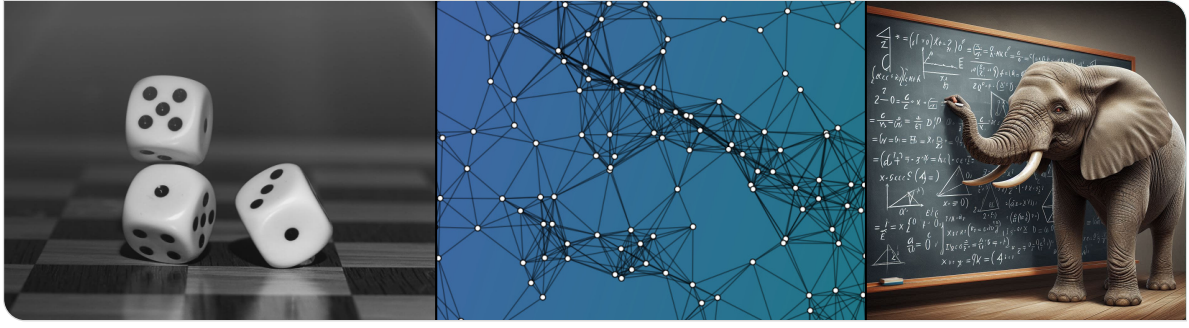
Contents

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Basic Notions and Notation

Stefan Walzer | WS 2025/2026



Basic Notions from Probability Theory

The following holds for *discrete* probability spaces.[1] Gray = typically supressed in notation.

German	English	standard notation & meaning
Ergebnismenge	sample space	set Ω
Ergebnis	outcome	element $\omega \in \Omega$
Wahrscheinlichkeitsfunktion	probability mass function	$p : \Omega \to [0,1]$ with $\sum_{\omega \in \Omega} p(\omega) = 1$
Wahrscheinlichkeitsraum	probability space	(Ω, p)
Ereignis	event	subset $E \subseteq \Omega$
Wahrscheinlichkeit	probability	$\Pr[E] = \sum_{\omega \in E} p(\omega)$ for event E
(reellwertige) Zufallsvariable	(real-valued) random variable	$X : \Omega \to \mathbb{R}$, a "random real number": "$X \geq 2$" means "$X(\omega) \geq 2$"
Erwartungswert	expectation	$\mathbb{E}[X] = \sum_{\omega \in \Omega} p(\omega) \cdot X(\omega) = \sum_x x \cdot \Pr[X = x]$
Varianz	variance	$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$
bedingte Wahrscheinlichkeit	conditional probability	$\Pr[E \mid C] = \Pr[E \cap C] / \Pr[C]$
bedingter Erwartungswert	conditional expectation	$\mathbb{E}[X \mid C] = \sum_x x \cdot \Pr[X = x \mid C]$
Verteilungsfunktion	cumulative distribution function (CDF)	$x \mapsto \Pr[X \leq x]$

[1] Continuous probability spaces are more complicated. The probability mass function is often a probability density function and sums become integrals. We use continuous probability spaces informally in this lecture.

Some Calculation Rules for Probabilities

Linearity of Expectation

$$\mathbb{E}[X_1 + \ldots + X_n] \overset{\text{lin.}}{=} \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n].$$

Tail Sum Formula

If $X : \Omega \to \mathbb{N}_0$ is a random variable assuming natural numbers then

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i].$$

Union Bound

For any events $E_1, \ldots, E_n \subseteq \Omega$

$$\Pr[E_1 \cup \ldots \cup E_n] \overset{\text{UB}}{\leq} \Pr[E_1] + \ldots + \Pr[E_n].$$

Law of Total Probability

If $E_1, \ldots, E_n \subseteq \Omega$ are disjoint events with $E_1 \cup \ldots \cup E_n = \Omega$ and F is any event then

$$\Pr[F] \overset{\text{LTP}}{=} \sum_{i=1}^{n} \Pr[F \mid E_i] \cdot \Pr[E_i].$$

Law of Total Expectation

If $E_1, \ldots, E_n \subseteq \Omega$ are disjoint events with $E_1 \cup \ldots \cup E_n = \Omega$ and X is a random variable then

$$\mathbb{E}[X] \overset{\text{LTE}}{=} \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \cdot \Pr[E_i].$$

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Probability and Computing – The Power of Randomness

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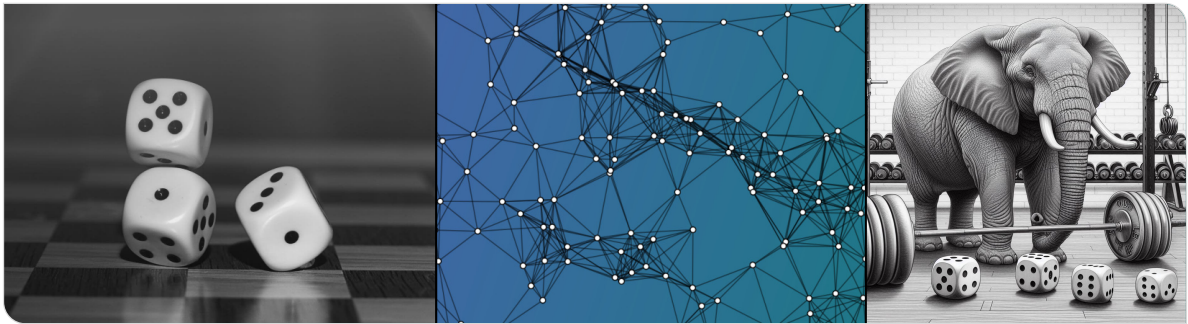

Content

Organisation

Lectures by

Dr. Stefan Walzer

likes elephants

- lectures every Thursday, 11:30
- exercises every second Tuesday, 9:45
- exception: switched in first week
- Website: `https://ae.iti.kit.edu/4994.php`
- Ilias: `https://ilias.studium.kit.edu/goto_produktiv_crs_2777317.html`
  - discuss exercises
  - ask questions
  - report typos / mistakes

Exercises by

Stefan Hermann

- oral exam
- literature:
  - Probability and Computing (Mitzenmacher + Upfal)
  - Randomised Algorithms (Motwani + Raghavan)
  - Modern Discrete Probability (Roch)

Exercises

Organisation

- one sheet published with each lecture
- one exercises session every two weeks
  $\Rightarrow$ two sheets per exercises session
- solutions provided after the exercise session
- optional, no hand-in, no grading. *But:*
- content of sheets relevant for exam
  - you may be asked to reproduce/rediscover solutions in the exam

Recommendation

- **You should**, prior to the exercise session
  - **think about** the exerices or
  - **discuss** them in your study group.
- **You should** do **at least one of** the following
  - **solve** the exercises
  - **attend** the exercise sessions and follow along
  - **work through** the provided solutions
- **We hope** that some of you will
  - **present** your own solutions during sessions

Content

Can Randomness Improve (Worst-Case) Running Time?

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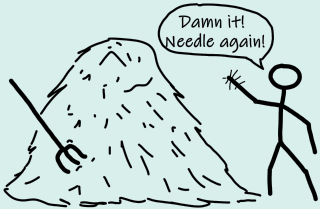
it depends on what you mean by “worst case”...

