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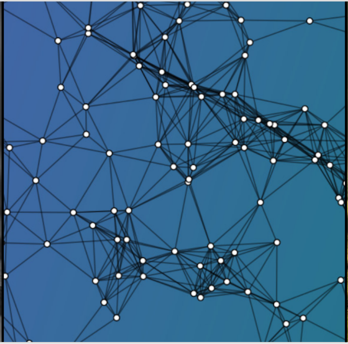
Probability and Computing

Coupling, Balls into Bins, Poissonisation and the Poisson Point Process

Stefan Walzer | WS 2025/2026


Content

An easy choice?

A Simple Game

You win if you get ≥ 5 heads in 10 coin tosses. Choose:

- i a fair coin with $\Pr[\text{“heads”}] = \frac{1}{2}$
- ii a biased coin with $\Pr[\text{“heads”}] = \frac{2}{3}$

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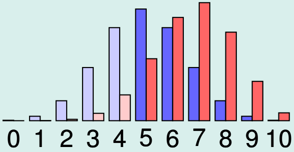
How to prove that (ii) is the better choice?



fair coin

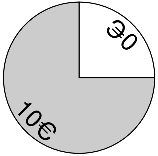
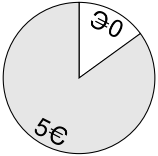
biased coin

$$\sum_{i=5}^{10} \binom{10}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{10-i} \overset{?}{<} \sum_{i=5}^{10} \binom{10}{i} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{10-i}$$

Shouldn't there be an answer that needs no calculation?

Which Lottery do you prefer?

Consider two “wheels of fortune”:

$X \sim$  and $Y \sim$ 

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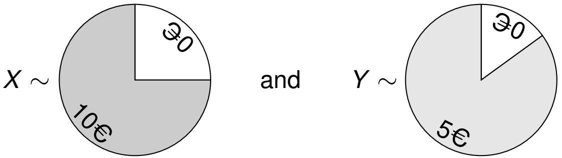

Both can be rationally preferred

- $\mathbb{E}[X] > \mathbb{E}[Y]$ // maximises expected reward
- $\Pr[Y \geq 5€] > \Pr[X \geq 5€]$ // maximises probability that you can afford ice cream

See `https://en.wikipedia.org/wiki/Von_Neumann%E2%80%93Morgenstern_utility_theorem` to get started on rational choice theory.

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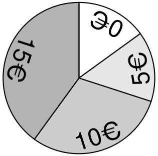
$X \sim$ 

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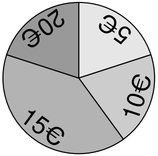
$Y \sim$ 

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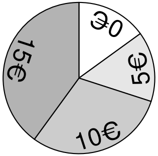
$X \sim$ 

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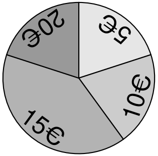
$Y \sim$ 

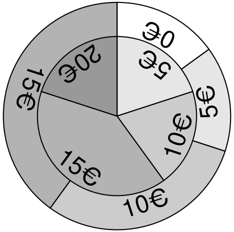
Formal Reason you should prefer Y

For every c we have:

$$\Pr[X \geq c] \leq \Pr[Y \geq c].$$

Intuitive Reason you should prefer Y

Glueing the wheels together guarantees $X < Y$.

$(X, Y) \sim$ 

Content

Equality in Distribution

Notation

We write $X \overset{\mathrm{d}}{=} X'$ for two random variables if X and X' have the same distribution.

