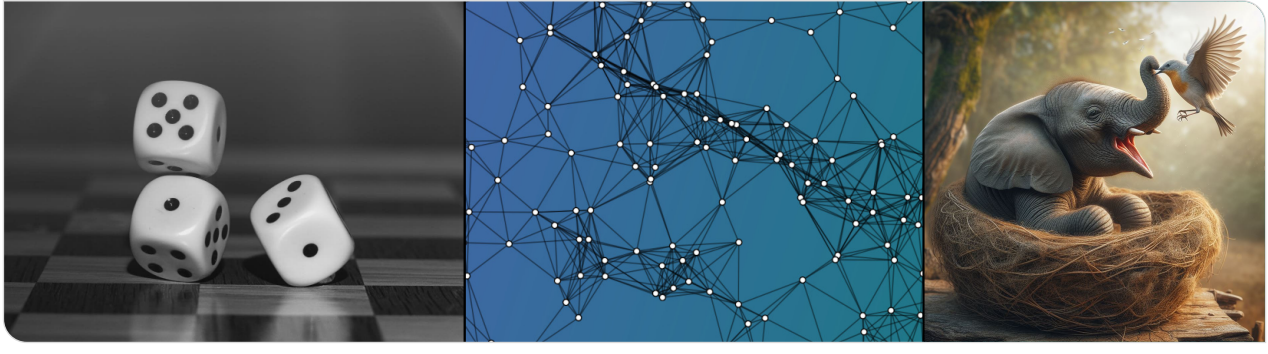
Karlsruhe Institute of Technology

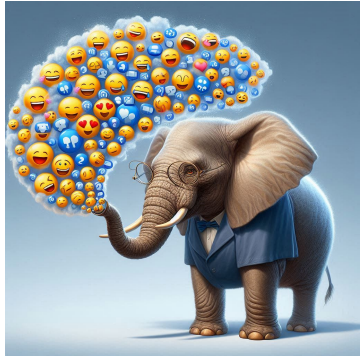
Algorithm Engineering

# Probability and Computing – Cuckoo Hashing

Stefan Walzer | WS 2025/2026



# Evaluation



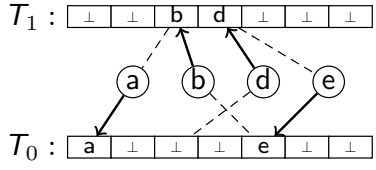
https://onlineumfrage.kit.edu/evasys/online.php?p=RF73W

# Content

# Classic Cuckoo Hashing

## Setup

| | |
|---|---|
| $S \subseteq D$ | key set of size $n$ |
| $T_0, T_1$ | two tables of size $m$ |
| $h_0, h_1 \sim \mathcal{U}([m]^D)$ | two hash functions (SUHA) |
| $\frac{n}{m} = 1 - \beta$ | for some $\beta > 0$ |
| | ($\triangle$! load factor $\alpha = \frac{n}{2m}$) |



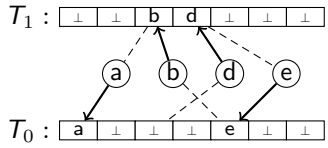

**Algorithm** lookup($x$):
  **return** $x \in \{T_0[h_0(x)], T_1[h_1(x)]\}$

**Algorithm** delete($x$):
  **if** $T_0[h_0(x)] = x$ **then**
    $T_0[h_0(x)] \leftarrow \perp$
  **else if** $T_1[h_1(x)] = x$ **then**
    $T_1[h_1(x)] \leftarrow \perp$

**Algorithm** insert($x$):
  **for** $i = 0$ **to** LIMIT **do**
    $b \leftarrow i \bmod 2$
    swap($x$, $T_b[h_b(x)]$)
    **if** $x = \perp$ **then**
      **return** SUCCESS
  **return** FAILURE

# Cuckoo Hashing Theorem

**Algorithm** insert($x$)**:**
- **for** $i = 0$ **to** LIMIT **do**
  - $b \leftarrow i \bmod 2$
  - swap($x$, $T_b[h_b(x)]$)
  - **if** $x = \perp$ **then**
    - **return** SUCCESS
- **return** FAILURE




## Theorem (Analysis with LIMIT $= \infty$)

Assume we insert all $x \in S$ and then another key $y$. Let $E$ be the event that this succeeds and

$$T = \begin{cases} \text{insertion time of } y & \text{if } E \text{ occurs} \\ 0 & \textit{otherwise} \end{cases}$$

Then (i) $\Pr[E] = 1 - \mathcal{O}(1/m)$ and (ii) $\mathbb{E}[T] = \mathcal{O}(1)$.

