{-b}.$$

Hint: Use the Chernoff bound $\Pr[X \geq (1+\delta)\mu] \leq (\frac{e^\delta}{(1+\delta)^{1+\delta}})^\mu$.

Solution 3

Following the hint ans setting $\delta = b/\mu - 1$, we obtain:

$$\begin{aligned}\Pr[X \geq b] = \Pr[X \geq (1+\delta)\mu] &\leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu} \\ &\leq \left(\frac{e^{b/\mu}}{(b/\mu)^{b/\mu}}\right)^{\mu} = \left(\frac{e}{b/\mu}\right)^{b} \leq \left(\frac{e}{6}\right)^{b} \leq 2^{-b}.\end{aligned}$$

Exercise 4 – Comparing Concentration Inequalities

For $n \in \mathbb{N}$ let X_n be the number of sixes when rolling a fair die n times. Let p_n be the probability that X_n exceeds its expectation by at least 10%. For each of the following, find an upper bound on p_n using...

(a) ... Markov's Inequality.

(b) ... Chebyshev's Inequality.

(c) ... the Chernoff bound (or a variant).

(d) Compare the asymptotic strength of the bounds.

Solution 4

Preparations:

- $\mu := \mathbb{E}[X] = n/6$.
- $\mathrm{Var}(X) = n \cdot (\frac{1}{6}(\frac{5}{6})^2 + \frac{5}{6}(\frac{1}{6})^2) = \frac{5}{36}n$.
- We bound $p_n = \Pr[X \geq 1.1\mu] \leq \Pr[|X-\mu| \geq 0.1\mu]$.

(a) $p_n = \Pr[X \geq 1.1\mu] \leq \mu/(1.1\mu) = \frac{10}{11} = \Theta(1)$.

(b) $p_n \leq \Pr[|X-\mu| \geq 0.1\mu] \leq \frac{\mathrm{Var}(X)}{(0.1\mu)^2} = \frac{5n/36}{0.01 \cdot n^2/36} = \frac{500}{n} = \Theta(1/n)$.

(c) $p_n = \Pr[X \geq (1+0.1)\mu] \leq \exp(-\frac{0.1^2}{2+0.1}\mu) = \exp(-\frac{1}{210}\mu) = \exp(-\frac{1}{1260}n) = \exp(-\Theta(n))$.

(d) Asymptotically, the Chernoff bound is the strongest and Markov the weakest.
Remark: While Markov and Chernoff give bounds < 1 for all $n \in \mathbb{N}$, Chebyshev becomes nontrivial only for $n \geq 501$.