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Equivalent Definitions

$$\begin{aligned} X \overset{\mathrm{d}}{=} X' &\Leftrightarrow \forall x : \Pr[X = x] = \Pr[X' = x] && \text{(for discrete R.V. } X \text{ and } X'\text{)} \\ &\Leftrightarrow \forall x : \Pr[X \le x] = \Pr[X' \le x] && \text{(for real-valued R.V. } X \text{ and } X'\text{)} \end{aligned}$$

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To Clarify:

If $X, Y \sim \mathcal{U}([0,1])$ are independent then

- $X \overset{\text{d}}{=} Y$
- $\Pr[X = Y] = 0$

Definition: Coupling of X and Y

A random variable X

$\downarrow$

A random variable Y

$\downarrow$

A Coupling of X and Y

A random variable (X', Y') with

- $X' \stackrel{d}{=} X$
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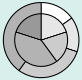
$(X', Y') \sim$ 

- $X' \stackrel{\text{d}}{=} X$ ✓
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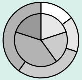
$(X', Y') \sim$ 

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Remarks

- No assumption on joint distribution of X and Y. Might be independent, correlated or undefined.
- X' and Y' should be correlated in an interesting/useful way.
- Example coupling shows:

$$\Pr[X \geq c] \overset{X \overset{d}{=} X'}{=} \Pr[X' \geq c] \overset{X' \leq Y'}{\leq} \Pr[Y' \geq c] \overset{Y \overset{d}{=} Y'}{=} \Pr[Y \geq c]$$

An easy choice!

A Simple Game (Generalised)

You win if your random variable exceeds $c \in \mathbb{N}$. Choose:

- i $X \sim \mathrm{Bin}(n, \frac{1}{2})$ // number of heads of fair coin
- ii $Y \sim \mathrm{Bin}(n, \frac{2}{3})$ // number of heads of biased coin

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Prove that Y is better than X using a Coupling

Let $R_1, \ldots, R_n \sim \mathcal{U}([6])$ be n fair dice rolls.

- $X' = |\{i \in [n] \mid R_i \in \{1,2,3\}\}|$
- $Y' = |\{i \in [n] \mid R_i \in \{1,2,3,4\}\}|$

Observe:

- $X' \overset{\mathrm{d}}{=} X$
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- $X' \leq Y'$ guaranteed

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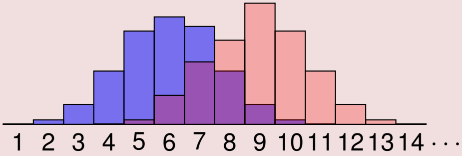
Hence: $\Pr[X \geq c] = \Pr[X' \geq c] \leq \Pr[Y' \geq c] = \Pr[Y \geq c]$.

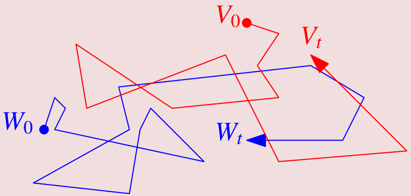
Exercises

Coupling and Total Variation Distance



The histograms of X and Y intersect in an area of A if and only if there exists a coupling (X', Y') of X and Y such that $\Pr[X' = Y'] = A$.

Coupling of Random Walks



For any pair of start points W_0 and V_0, intuitively:

- The distributions of W_t and V_t are asymptotically indinstinguishable for large t.
- Random walks "forget" where they started.
- There is a coupling $(W'_t, V'_t)_{t\geq 0}$ of $(W_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ such that $\lim_{t\to\infty} \Pr[W'_t = V'_t] = 1$.

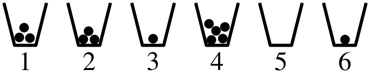
Content

Balls Into Bins

General Terminology

- m balls are randomly distributed among n bins
- the *load* of a bin is the number of balls in it

$$m = 13, \qquad n = 6$$



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Fully Random Allocation

- $X_1, \dots, X_m \sim \mathcal{U}([n])$ independent
- $L_i := |\{j \in [m] \mid X_j = i\}|$ is the load of bin $i \in [n]$
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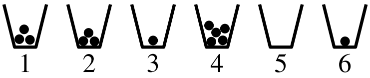

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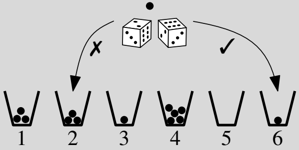
Example for Partially Random Allocation (not in this lecture)