Worst Input & Worst Luck

Any random decision is the worst decision.
↪ randomness is useless.

Finding Hay According to This View



Worst Input & Average Luck

Randomness *can* help. See next slide.

↑ this is what we mean in the following

In other words:

1. *We* fix a randomised algorithm.
2. *Adversary* fixes an input.
3. Random choices made independently.

Example 1: Finding an Empty Slot

Task

Input: array $A[1..n]$ where $n/2$ slots are empty
Output: $i \in [n]$ with $A[i] = \text{EMPTY}$

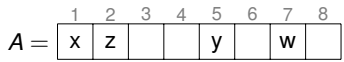
Observation

For any *deterministic* algorithm D there exists an input A such that D inspects $\geq n/2$ entries of A.

Observation

The randomised algorithm R that inspects slots of A at random finds an empty slot after X attempts where

$$\mathbb{E}[X] \overset{\text{TSF}}{=} \sum_{i \in \mathbb{N}_0} \Pr[X > i] = \sum_{i \in \mathbb{N}_0} 2^{-i} = 2.$$



Note

- the analysis of R holds for *any input*
- "$\mathbb{E}$" relates to choices of R (not to input)
- input is fixed *before* random choices

Example 2 and 3: Verifying Identities

Exercise: Verifying Polynomial Identities

Let f and g be two polynomial functions over a field $\mathbb{F}$. For instance:

$$f(x) = (x^3 + 2x^2 - 5x - 6)(x^2 + x - 20)(x - 6) \text{ and } g(x) = x^6 - 7x^3 + 25.$$

Check whether $f \equiv g$ with a randomised algorithm![1]

Exercise: Verifying Matrix Identities (Freivalds' Algorithm)

Let $A, B, C \in \mathbb{F}^{n \times n}$ be matrices over the field $\mathbb{F}$. Check whether $A \cdot B = C$ with a randomised algorithm![1]

[1] The algorithm may occasionally accept incorrect identities. Precise statements on the exercise sheet.

Example 4: Evaluating Games without Draws

Three Types of Game States

value(S) $= 1 =$ W // active player has winning strategy
value(S) $= 0 =$ L // inactive player has winning strategy
~~value(S) $= D$ // draw in optimal play~~

Task: Evaluating a Game

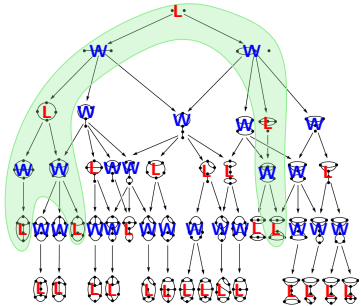
Input: (Implicit representation of) a game.
Output: value of start state.

Observation

A state S is winning if and only if some successor state is losing.

$$\text{value}(S) = \overline{\bigwedge_{S' \text{ successor of } S}} \text{value}(S').$$

Game of Sprouts (see wikipedia)



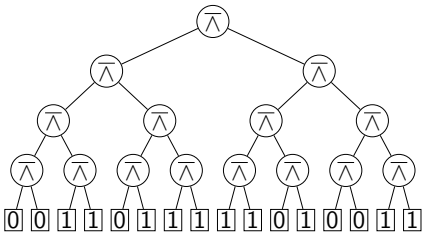
Observation

May not have to inspect entire tree to derive value at root.

Example 4 Simplified: Evaluating $\overline{\wedge}$-Trees

Problem

Input: $I \in \{0,1\}^n$ for $n = 2^d$.
Output: Value of complete binary $\overline{\wedge}$-tree with leaf values from I.
Cost Model: Number of inspected entries of I.



Exercise

For any deterministic algorithm A there exists an input $I_A \in \{0,1\}^n$ such that A inspects all n entries of I.

Our Goal

Randomised algorithm that, for any input, inspects only X entries with

$$\mathbb{E}[X] = \mathcal{O}(n^{0.793}).$$

Example 4 Simplified: Evaluating $\overline{\wedge}$-Trees

Algorithm randEval(T)**:**

- **if** $T = \text{Leaf}(b)$ **then**
  - **return** b
- $(T_0, T_1) \leftarrow T$
- // coin flip:
- sample $r \sim \mathcal{U}(\{0,1\})$
- $b_r \leftarrow \text{randEval}(T_r)$
- **if** $b_r = 0$ **then**
  - **return** 1
- **return** $1 - \text{randEval}(T_{1-r})$

Lemma

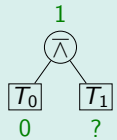
Assume randEval is excecuted for a tree T of depth $d \geq 2$. Let X be the number of resulting calls with subtrees of depth $d-2$. Then $\mathbb{E}[X] \leq 3$.

Proof.

Let $T = (T_0, T_1) = ((T_{00}, T_{01}), (T_{10}, T_{11}))$.

Case 1: $\text{value}(T) = 1$.

- Then $\text{value}(T_0) = 0$ or $\text{value}(T_1) = 0$ (or both).
- Assume (wlog) $\text{value}(T_0) = 0$.
- With probability $1/2$ we select $r = 0$ and T_1 need not be evaluated.



$$\Rightarrow \mathbb{E}[X] \leq \tfrac{1}{2} \cdot 2 + \tfrac{1}{2} \cdot 4 = 3.$$

□

Example 4 Simplified: Evaluating $\overline{\wedge}$-Trees

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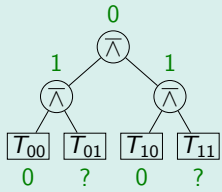
Proof.

Let $T = (T_0, T_1) = ((T_{00}, T_{01}), (T_{10}, T_{11}))$.

Case 2: $\text{value}(T) = 0$.

- Then $\text{value}(T_0) = \text{value}(T_1) = 1$.
- Like before: T_{01} and T_{11} only evaluated with probability $1/2$ each.

$$\Rightarrow \mathbb{E}[X] \leq 2 + \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 1 = 3.$$



□

Example 4 Simplified: Evaluating $\overline{\wedge}$-Trees

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Assume randEval is excecuted for a tree T of depth $d \geq 2$. Let X be the number of resulting calls with subtrees of depth $d-2$. Then $\mathbb{E}[X] \leq 3$.

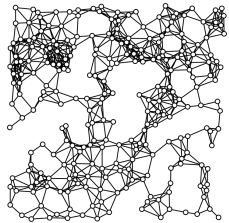
Corollary

Let T be a tree of depth $d \in \{0, 2, 4, \dots\}$, i.e. $n = 2^d$.
The number L of leafs visited by randEval(T) satisfies

$$\underbrace{\mathbb{E}[L] \leq 3^{d/2}}_{\text{proof on blackboard}} = 4^{\log_4(3^{d/2})} = 4^{d/2 \log_4(3)} = 2^{d \log_4(3)} = n^{\log_4(3)}.$$

Content

Average Case Analysis



Theory-Practice Gap

SAT is NP-complete $\overset{???}{\longleftrightarrow}$ modern SAT-solvers handle relevant instances with millions of clauses

Similar observations for NP-hard graph problems on relevant graph classes, e.g. social networks.