## Theorem (full analysis, not here)

If we
- set LIMIT $= \Omega(\log n)$ appropriately
- rebuild the table with fresh hash functions when LIMIT is reached

we obtain a hash table where lookup and delete take $\mathcal{O}(1)$ time and insert takes *expected* $\mathcal{O}(1)$ time.

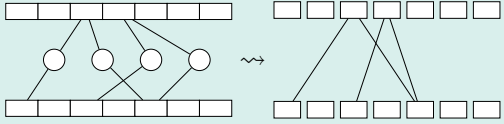
# Proof of i: Success probability is $1 - \mathcal{O}(1/m)$

## The Cuckoo Graph

Consider the bipartite *cuckoo graph*

$$G = ([m], [m], \{(h_0(x), h_1(x)) \mid x \in S\})$$

the key $x$ corresponds to the edge $(h_0(x), h_1(x))$ and each table position to a vertex.



// Duplicate edges possible, don't worry about it.

## Connection to Erdős-Renyi Graphs

$G$ is a bipartite Erdős-Renyi variant

- much like $G^{\text{UE}}(2m, n)$ // uniform endpoint model
- a bit like $G(2m, n)$ // original Erdős-Renyi
- a bit like $G(2m, n/(2m^2))$ // Gilbert

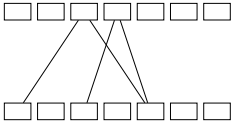
Confusing: $n$ is a number of edges and $2m$ a number of vertices.

## Exercise

Design a variant of cuckoo hashing such that the "Sudden Emergence" result for the $G^{\text{UE}}(m, n)$ model implies success for load factor $\alpha < \frac{1}{2}$.

Next: Completely self-contained analysis without reference to Erdős-Renyi.

# Proof of i: Success probability is $1 - \mathcal{O}(1/m)$



## Keys and buckets in the infinite loop

Assume $\bar{E}$ occurs, i.e. an insertion fails due to an infinite loop. Let $G^* = (V^*, E^*)$ be the subgraph of $G$ with

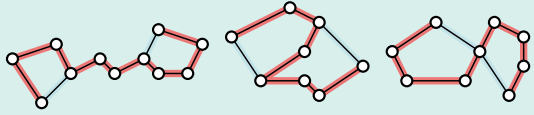
- $V^*$: table positions touched in the infinite loop
- $E^*$: keys touched in the infinite loop.

Properties of $G^*$:

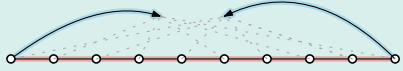
- connected
- $|E^*| = |V^*| + 1$ // can you see why?
- $\deg_{E^*}(v) \geq 2$ for $v \in V^*$.

## Possibilities for $G^*$

There are three options:



In all three cases: Simple path through $|V^*|$ and two extra edges connecting inwards:



# Proof of i: Success probability is $1 - \mathcal{O}(1/m)$

$$\Pr[\bar{E}] = \Pr[\exists \text{path as shown}]$$

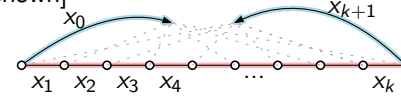
$$= \Pr[\exists k \in \mathbb{N} : \exists x_0, \ldots, x_{k+1} \in S : x_0, \ldots, x_{k+1} \text{ form a path as shown}]$$

$$\overset{\text{union bound}}{\leq} \sum_{k=1}^{n} \sum_{x_0,\ldots,x_{k+1} \in S} \Pr[x_0, \ldots, x_{k+1} \text{ form a path as shown}]$$

$$\leq \sum_{k=1}^{n} \underbrace{n^{k+2}}_{\text{a}} \cdot \underbrace{2}_{\text{b}} \cdot \underbrace{\tfrac{1}{m^{k+1}}}_{\text{c}} \cdot \underbrace{\left(\tfrac{k+1}{2m}\right)^2}_{\text{d}}$$

$$\leq \tfrac{1}{2} \sum_{k=1}^{n} m^{k+2-k-1-2}(1-\beta)^{k+2}(k+1)^2$$

$$\leq \tfrac{1}{2m} \sum_{k=1}^{\infty} (1-\beta)^{k+2}(k+1)^2 = \tfrac{1}{m} \cdot \mathcal{O}\left(\tfrac{1}{\beta^3}\right) = \mathcal{O}\left(\tfrac{1}{m}\right) \quad \square$$