- balls are placed sequentially
- each ball chooses the *least loaded* among two randomly chosen bins (ties broken randomly)



Balls into Bins: Many Interesting Questions

- What is the distribution/expectation/concentration of

random variable	interpretation
$\max_{i\in[n]} L_i$	most loaded bin
$\max_{i\in[n]} L_i - \frac{m}{n}$	most loaded bin relative to mean
$\lvert\{i \in [n] \mid L_i = 0\}\rvert$	number of empty bins

- Can we make the allocation more balanced by intervening in some way?
  - e.g. with partially random allocation (last slide) $\lim\limits_{m\to\infty} \mathbb{E}[\max_{i\in[n]} L_i - \frac{m}{n}] = \Theta(\log\log n)$ for any fixed n.

Countless variants exist...

"Balls into Bins" is everywhere

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Hashing with Chaining $\longleftrightarrow$ n Balls into m Bins

length of the list in bucket i $\longleftrightarrow$ number of balls in bin i

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Hashing with Chaining $\longleftrightarrow$ n Balls into m Bins

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a filter bit is set to 0 $\longleftrightarrow$ ith bin is empty

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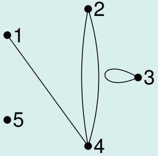
Degree Sequence of Random (Multi-)Graph $\longleftrightarrow$ $2m$ Balls into n bins

Given independent $v_1, \ldots, v_{2m} \sim \mathcal{U}([n])$ let
$G = (V = [n], E = \{\{v_1, v_2\}, \ldots, \{v_{2m-1}, v_{2m}\}\})$
(we allow multi-edges and loops in G)

degree of vertex i $\longleftrightarrow$ load of bin i



$n = 5, \quad m = 4$

$$\begin{aligned} v_1 &= 1, & v_2 &= 4 \\ v_3 &= 4, & v_4 &= 2 \\ v_5 &= 3, & v_6 &= 3 \\ v_7 &= 2, & v_8 &= 4 \end{aligned}$$

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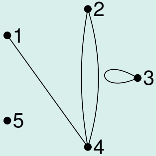

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“Balls into Bins” is the standard language for discussing underlying mathematical questions.

Content

Load of a Single Bin

Setting: Expected Constant Load per Bin

- fully random allocation
- $m = \lambda n$ balls n bins for large n
- λ fixed constant

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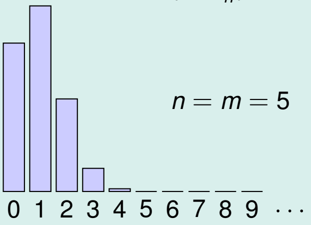
Consider $L^{(n)} \sim \mathrm{Bin}(\lambda n, \frac{1}{n})$. For $\lambda = 1$:



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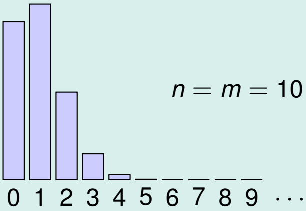

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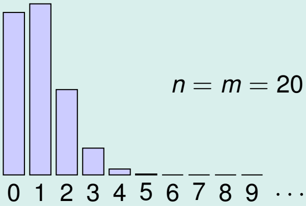

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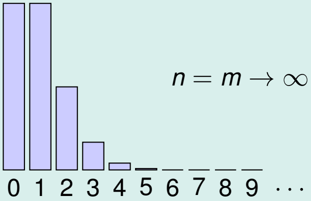

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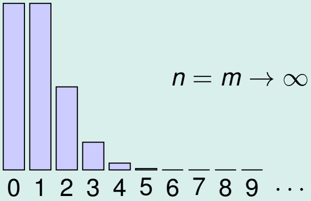

Poisson Distribution

For $\lambda \in \mathbb{R}_{\geq 0}$, $X \sim \mathrm{Pois}(\lambda)$ is a random variable with

$$\Pr[X = i] = e^{-\lambda}\frac{\lambda^i}{i!}$$ // note: probabilities sum to 1

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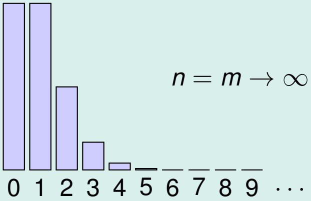

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Theorem (proof on blackboard)

$\lim_{n\to\infty} \Pr[L^{(n)} = i] = \Pr[X = i]$.