Explaining the Gap

1. Define a distribution $\mathcal{I}$ on inputs.
   - $\mathcal{I}$ should be realistic, i.e. model real world instances
   - $\mathcal{I}$ should have simple mathematical structure
2. Show that time to solve $I \sim \mathcal{I}$ is small *in expectation*.

Goals

- model real world instances
- identify useful properties of these instances
- build algorithms exploiting these properties

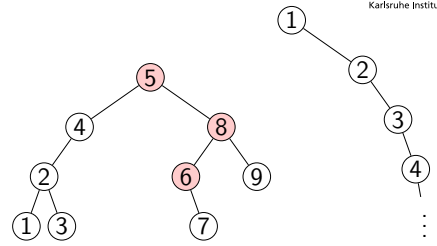
Toy Example: Unbalanced Search Trees

Setting

Inserted $1, \ldots, n$ into search tree *in some order*.
Consider: Depth of Element $y \in \{1, \ldots, n\}$.

Worst Case

Sorted order: $\text{depth}(y) = y$.



Possible Observation

- Alice sees good performance in her setting.
- Can we explain why that might be?

Average Case Analysis

1. Model: Elements of $\{1, \ldots, n\}$ are inserted in random order.
   ↪ Note: May or may not reflect Alice's setting...
2. Goal: Show that $y \in \{1, \ldots, n\}$ has expected depth $\mathcal{O}(\log n)$.
   ↪ Proved on next slide!

Toy Example: Unbalanced Search Trees – Analysis

Lemma

For any $x, y \in [n]$: $\Pr[E_{xy}] = \frac{1}{|y-x|+1}$.

Proof.

Assume wlog $x < y$.
Let v be the element of $\{x, \dots, y\}$ inserted first.
Note: All elements of $\{x, \dots, y\}$ are descendents of v.

Case 1: $v = x$. Then x is ancestor of y.

Case 2: $v = y$. Then y is ancestor of x.

Case 3: $v \notin \{x, y\}$. Then x is in left subtree of v and y in right subtree of v.

Hence E_{xy} occurs $\Leftrightarrow$ $x = v \Leftrightarrow$ Case 1.
Therefore: $\Pr[E_{xy}] = \Pr[\text{Case 1}] = \frac{1}{|\{x,\dots,y\}|} = \frac{1}{y-x+1}$. □

Context

Elements $\{1, \dots, n\}$ inserted into search tree in uniformly random order.

Definition

Event $E_{xy} = \{x \text{ is ancestor of } y\}$

// x counts as ancestor of x

Toy Example: Unbalanced Search Trees – Analysis

Lemma

For any $x, y \in [n]$: $\Pr[E_{xy}] = \frac{1}{|y-x|+1}$.

Theorem

Let $y \in [n]$ and ℓ_y the depth y. Then $\mathbb{E}[\ell_y] \leq 2\ln(n) + 2$.

Proof.

We have $\ell_y = \sum_{x \in [n]} \mathbb{1}_{E_{xy}}$. Hence:

$$\mathbb{E}[\ell_y] \overset{\text{lin.}}{=} \sum_{x \in [n]} \mathbb{E}[\mathbb{1}_{E_{xy}}] = \sum_{x \in [n]} \Pr[E_{xy}] = \sum_{x \in [n]} \frac{1}{|y-x|+1}$$

$$\leq 2 \sum_{i=1}^{n} \tfrac{1}{i} = 2 \cdot H_n \leq 2(\ln(n)+1). \qquad \square$$

Context

Elements $\{1, \ldots, n\}$ inserted into search tree in uniformly random order.

Definition

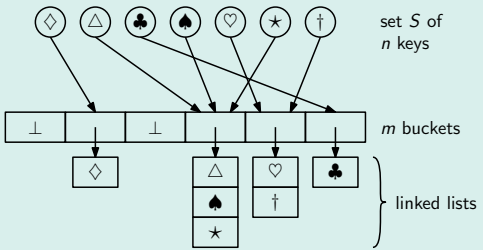
Event $E_{xy} = \{x \text{ is ancestor of } y\}$

// x counts as ancestor of x

Content

Achieve Load Balancing with Pseudorandomness

Hashing with Chaining:



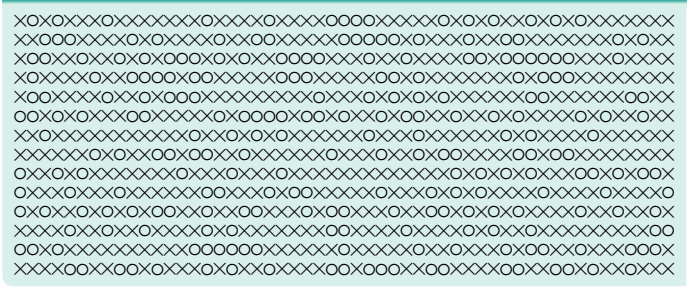
Stay Tuned!

- Linear Probing
- Cuckoo Hashing
- Bloom Filters
- Retrieval
- Perfect Hashing

Content

Approximate in Sublinear Time using Random Sampling

More $\times$ or more $\circ$?



Stay Tuned!

Approximation algorithms can estimate quantities by random sampling.

Content

Semester Outline

Tools from Probability Theory

- Concentration Bounds
- Random Coupling
- Yao's Principle
- Method of Bounded Differences

Random Graph Models

- Erdős-Renyi Random Graphs
- Branching Processes
- Random Geometric Graphs

Other Stuff

- Randomised Complexity Classes
- Probabilistic Method

Algorithm Design

- Random Sampling
- Approximation Algorithms
- Streaming Algorithms
- Probability Amplification

Randomised Data Structures

- Classic Hash Tables
- Cuckoo Hashing
- Bloom Filters
- Retrieval Data Structures
- Perfect Hash Functions

Conclusion

Avoiding the Worst Case with Randomness – Example: $\overline{\wedge}$-Tree Evaluation

Deterministic Algorithms:

- $\forall$Algo : $\exists$Input : Algo slow on Input.
- every algorithm is vulnerable to adversarial inputs

Our Randomised Algorithm:

- On *any* input: fast in expectation.
  on any input: slow if unlucky.
- not vulnerable to adversarial inputs

Average Case Analysis

- Model real world using probability distribution over inputs.
- In many cases random instances . . .
  - . . . are easier to solve than worst-case instances
    $\hookrightarrow$ NP-hard problems may be easy *on average*
  - . . . admit simpler algorithms and data structures
    $\hookrightarrow$ e.g. search trees with random insertion order need no load balancing

Anhang: Mögliche Prüfungsfragen I

- Can we use randomness to improve worst-case running times?
  - In what sense?
  - What is an example?
- How can a randomized algorithm be used to verify a polynomial equation?
- How can a randomized algorithm be used to verify a matrix multiplication?
- With respect to the evaluation of $\overline{\wedge}$-trees:
  - What was our optimization goal?
  - What can be achieved with deterministic algorithms?
  - How does our randomized approach work?
  - What is its running time and why?
- What is average-case analysis, and what is it supposed to achieve?
- How do search trees behave under insertions in random order?
  - What holds for the expected depth of a node, and why?