- a Choose sequence of $k+2$ keys.
- b Choose to start in top or bottom table.
- c Neighbouring keys share a hash.
- d Two bordering keys connect back inward.

# Proof of i: Success probability is $1 - \mathcal{O}(1/m)$

$$\Pr[\bar{E}] = \Pr[\exists \text{path as shown}]$$

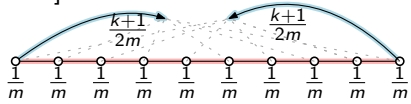
$$= \Pr[\exists k \in \mathbb{N} : \exists x_0, \ldots, x_{k+1} \in S : x_0, \ldots, x_{k+1} \text{ form a path as shown}]$$

$$\overset{\text{union bound}}{\leq} \sum_{k=1}^{n} \sum_{x_0,\ldots,x_{k+1} \in S} \Pr[x_0, \ldots, x_{k+1} \text{ form a path as shown}]$$

$$\leq \sum_{k=1}^{n} \underbrace{n^{k+2}}_{\text{a}} \cdot \underbrace{2}_{\text{b}} \cdot \underbrace{\tfrac{1}{m^{k+1}}}_{\text{c}} \cdot \underbrace{\left(\tfrac{k+1}{2m}\right)^2}_{\text{d}}$$

$$\leq \tfrac{1}{2} \sum_{k=1}^{n} m^{k+2-k-1-2}(1-\beta)^{k+2}(k+1)^2$$

$$\leq \tfrac{1}{2m} \sum_{k=1}^{\infty} (1-\beta)^{k+2}(k+1)^2 = \tfrac{1}{m} \cdot \mathcal{O}\left(\tfrac{1}{\beta^3}\right) = \mathcal{O}\left(\tfrac{1}{m}\right) \quad \square$$




- a Choose sequence of $k + 2$ keys.
- b Choose to start in top or bottom table.
- c Neighbouring keys share a hash.
- d Two bordering keys connect back inward.

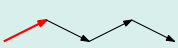
# Proof of ii: Expected insertion time is $\mathcal{O}(1)$

Karlsruhe Institute of Technology

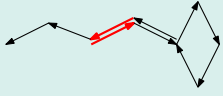
## Lemma

*If the insertion of $y$ takes $t \in \mathbb{N}$ steps then the cuckoo graph $G$ contained (previously) a path of length $\lceil (t-2)/3 \rceil$ starting from $h_0(y)$ or from $h_1(y)$.*

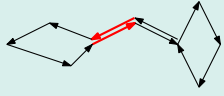
## Proof.



no turning back
$\rightsquigarrow$ path of length $t-1$
starting from $h_0(y)$



turn back once
$\rightsquigarrow$ path of length $\lceil (t-2)/3 \rceil$
starting from $h_0(y)$ or $h_1(y)$



turn back twice
impossible: insertion would fail

□

# Proof of ii: Expected insertion time is $\mathcal{O}(1)$ (continued)

$$\mathbb{E}[T] = \sum_{t\geq 1} \Pr[T \geq t] \qquad \text{tail sum formula}$$

$$\leq \sum_{t\geq 1} \Pr[\exists \text{path of length } \lceil (t-2)/3 \rceil \text{ starting from } h_0(y) \text{ or } h_1(y)] \qquad \text{by Lemma}$$

$$\leq 2 \cdot \sum_{t\geq 1} \Pr[\exists \text{path of length } \lceil (t-2)/3 \rceil \text{ starting from } h_0(y)] \qquad \text{union bound + symmetry}$$