Remarks

- we say "$L^{(n)}$ converges in distribution to X"
- we write $L^{(n)} \xrightarrow{d} X$
- this formally refers to convergence of CDFs

Properties of the Poisson Distribution

Exercise: $X \sim \text{Pois}(\lambda)$ has Nice Properties

- i $\mathbb{E}[X] = \lambda$.
- ii $\text{Var}(X) = \lambda$.
- iii Let $Y \sim \text{Pois}(\rho)$ be independent of X. Then $X + Y \sim \text{Pois}(\lambda + \rho)$.
- iv Let $X' \sim \text{Bin}(X, p)$. Then $X' \sim \text{Pois}(\lambda p)$.

Poissonised Balls into Bins

$(L_i): \; 3 \quad 3 \quad 1 \quad 5 \quad 0 \quad 1$

λn Balls into n Bins Model

- $X_1, \ldots, X_{\lambda n} \sim \mathcal{U}([n])$
- $L_i := |\{j \in [m] \mid X_j = i\}| \sim \mathrm{Bin}(\lambda n, \frac{1}{n})$
- $(L_i)_{i \in [n]}$ *not independent*
  - e.g. large L_1 is (weak) evidence for small L_2
  - annoying in analysis
- number λn of balls *fixed*

"Poissonised" Model

- $L_1, \ldots, L_n \sim \mathrm{Pois}(\lambda)$ independent
  - extremely convenient for analysis
- $\mathbb{E}[L_1 + \cdots + L_n] = \lambda n$
- number of balls *random* $\sim \mathrm{Pois}(\lambda n)$
  - unusual setting in practice

Wouldn't it be nice…

… if we could switch between the models whenever convenient?

Connection: Poissonised and Regular Balls into Bins

Lemma 1

Let $n \in \mathbb{N}$ and $\lambda > 0$. Consider two variants of Poissonised balls into bins:

Regular Variant:

- sample $L_1, \ldots, L_n \sim \text{Pois}(\lambda)$

Sum-First-Variant:

- sample $M \sim \text{Pois}(\lambda n)$
- perform a regular M balls into n bins experiment
  - sample $X_1, \ldots, X_M \sim \mathcal{U}([n])$
  - let $L'_i := |\{j \in [M] \mid X_j = i\}|$

Both variants are equivalent, i.e. $(L_1, \ldots, L_n) \stackrel{\text{d}}{=} (L'_1, \ldots, L'_n)$.

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Both variants are equivalent, i.e. $(L_1, \dots, L_n) \overset{\mathrm{d}}{=} (L'_1, \dots, L'_n)$.

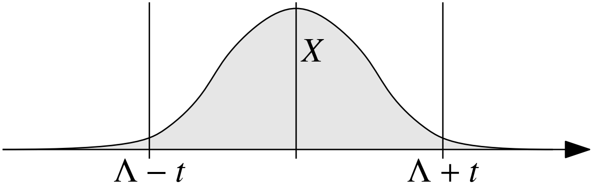
What we need to show (calculation on blackboard):

For arbitrary $(\ell_1, \dots, \ell_n) \in \mathbb{N}^n$: $\Pr[(L_1, \dots, L_n) = (\ell_1, \dots, \ell_n)] = \Pr[(L'_1, \dots, L'_n) = (\ell_1, \dots, \ell_n)]$.