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Probability and Computing – Important Random Variables and How to Sample Them

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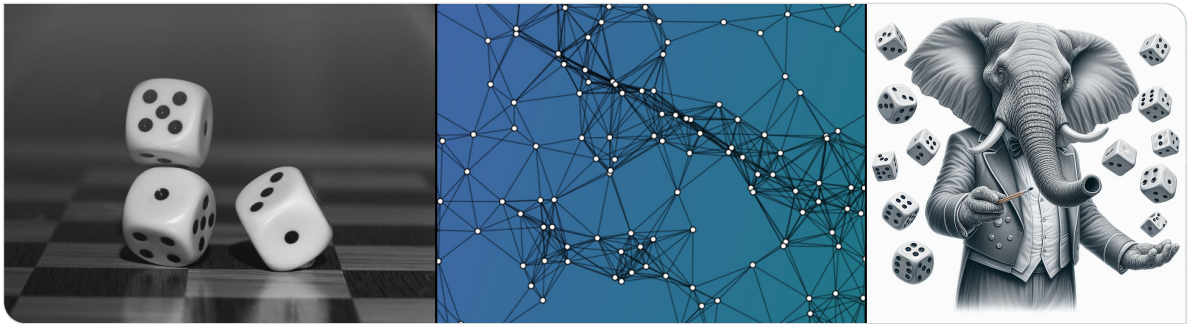


Content

What is a Probability?

Physical Accounts

Probabilities are persistent rates of outcomes when observing the same (random) process over and over again.

It's about **objective stuff**:

"The probability that the coin comes up heads is 50%*."*

Evidential / Bayesian Accounts

Probabilities reflect how much a rational agent believes in a proposition and about how much they are willing to bet on it.

It's about **what I subjectively know**:

"The probability that it is going to rain tomorrow is 33%*."*

See `https://en.wikipedia.org/wiki/Probability_interpretations`.
In this lecture, we use a naive notion.

Bernoulli Distribution

Definition: $\mathrm{Ber}(p)$ for $p \in [0, 1]$

$B \sim \mathrm{Ber}(p)$ is a random variable with

$$\Pr[B = 1] = p \text{ and } \Pr[B = 0] = 1 - p.$$

Standard Assumption: Access to Coin Flips

Algorithms have access to a sequence $B_1, B_2, \ldots \sim \mathrm{Ber}(1/2)$ in independent uniformly random bits.

Exercise: $\mathrm{Ber}(1/3)$ from $\mathrm{Ber}(1/2)$

Design an algorithm that outputs B such that $B \sim \mathrm{Ber}(1/3)$.

Uniform Distribution

Definition: $\mathcal{U}(D)$ on finite D

If $|D| < \infty$, then $X \sim \mathcal{U}(D)$ is a random variable with

$$\Pr[X = x] = \frac{1}{|D|} \text{ for all } x \in D.$$

Definition: $\mathcal{U}(D)$ on infinite D

If D is infinite but has finite measure[a] then $X \sim \mathcal{U}(D)$ is a random variable with uniform density function on D. Important example:

$$X \sim \mathcal{U}([0,1]) \Leftrightarrow \forall x \in [0,1] : \Pr[X < x] = x.$$

[a] Formal details: Not in this lecture.

Standard Assumption

Algorithms have access to $X_1, X_2, \ldots \sim \mathcal{U}([0,1])$.
In practice: Initialise the significand[a] of floating point number with random bits.

[a] Deutsch: Mantisse.

Exercise: $\mathcal{U}(\{1,\ldots,n\})$ from $\mathcal{U}([0,1])$

Design an algorithm that outputs X such that $X \sim \mathcal{U}(\{1,\ldots,n\})$.

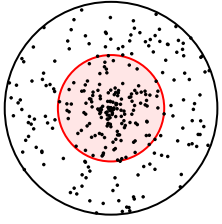
Uniform Distribution on a Disc

Task

Sample $P \sim \mathcal{U}(D)$ for $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Flawed Attempt

sample $\Phi \sim \mathcal{U}([0, 2\pi])$
sample $R \sim \mathcal{U}([0, 1])$
return $(R \cdot \cos \Phi, R \cdot \sin \Phi)$



Issue

Disc of half the radius is hit 50% of the time but makes up only $1/4$ of the area!

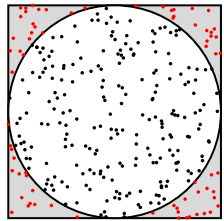
Uniform Distribution on a Disc with Rejection Sampling

Task

Sample $P \sim \mathcal{U}(D)$ for $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Solution with Rejection Sampling

repeat
  sample $X \sim \mathcal{U}([-1, 1])$
  sample $Y \sim \mathcal{U}([-1, 1])$
until $X^2 + Y^2 \leq 1$
return (X, Y)



- Idea: $P \sim \mathcal{U}([-1, 1]^2)$ *conditioned* on $P \in D$ is uniform on D.
- Each sample is accepted with probability $\pi/4$.
- Expected number of rounds is $1/(\pi/4) = \mathcal{O}(1)$.

Spoiler alert: We'll get worst-case constant time with inverse transform sampling later.

Rejection Sampling in General Discrete Distributions

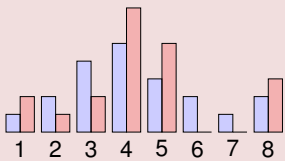
Exercise

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be distributions on a finite[a] set D. Assume

1. We can sample in constant time from $\mathcal{D}_1$.
2. There exists $C > 0$ such that for any $x \in D$ we have

$$\Pr_{X \sim \mathcal{D}_2}[X = x] \leq C \cdot \Pr_{X \sim \mathcal{D}_1}[X = x].$$

Design an algorithm that generates a sample from $\mathcal{D}_2$ in expected time $\mathcal{O}(C)$.



[a] This can be generalised.

Inverse Transform Sampling

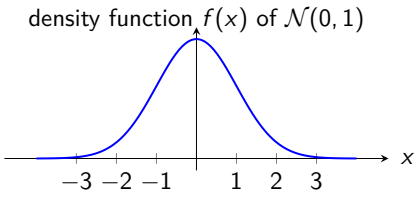
- Let $\mathcal{D}$ be a distribution on $\mathbb{R}$.
  ↪ e.g. $\mathcal{D} = \mathcal{N}(0,1)$
- Let $X \sim \mathcal{D}$ and $F_X(x) = \Pr[X \leq x]$.
  ↪ F_X is the *cumulative distribution function* of X
  ↪ the CDF of the normal distribution is called Φ
- Let $F_X^{-1}(u) := \inf\{x \in \mathbb{R} \mid F_X(x) \geq u\}$.
  ↪ ordinary inverse for strictly monotone F_X

Theorem (Inverse Transform Sampling)

If $U \sim \mathcal{U}([0,1])$ then $F_X^{-1}(U) \overset{\text{d}}{=} X$, i.e. $F_X^{-1}(U) \sim \mathcal{D}$.

("$\overset{\text{d}}{=}$" means: "has the same distribution as")

Reason: $\Pr[F_X^{-1}(U) \leq x] = \Pr[U \leq F_X(x)] = F_X(x)$.