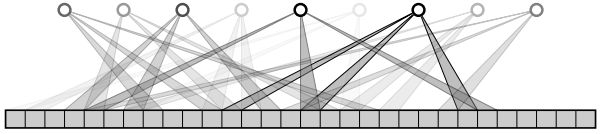
$$\leq 2\left(2 + 3 \cdot \sum_{t\geq 1} \Pr[\exists \text{path of length } t \text{ starting from } h_0(y)]\right) \qquad \sum_{i\geq 1} f(\lceil t/3 \rceil) = 3 \cdot f(1) + 3 \cdot f(2) + \ldots$$

$$\leq 4 + 6 \cdot \sum_{t\geq 1} \sum_{x_1,\ldots,x_t \in S} \Pr[x_1, \ldots, x_t \text{ form path starting from } h_0(y)] \qquad \text{union bound}$$

$$\leq 4 + 6 \cdot \sum_{t\geq 1} n^t m^{-t} \leq 6 \sum_{t\geq 0} (1-\beta)^t = 6/\beta = \mathcal{O}(1). \quad \square$$

# Content

# Cuckoo Hashing with one table and $k$ hash functions



$n \in \mathbb{N}$ keys
$m \in \mathbb{N}$ table size
$\alpha = \frac{n}{m}$ load factor
$h_1, \ldots, h_k \sim \mathcal{U}([m]^D)$ hash functions
↪ Could also use a separate table per hash function.

## randomWalkInsert($x$)

**while** $x \neq \perp$ **do** // TODO: limit
  sample $i \sim \mathcal{U}([k])$
  swap($x, T[h_i(x)]$)

(some improvements possible)

## Theorem (without proof)

For each $k \in \mathbb{N}$ there is a **threshold** $c_k^*$ such that:

- if $\alpha < c_k^*$ all keys can be placed with probability $1 - \mathcal{O}(\frac{1}{m})$.
- if $\alpha > c_k^*$ **not** all keys can be placed with probability $1 - \mathcal{O}(\frac{1}{m})$.

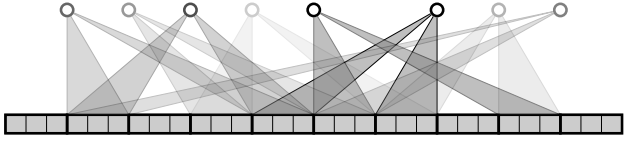
$c_2^* = \frac{1}{2}$, $c_3^* \approx 0.92$, $c_4^* \approx 0.98, \ldots$

## Theorem (Bell, Frieze, 2024; retracted in 2025; re-announced for 2026)

If $k \geq 3$ and $\alpha < c_k^*$ then, conditioned on a high probability event[a], the expected insertion time is $\mathcal{O}(1)$.

[a] Without this conditioning, randomWalkInsert might be trapped in an infinite loop.

# Cuckoo Hashing with $k$ buckets of size $\ell$

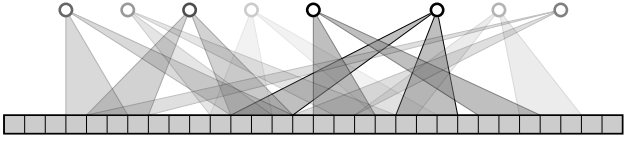


- picture illustrates $k = 2, \ell = 3$
- $k = 2$ has best cache efficiency
- larger $\ell$ improves space efficiency

## Thresholds for the load factor

| $\ell \backslash k$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.9179352767 | 0.9767701649 | 0.9924383913 | 0.9973795528 | 0.9990637588 |
| 2 | 0.8970118682 | 0.9882014140 | 0.9982414840 | 0.9997243601 | 0.9999568737 | 0.9999933439 |
| 3 | 0.9591542686 | 0.9972857393 | 0.9997951434 | 0.9999851453 | 0.9999989795 | 0.9999999329 |
| 4 | 0.9803697743 | 0.9992531564 | 0.9999720661 | 0.9999990737 | 0.9999999721 | 0.9999999992 |
| 5 | 0.9895513619 | 0.9997746588 | 0.9999958681 | 0.9999999374 | 0.9999999992 | $\approx 1$ |
| 6 | 0.9940727066 | 0.9999281468 | 0.9999993570 | 0.9999999956 | $\approx 1$ | $\approx 1$ |