Some Concentration Bounds

Lemma 2

- Let $\Lambda > 0$ and $X \sim \mathrm{Pois}(\Lambda)$. Then $\Pr[|X - \Lambda| > t] \leq \frac{\Lambda}{t^2}$ for any $t > 0$. // Chebyschev



Some Concentration Bounds

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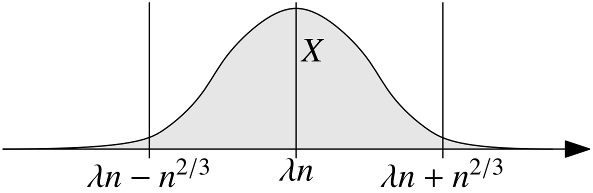

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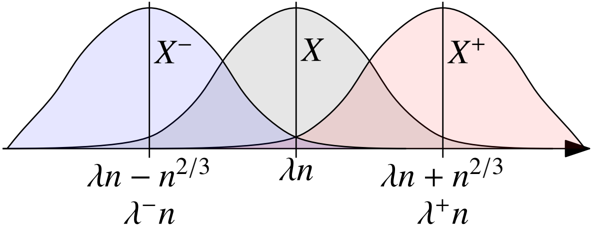
- [iv] Let $\lambda = \Theta(1)$, $\lambda^- := \lambda - n^{-1/3}$ and $X^- \sim \mathrm{Pois}(\lambda^- n)$ then $\Pr[X^- \leq \lambda n] = 1 - o(1)$.



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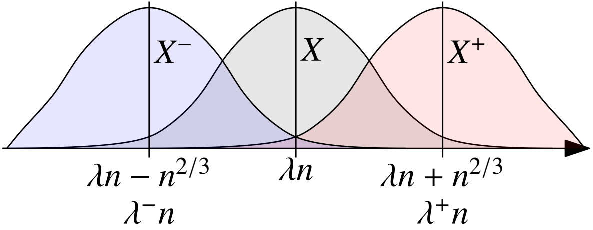
- v In particular: $\Pr[X^- \leq \lambda n \leq X^+] = 1 - o(1)$.



Coupling of Poissonised and Regular Balls into Bins

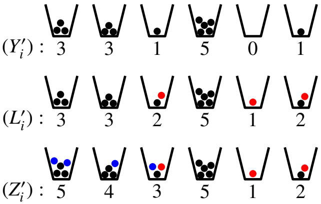
Theorem

Let $n, \lambda, \lambda^+, \lambda^-$ be as before. Consider three "balls into bins" models:

1. $Y_1, \ldots, Y_n \sim \text{Pois}(\lambda^-)$ // poissonised with slightly reduced λ
2. $L_1, \ldots, L_n$ arising from regular $m = \lambda n$ balls into n bins
3. $Z_1, \ldots, Z_n \sim \text{Pois}(\lambda^+)$ // poissonised with slightly increased λ

There is a coupling $(Y_i', L_i', Z_i')_{i\in[n]}$ of $(Y_i)_{i\in[n]}, (L_i)_{i\in[n]}, (Z_i)_{i\in[n]}$ such that

$$\text{with probability } 1 - o(1)\text{: } Y_i' \le L_i' \le Z_i' \text{ for all } i \in [n].$$



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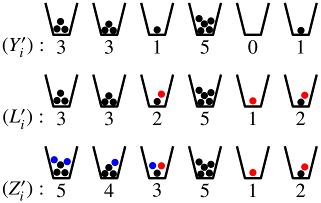

Proof.

Let $X_1, X_2, \dots \sim \mathcal{U}([n])$, $M^- \sim \mathrm{Pois}(\lambda^- n)$, $M^+ \sim \mathrm{Pois}(\lambda^+ n)$.

- $Y_i' := |\{j \in [M^-] \mid X_j = i\}|$ for $i \in [n]$.
- $L_i' := |\{j \in [m\ \ ] \mid X_j = i\}|$ for $i \in [n]$.
- $Z_i' := |\{j \in [M^+] \mid X_j = i\}|$ for $i \in [n]$.