Uniform Distribution on a Disc with Inverse Transform Sampling

Task

Sample $P \sim \mathcal{U}(D)$ for $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

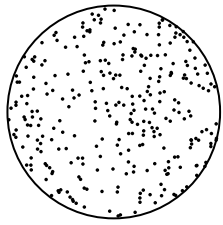
Preparation

If $(x, y) \sim \mathcal{U}(D)$ then $R = \sqrt{x^2 + y^2}$ satisfies

$$F_R(r) = \Pr[R \leq r] = r^2\pi/\pi = r^2 \text{ hence } F_R^{-1}(u) = \sqrt{u}.$$

Solution with Inverse Transform Sampling

sample $\Phi \sim \mathcal{U}([0, 2\pi])$
sample $U \sim \mathcal{U}([0, 1])$
$R \leftarrow \sqrt{U}$
return $(R \cdot \cos\Phi, R \cdot \sin\Phi)$



Geometric Distribution

Definition: $G \sim \text{Geom}_1(p)$ and $G' \sim \text{Geom}_0(p)$

Let $p \in (0, 1]$ and $B_1, B_2, \ldots \sim \text{Ber}(p)$.
Then we define the geometric random variables

$$G := \min\{i \in \mathbb{N} \mid B_i = 1\}$$

$\hookrightarrow$ number of $\text{Ber}(p)$ trials until (and including) the first success

$$G' := G - 1$$

$\hookrightarrow$ number of $\text{Ber}(p)$ failures before the first success

We write $G \sim \text{Geom}_1(p)$ and $G' \sim \text{Geom}_0(p)$.[a]

[a] In the literature Geom is used inconsistently.

Sampling $G \sim \text{Geom}_1(p)$ in time $\mathcal{O}(G)$

$i \leftarrow 0$
repeat
  $i \leftarrow i + 1$
  sample $X \sim \text{Ber}(p)$
until $X = 1$
return i

Quite bad: $\mathbb{E}[G] = 1/p$ might be large.

Exercise

Use inverse transform sampling to sample $G \sim \text{Geom}_1(p)$ in time $\mathcal{O}(1)$.

Sampling Without Replacement

Exercise

Design an algorithm that, given $k, n \in \mathbb{N}$ with $0 \leq k \leq n$ outputs a set $S \subseteq [n]$ of size $|S| = k$ uniformly at random.

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

Algorithm init(k)**:**

allocate `reservoir`$[1..k]$

$n \leftarrow 0$

Algorithm observeItem(x)**:**

$n \leftarrow n + 1$

if $n \leq k$ **then**

`reservoir`$[n] \leftarrow x$

else

sample $I \sim \mathcal{U}(\{1, \dots, n\})$

if $I \leq k$ **then**

`reservoir`$[I] \leftarrow x$

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

Conclusion

General Techniques

- rejection sampling
- inverse transform sampling

Distributions

- Bernoulli distribution
- uniform distribution
- geometric distribution

Other Stuff

- sampling from a set without replacement
- reservoir sampling

Appendix: Possible Exam Questions I

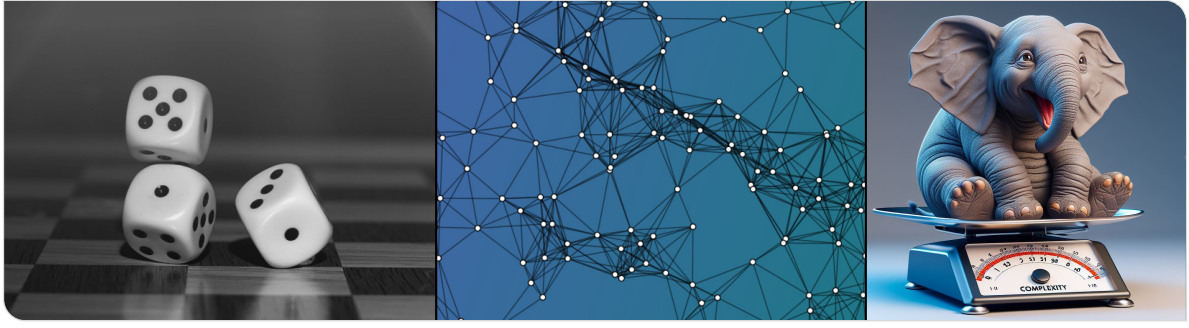
- How can one sample $B \sim \mathrm{Ber}(p)$? What about $X \sim \mathcal{U}(\{1, \ldots, n\})$? Under which assumptions?
- How does rejection sampling work in general? Under which conditions does rejection sampling lead to an efficient algorithm?
- How does inverse transform sampling work in general? Under which conditions does inverse transform sampling lead to an efficient algorithm?
- How can one sample a random point from a disk? Name two techniques and state their advantages and disadvantages.
- Given a set of size n. How can I determine a random subset of size $k \leq n$ and how long does that take?
- Explain reservoir sampling. Isn't that just a slower algorithm for "sampling without replacement"?

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Randomised Complexity Classes

Stefan Walzer | WS 2025/2026



Lecture Notes

Lecture notes by Thomas Worsch available:

Today: Decision Problems Only

- ~~approximation algorithms~~
- ~~average case analysis~~
- ~~data structures~~
- ~~optimisation problems~~
- **decision problems**
  - for some language L such as $L = \text{PRIMES}$
  - decide for input x the question "is $x \in L$?"
  - can you do it in polynomial time?
  - does randomisation help?

Turing machines

(Non-) deterministic Turing machine

- S: finite state set
- B: finite tape alphabet including blank symbol $\square$
- $A \subseteq B - \{\square\}$: input alphabet
- one tape, one head
- transition functions
  - *deterministic:* one
    $\delta : S \times B \to (S \cup \{\text{YES}, \text{NO}\}) \times B \times \{-1, 0, 1\}$
  - *non-deterministic* two (or more)
    $\delta_0, \delta_1 : S \times B \to (S \cup \{\text{YES}, \text{NO}\}) \times B \times \{-1, 0, 1\}$
    (alternatively: general transition *relation*)
  - in states YES and NO: "T halts"
- accepted language
  $L(T) = \{w \in A^+ \mid \exists \text{YES-computation for } w\}$

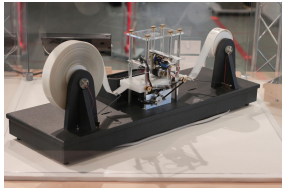


Photo: Rocky Acosta

Probabilistic Turing machine

- definition like non-deterministic TM
- uses δ_0 or δ_1 with probability $1/2$ in each step
- output $T(w)$ is random variable
- difference to NTM:
  - *quantified* non-determinism
  - can study e.g. *probability* of acceptance

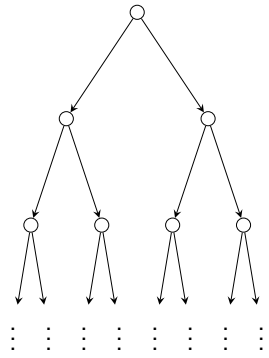
When is a PTM polynomial time?

Annoying

Running time for input x is random variable $T(x) \in \mathbb{N} \cup \{\infty\}$.