# Cuckoo Hashing with $k$ unaligned blocks of size $\ell$



- picture illustrates $k = 2, \ell = 3$
- note: unaligned block may cross cache line boundary

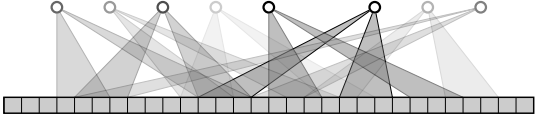
## Thresholds for the load factor (slightly better than for buckets)

| $\ell \backslash k$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| 1 | 0.5 | 0.9179352767 | 0.9767701649 | 0.9924383913 | 0.9973795528 | 0.9990637588 |
| 2 | 0.9649949234 | 0.9968991072 | 0.9996335076 | 0.9999529036 | 0.9999937602 | 0.9999991631 |
| 3 | 0.9944227538 | 0.9998255112 | 0.9999928198 | 0.9999996722 | 0.9999999843 | 0.9999999992 |
| 4 | 0.9989515932 | 0.9999896830 | 0.9999998577 | 0.9999999977 | $\approx 1$ | $\approx 1$ |
| 5 | 0.9997922174 | 0.9999993863 | 0.9999999972 | $\approx 1$ | $\approx 1$ | $\approx 1$ |
| 6 | 0.9999581007 | 0.9999999635 | 0.9999999999 | $\approx 1$ | $\approx 1$ | $\approx 1$ |

# Back to Approximate Membership: Cuckoo Filters

## General Idea

- construct cuckoo table for key set $S \subseteq D$
- store $\text{fp}(x)$ instead of $x \in S$ for random *fingerprint function* $\text{fp} : D \to \{0,1\}^r$
- query$(x)$ checks for $\text{fp}(x)$ in positions associated with $x$



## State of the Art Variant[1]

- uses $k = 2$ unaligned blocks of size $\ell = 2$
  - threshold $\approx 0.965$
  - queries check 4 positions
- false positive probability $\varepsilon \leq 4 \cdot 2^{-r} = 2^{-r+2}$
- space $\approx \frac{r}{0.965} n = 1.04(\log_2(1/\varepsilon) + 2)n$ bits
  // compared to Bloom $\approx 1.44 \log_2(1/\varepsilon)n$ bits

## Supporting Insertions and Deletions

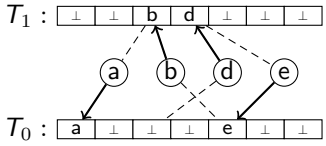
**Complication:** Need to evict fingerprints $\text{fp}(x)$ without knowing $x$.
**Solution:** Positions and fingerprints are related. Tricky details.

[1] Schmitz, Zentgraf, Rahman: Smaller and More Flexible Cuckoo Filters, ALENEX 2026.

# Conclusion

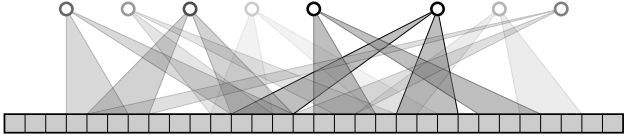
## Classic Cuckoo Hashing

- hash table with *worst case* constant access times
- analysis considers paths in graphs similar to the Erdős-Renyi model




## Practical Cuckoo Hashing

- uses buckets (or unaligned blocks) e.g. $k = 2$ buckets of size $\ell = 8$
- better than conventional hash tables, if high load factors are needed
- cuckoo filters are state of the art dynamic AMQ data structures
- *very* difficult to analyse



# Anhang: Mögliche Prüfungsfragen I

- Was ist und was kann Cuckoo Hashing?
  - Was ist die Grundidee? Wie funktionieren die Operationen?
  - Worauf ist bei der Wahl der Tabellengröße / beim Load Factor zu achten?
  - Was kann man über die Laufzeit der Operationen sagen?
  - Welche Vorteile und Nachteile ergeben sich im Vergleich zu anderen Techniken wie linearem Sondieren?
- Analyse:
  - Eine Einfügung, die fehlschlägt, entspricht gewissen Strukturen im Cuckoo-Graphen. Welchen?
  - Wie haben wir gezeigt, dass solche Strukturen unwahrscheinlich sind?
  - Wie haben wir die erwartete Einfügezeit abgeschätzt?