This is indeed a coupling as claimed:

- $(Y_i')_{i\in[n]} \overset{\mathrm{d}}{=} (Y_i)_{i\in[n]}$ by Lemma 1.
- $(L_i')_{i\in[n]} \overset{\mathrm{d}}{=} (L_i)_{i\in[n]}$ by construction.
- $(Z_i')_{i\in[n]} \overset{\mathrm{d}}{=} (Z_i)_{i\in[n]}$ by Lemma 1.

By Lemma 2 (v) we have $M^- \le m \le M^+$ with probability $1 - o(1)$. In that case clearly $Y_i' \le L_i' \le Z_i'$ for all $i \in [n]$. $\square$

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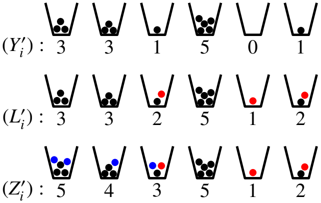
with probability $1 - o(1)$: $Y'_i \leq L'_i \leq Z'_i$ for all $i \in [n]$.



Application involving Monotonous Functions

Let $f : \mathbb{N}_0^n \to \mathbb{R}$ be non-decreasing in each argument.
Examples:

- maximum load of a bin
- longest run of non-empty bins
- collision number // numbers of pairs of co-located balls

For some bound $B \in \mathbb{R}$ let

- $p^- := \Pr[f((Y_i)_{i\in[n]}) \geq B]$ // easier to compute
- $p := \Pr[f((L_i)_{i\in[n]}) \geq B]$ // what we want
- $p^+ := \Pr[f((Z_i)_{i\in[n]}) \geq B]$ // easier to compute

Then $p \in [p^- - o(1), p^+ + o(1)]$.

Back to Bloom Filters

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Exercise:

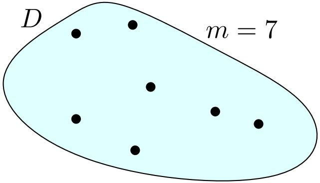
Analyse Bloom filters in a “Poissonised” model and discuss how the results can be transferred to the exact model.

Content

What about “Balls into Continuous Domain”?

Setting

- D is space of finite measure
- $m \in \mathbb{N}$ // number of balls
- $X_1, \ldots, X_m \sim \mathcal{U}(D)$ // randomly thrown into D



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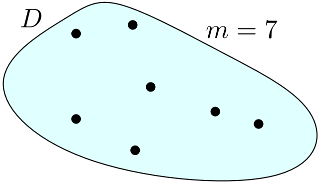

Note: If $D = \{1, \ldots, n\}$ we have discrete balls into bins.

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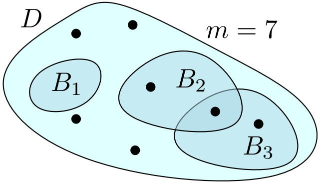

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Same annoying issue

If $B_1, B_2 \subseteq D$ with $B_1 \cap B_2 = \varnothing$ are two "bins" then the numbers L_1 and L_2 of "balls" in B_1 and B_2 are correlated.

Similar elegant solution

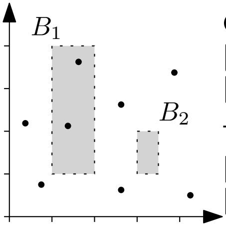
- We can "Poissonise" the setting.
- But we drop "balls into bins" terminology:
  - we allow infinite domains D
  - we allow infinite number of balls

Definitions of the Poisson Point Process

General Definition

Let D be a measurable space with measure μ. // e.g. $D = \mathbb{R}^2$ and $\mu =$“area”
The Poisson point process with parameter $\lambda \in \mathbb{R}_{\geq 0}$ is a random set $P \subseteq D$ such that