Simplification for Today: PTM in normal form

- For all inputs of length n, the PTM *halts* and does so after the *same number of steps* $t(n)$.
  $\hookrightarrow$ this is without loss of generality under weak conditions
- computation tree of PTM in normal form is complete binary tree of depth $t(n)$.
- call $t(n)$ the *running time*
- PTM runs in *polynomial time*, if $t(n) \leq p(n)$ for a polynomial $p(n)$.
- acceptance probability is the $\frac{\text{number of accepting computations}}{2^{t(n)}}$.



“Classic” Complexity Classes

class $\mathcal{C}$	requirement for $L \in \mathcal{C}$
P	polynomial time DTM can decide L
NP	polynomial time NTM can decide L
PSPACE	polynomial space TM can decide L

Complement Classes

For class $\mathcal{C}$ let $\text{co}-\mathcal{C} = \{L \mid \bar{L} \in \mathcal{C}\} = \{\bar{L} \mid L \in \mathcal{C}\}$, e.g.

- $\mathbf{P} = \text{co}-\mathbf{P}$
- $\mathbf{P} \subseteq \mathbf{NP} \cap \text{co}-\mathbf{NP}$
- relationship between **NP** and $\text{co}-\mathbf{NP}$ unknown
- $\mathbf{NP} \cup \text{co}-\mathbf{NP} \subseteq \mathbf{PSPACE}$

Polynomial time reduction from L_1 to L_2

- in polynomial time computable function $f : A^+ \to A^+$, such that
- $\forall w \in A^+$: $w \in L_1 \iff f(w) \in L_2$.

$\hookrightarrow$ then e.g. $L_2 \in \mathbf{NP}$ implies $L_1 \in \mathbf{NP}$.

Hardness

- A language H is $\mathcal{C}$-*hard*, if every language $L \in \mathcal{C}$ can be reduced to H in polynomial time.
- A language is $\mathcal{C}$-*complete*, if it is $\mathcal{C}$-hard and in $\mathcal{C}$.

Probabilistic Complexity Classes

A language L is in class **P/RP/BPP/PP**, if there exists a probabilistic polynomial time turing machine T such that...

class	name	requirement	visualisation	
P	polynomial time	$\forall w \notin L : \Pr[T(w) = \text{YES}] = 0$ $\forall w \in L : \Pr[T(w) = \text{YES}] = 1$		no error
RP	randomised polynomial time	$\forall w \notin L : \Pr[T(w) = \text{YES}] = 0$ $\forall w \in L : \Pr[T(w) = \text{YES}] \geq 1/2$		one-sided error
BPP	bounded-error probabilistic polynomial time	$\forall w \notin L : \Pr[T(w) = \text{YES}] < 1/4$ $\forall w \in L : \Pr[T(w) = \text{YES}] > 3/4$		two-sided error
PP	probabilistic polynomial time	$\forall w \notin L : \Pr[T(w) = \text{YES}] \leq 1/2$ $\forall w \in L : \Pr[T(w) = \text{YES}] > 1/2$		two-sided error

ZPP $:=$ **RP** $\cap$ co$-$**RP**. zero error probabilistic polynomial time

↪ requires *two* Turing machines, one for **RP**, one for co$-$**RP**.

We say a polynomial time PTM is an **RP**-PTM, **BPP**-PTM or **PP**-PTM if it is of the corresponding form.

Probability Amplification

Theorem

Instead of "$1/2$" we can use "$1 - 2^{-q(n)}$" in the definition of $\mathbf{RP}$ without affecting the class.

Proof.

Let T be the Turing machine witnessing $L \in \mathbf{RP}$.
By running T independently $q(n)$ times the error probability is $2^{-q(n)}$.
Running time increases by polynomial factor $q(n)$.

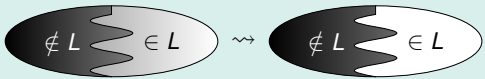
for $i = 1$ *to* $q(n)$ **do**
  if $T(w) =$ YES **then**
    return YES
return NO

□

Probability Amplification (2)

Theorem

Instead of "$1/4$" and "$3/4$" we can use "$2^{-q(n)}$" and "$1 - 2^{-q(n)}$" in the definition of **BPP** without affecting the class.



Proof.

Repeat $\mathcal{O}(q(n))$ times and take the majority answer.
See exercise sheet on probability amplification. □

ZPP: Zero-Error-Probabilistic Polynomial Time

Theorem: $L \in \mathbf{ZPP} \Rightarrow$ Las-Vegas Algorithm for L

If $L \in \mathbf{ZPP} := \mathbf{RP} \cap \mathrm{co}-\mathbf{RP}$ then there exists a PTM that

- decides L with no error
- has *expected* polynomial running time
  ↪ this PTM is not in normal form

Las Vegas Algorithm

Randomised Algorithm that never outputs an incorrect result.
Some definitions allow the algorithm to "give-up", reporting failure.

Proof

Let T be an $\mathbf{RP}$-PTM for L with running time $p(n)$.
↪ never errs for $x \notin L$

Let $\bar{T}$ be an $\mathbf{RP}$-PTM for $\bar{L}$ with running time $p(n)$.
↪ never errs for $x \notin \bar{L}$

repeat
  $r_1 \leftarrow T(w)$
  $r_2 \leftarrow \mathbf{not}\,\bar{T}(w)$
until $r_1 = r_2$
return r_1

- T and $\bar{T}$ never *both* answer incorrectly $\Rightarrow$ we always answer correctly.
- Every round gives $r_1 = r_2$ with probability $\geq 1/2$.

$$\mathbb{E}[\text{running time}] \leq 2p(|w|) \cdot \mathbb{E}[\#\text{rounds}] \overset{\text{TSF}}{=} 2p(|w|) \cdot \sum_{i \geq 1} \Pr[\#\text{rounds} \geq i] \leq 2p(n) \cdot \sum_{i \geq 1} 2^{-(i-1)} = 2p(n) \cdot \sum_{i \geq 0} 2^{-i} = 4p(n). \quad \square$$

Complete Problems?

Remark

The classes $\mathbf{RP}$, $\mathrm{co}-\mathbf{RP}$ and $\mathbf{BPP}$ are not believed to have complete problems unless, e.g. $\mathbf{BPP} = \mathbf{P}$.

A complete problem for $\mathbf{NP}$

$$L = \{(T, x) \mid T \text{ is an } \mathbf{NP}\text{-NTM in normal form and } T \text{ accepts } x\}$$

- L is $\mathbf{NP}$-hard ✓

  Assume $L' \in \mathbf{NP}$
  $\Rightarrow$ there exists $\mathbf{NP}$-NTM T for L' in normal form
  $\Rightarrow$ reduction: $x \in L' \Leftrightarrow (T, x) \in L$

- $L \in \mathbf{NP}$ ✓
  - check if T is $\mathbf{NP}$-NTM in normal form // $\in \mathbf{P}$
  - check if T accepts x // simulate

A complete problem for $\mathbf{RP}$?

$$L = \{(T, x) \mid T \text{ is an } \mathbf{RP}\text{-PTM in normal form and } \Pr[T \text{ accepts } x] \geq 1/2\}$$

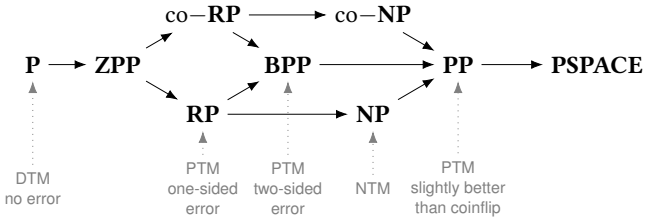
- L is $\mathbf{RP}$-hard ✓

  Assume $L' \in \mathbf{RP}$
  $\Rightarrow$ there exists $\mathbf{RP}$-PTM T for L' in normal form
  $\Rightarrow$ reduction: $x \in L' \Leftrightarrow (T, x) \in L$

- $L \in \mathbf{RP}$ ✗
  - check if T is $\mathbf{RP}$-PTM in normal form ✗
    ⚠ **undecidable**!
  - check if T accepts x // simulate

Content

Beziehungen zwischen Komplexitätsklassen



Exercise

- $\mathbf{P} \subseteq \mathbf{ZPP}$
- $\mathbf{ZPP} \subseteq \mathbf{RP}$ and $\mathbf{ZPP} \subseteq \mathrm{co}-\mathbf{RP}$
- $\mathbf{RP} \subseteq \mathbf{NP}$ and $\mathrm{co}-\mathbf{RP} \subseteq \mathrm{co}-\mathbf{NP}$
- $\mathbf{RP} \subseteq \mathbf{BPP}$ and $\mathrm{co}-\mathbf{RP} \subseteq \mathbf{BPP}$
- $\mathbf{BPP} \subseteq \mathbf{PP}$

Following Slides

- $\mathbf{NP} \subseteq \mathbf{PP}$ and $\mathrm{co}-\mathbf{NP} \subseteq \mathbf{PP}$
- $\mathbf{PP} \subseteq \mathbf{PSPACE}$

“Typecasting” Turing Machines

DTM as NTM

Given DTM T with transition function δ, consider NTM T' with transition functions $\delta_0 = \delta_1 = \delta$.
$\hookrightarrow$ No change in behaviour: $T(w) = \text{YES} \Leftrightarrow T'(w) = \text{YES}$.

NTM as PTM

Given NTM T, we can reinterpret it as a PTM T':

$$T(w) = \text{YES} :\Leftrightarrow \exists \text{YES-computation for } T \text{ and } w \Leftrightarrow \Pr[T'(w) = \text{YES}] > 0$$
$$T(w) = \text{NO} :\Leftrightarrow \nexists \text{YES-computation for } T \text{ and } w \Leftrightarrow \Pr[T'(w) = \text{YES}] = 0$$

PTM as DTM

Given PTM T, we can view it as DTM T' with random bitstring $b = b_1 b_2 \ldots$ as additional input.
In step i transition function δ_{b_i} is used.

$$\Pr[T(w) = \text{YES}] = \Pr_{b_1, b_2, \ldots \sim \text{Ber}(1/2)}[T'(w, b) = \text{YES}].$$

Theorem: $\mathbf{NP} \subseteq \mathbf{PP}$ (analogously $\mathrm{co}-\mathbf{NP} \subseteq \mathbf{PP}$)

i.e. show that each $L \in \mathbf{NP}$ satisfies $L \in \mathbf{PP}$

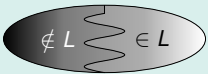
Have: NTM T certifying that $L \in \mathbf{NP}$

$w \in L \Leftrightarrow \exists$YES-computation for T and w

Use the NTM T as a PTM T':

$\forall w \notin L : \Pr[T'(w) = \text{YES}] = 0$
$\forall w \in L : \Pr[T'(w) = \text{YES}] > 0$

Want: PTM T'' certifying that $L \in \mathbf{PP}$



$\forall w \notin L : \Pr[T''(w) = \text{YES}] \leq 1/2$
$\forall w \in L : \Pr[T''(w) = \text{YES}] > 1/2$

T'' achieves this shift with a simple trick

$r \leftarrow T'(w)$ // T' is T as PTM
if $r = \text{YES}$ **then**
  return YES
else
  sample $b \sim \mathcal{U}(\{\text{YES}, \text{NO}\})$ // coinflip
  return b

Theorem: $\mathbf{PP} \subseteq \mathbf{PSPACE}$

i.e. show that each $L \in \mathbf{PP}$ satisfies $L \in \mathbf{PSPACE}$

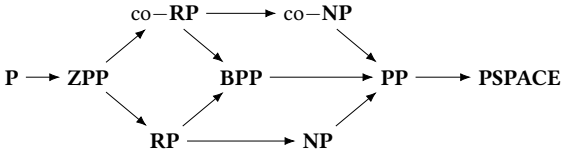
Proof

- Let T a $\mathbf{PP}$-PTM for L with running time $p(n)$.
- Consider DTM T' that simulates T for given w and random choices $b_1 b_2 \dots b_{p(n)}$.
- Consider DTM T'' that for input w runs $T'(w, b_1 b_2 \dots b_{p(n)})$ for all $2^{p(n)}$ possible $b_1 b_2 \dots b_{p(n)}$. Return YES if T' returns YES in majority of cases.
- space complexity:
  - $p(n)$ bits for counter a
  - $p(n)$ bits for $b_1, \dots, b_k$
  - $\mathcal{O}(p(n))$ space for simulating T (can only use $p(n)$ space in its $p(n)$ steps)

$\hookrightarrow$ T'' decides L in space $\mathcal{O}(p(n))$ (and time $\Omega(2^{p(n)})$). $\square$

$n \leftarrow |w|$
$k \leftarrow p(n)$
$a \leftarrow 0$ // k-bit counter
for $b_1 \dots b_k \leftarrow 00\dots0$ **to** $11\dots1$ **do**
  $r \leftarrow T'(w, b_1 \dots b_k)$
  if $r =$ YES **then**
    $a \leftarrow a + 1$
if $a > 2^{k-1}$ **then**
  return YES
else
  return NO

Conclusion



What we learned – not much

- Only “obvious” inclusions known
  ↪ e.g. one-sided error vs. two-sided error
- since $\mathbf{P} \stackrel{?}{=} \mathbf{PSPACE}$ is unsolved, none of the inclusions are known to be strict.
- Remark: History of PRIMES:
  - obviously: in $\mathrm{co}-\mathbf{NP}$.
  - 1976: in $\mathrm{co}-\mathbf{RP}$ (Rabin).
  - 1987: in $\mathbf{RP}$, hence in $\mathbf{ZPP}$ (Adleman, Huang).
  - 2002: in $\mathbf{P}$ (Agrawal, Kayal, Saxena).

A boring topic?