1. $|P \cap B| \sim \text{Pois}(\lambda\mu(B))$ for any $B \subseteq D$ with $\mu(B) < \infty$
2. $|P \cap B_1|$ and $|P \cap B_2|$ are independent whenever $B_1 \cap B_2 = \varnothing$



Generally:
$|B_1 \cap P| \sim \text{Po}(3\lambda)$
$|B_2 \cap P| \sim \text{Po}(\frac{1}{2}\lambda)$

This outcome:
$|B_1 \cap P| = 2$
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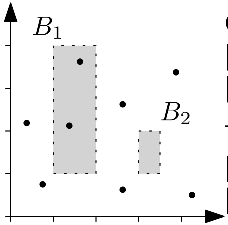

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Equivalent "Sum First" variant if $\mu(D) < \infty$

- sample $M \sim \text{Pois}(\lambda\mu(D))$
- sample $X_1, \dots, X_M \sim \mathcal{U}(D)$

Then $P \stackrel{d}{=} \{X_1, X_2, \dots, X_M\}$.

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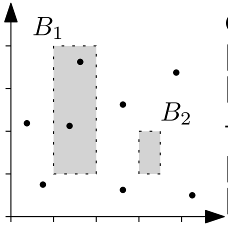

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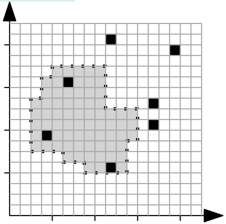
Construction as a limit

- subdivide D into pieces of measure ε
- let each piece contain a point with probability $\varepsilon\lambda$
- consider the limit for $\varepsilon \to 0$



$$|B \cap P| \sim \text{Bin}(\mu(B)/\epsilon, \lambda\epsilon) \xrightarrow{\epsilon \to 0} \text{Po}(\lambda\mu(B))$$

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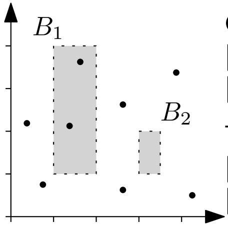

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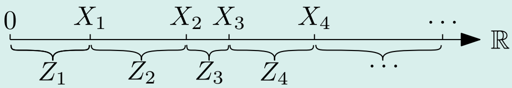
Equivalent Definition if $D = \mathbb{R}_{\geq 0}$ (where μ is the Borel measure)

- sample $Z_1, Z_2, \ldots \sim \text{Exp}(\lambda)$
- define $X_i = \sum_{j=1}^{i} Z_j$



Then $P \overset{\text{d}}{=} \{X_1, X_2, \ldots\}$.

Proof idea: $\Pr[\min P > t] = \Pr[|P \cap [0,t]| = 0] = \Pr_{X \sim \text{Pois}(\lambda t)}[X = 0] = e^{-\lambda t} \overset{\text{def}}{=} \Pr[Z_1 > t] = \Pr[X_1 > t]$.

Conclusion

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Coupling

- embedding of two random variables X and Y into a common probability space
- relationships between distributions of X and Y become visible as relationships between outcomes of X' and Y'

Balls into Bins

- standard language when m objects are randomly assigned to n other objects

Poissonisation

- the act of replacing multinomially distributed $(L_1, \ldots, L_n)$ with independent Poisson random variables $(L'_1, \ldots, L'_n)$
- often much easier to think about
- often formally justifiable

Poisson Point Process

- important model where points from a continuous space occur independently from each other with fixed density λ

Appendix: Possible Exam Questions I

- What is a coupling?
  - Give examples in which a coupling can be useful.
  - What does equality in distribution mean?
- Where in the lecture have we (implicitly or explicitly) considered balls-into-bins processes?
- Poissonisation:
  - Which annoying property does the load distribution in balls-into-bins processes have? What is different in a Poissonised model?
  - How do Poisson distributions arise in a balls-into-bins setting?
  - How did we relate the Poissonised and the regular balls-into-bins models? How can switching between the models be formally justified?
- Poisson point processes
  - How are Poisson point processes defined?