- People believe $\mathbf{BPP} = \mathbf{P}$
  ↪ “each $\mathbf{BPP}$ algorithm can be fully derandomised”
- $\mathbf{PP}$ is somewhat esoteric
  ↪ no interesting randomised classes remain?
- quantum computing may change the story.
  People suspect $\mathbf{NP} \not\subseteq \mathbf{BQP} \not\subseteq \mathbf{NP}$
  ↪ https://en.wikipedia.org/wiki/BQP

Appendix: Possible Exam Questions

- Define: What is a PTM? What is the difference compared to an NTM?
- Define the complexity classes $\mathbf{RP}$, $\mathrm{co}{-}\mathbf{RP}$, $\mathbf{BPP}$, $\mathbf{PP}$, $\mathbf{ZPP}$.
- In what sense do the constants $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$ that appear in the definitions matter? In what sense are they irrelevant?
- How is the class $\mathbf{ZPP}$ related to the concept of a Las Vegas algorithm? What do the transformations in one direction (lecture) and in the other direction (exercise) look like?
- Which inclusion relationships between these complexity classes are known?
- Justify each of these inclusion relationships. (In the actual exam, only one or two would be selected due to time constraints.)
- Are there inclusion relationships known to be strict? Are there classes that experts suspect may actually be identical?

Karlsruhe Institute of Technology

Probability & Computing

Probability Amplification

Probability Amplification

Definition: A **Monte Carlo Algorithm** is a randomized algorithm with bounded running time that, for each input, answers correctly with probability at least $p \in (0, 1)$.

- In decision problems p is the probability of giving the correct answer
  - **One-sided error**: either *false-biased* or *true-biased*
  - **Two-sided error**: *no bias*
- In optimization problems p is the probability of finding the optimum

Algo Output \ Correct Answer	✗	✓
✗	true neg	false neg
✓	false pos	true pos

Definition: **Probability amplification** is the process of increasing the success probability of a Monte Carlo algorithm by using multiple runs.

Probability Amplification for true-biased algorithms

Exercise: For two-sided error.

- Execute independently t times.
  - If ✓at least once: Return ✓. (surely correct)
  - Otherwise: Return ✗. $\Pr[\text{“correct”}] \geq 1 - (1-p)^t \geq 1 - e^{-pt}$

$1 + x \leq e^x$ for $x \in \mathbb{R}$

Probability Amplification

Definition: A **Monte Carlo Algorithm** is a randomized algorithm with bounded running time that, for each input, answers correctly with probability at least $p \in (0, 1)$.

- In decision problems p is the probability of giving the correct answer
  - **One-sided error**: either *false-biased* or *true-biased*
  - **Two-sided error**: *no bias*
- In optimization problems p is the probability of finding the optimum

Algo Output \ Correct Answer	✗	✓
✗	true neg	false neg
✓	false pos	true pos

Definition: **Probability amplification** is the process of increasing the success probability of a Monte Carlo algorithm by using multiple runs.

Probability Amplification for optimization algorithms

- Execute independently t times.
  - output best result

$$\Pr[\text{“optimal”}] \geq 1 - (1-p)^t \geq 1 - e^{-pt}$$

The Clustering Problem

Input

- Set P of points in a feature space (e.g., $\mathbb{R}^d$)
- Similarity measure $\sigma\colon P \times P \mapsto \mathbb{R}_+$

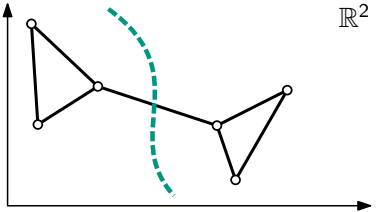
Output: $P_1, \ldots, P_k$ such that

- Points within a P_i have high similarity
- Points in distinct P_i, P_j have low similarity

Applications: Compression, medical diagnosis, etc.

Approach: Model as graph

- Each point is a node
- Edges between all node pairs, with the weight given by the similarity of the two nodes
- Find *cut-set* (edges to remove) of minimal weight such that the graph decomposes into k components.



Example

- six points in $\mathbb{R}^2$
- σ is the inversed Euclidean distance
- partition into two sets

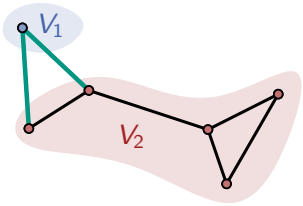
Today

$k = 2$ and $\sigma\colon P \times P \mapsto \{0, 1\}$

Computing Min Cuts

Cuts

- $G = (V, E)$ an unweighted, undirected, connected graph
- *Cut*: partition of V into non-empty parts V_1, V_2.
- *Cut-set*: set of edges with endpoints in V_1 and V_2
- *Weight of a cut*: size of the cut-set (or sum of weights in a weighted graph)



Today Goal: Compute a Min-Cut

i.e. a cut of minimum weight or cut-set of minimum size
the weight of the min-cut is known as the edge-connectivity of G

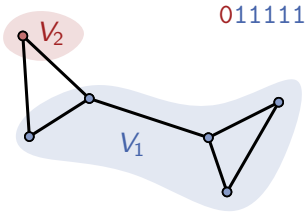
- Known deterministic strategies have worst case running time $\Omega(n^3)$.
- We'll see randomised algorithm with running time $O(n^2 \cdot \log^3(n))$.

A Trivial Algorithm: Random Cut

Observation: There are $2^{n-1} - 1$ cuts in a graph with n nodes.

$(2^n - 2)/2$

- Number of possible assignments of n nodes to 2 parts
- Partitions with empty parts that do not represent cuts
- Swapping parts does not yield a new partition



Algorithm: Random Cut

- Return a uniformly random cut.
- Minor challenge: How to uniformly sample cuts?
  - Represent cut using bit-string
  - Have to uniformly sample bit-string *while avoiding* `11...1` and `00...0`?
    - intution: sample from $\mathcal{U}(\{0,1\}^n)$ and use rejection sampling
    - actually for bounded running time: declare failure rather than sampling again
    - samples each cut with probability $1/2^{n-1}$

Random Cut: Analysis

Running time: $O(n)$ much better than the $\Omega(n^3)$ in the deterministic setting , but...

Success probability: $\geq 1/2^{n-1}$ "$=$" if there is only one min-cut.

$\rightarrow$ exponentially small!

Amplification

- Repeat the algorithm to obtain t independent random cuts, return the smallest

$$\Pr[\text{"min cut found"}] \geq 1 - \left(1 - 1/2^{n-1}\right)^t \geq 1 - e^{-t/2^{n-1}}$$

$1 + x \leq e^x$ for $x \in \mathbb{R}$

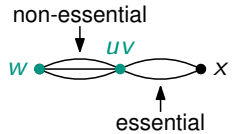
- For $t = 2^{n-1}$ min cut found with constant probability $1 - 1/e \approx 0.63$
- For $t = 2^{n-1} \cdot \ln(n)$ min cut found with high probability $1 - 1/n$

this is terrible so far...

Karger's Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops
- A (multi) graph with two nodes has a unique cut-set



Contraction Algorithm

- Motivation: distinguish *non-essential* (not part of a min-cut) as well as *essential* (part of a min-cut) edges & hope there are few essential ones

Karger$(G_0 = (V_0, E_0))$

for $i = 1$ to $n-2$ **do** // $O(n)$

  sample $e \sim \mathcal{U}(E_{i-1})$ // $O(1)$

  $G_i \leftarrow G_{i-1}$.**contract**(e) // $O(n)$

return unique cut-set in G_{n-2}

- Running time in $O(n^2)$
- Can be implemented to run in $O(m)$

Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i =$ "C in G_i"

$$\begin{aligned}\Pr[\mathcal{E}_1] &= 1 - \tfrac{k}{m} \\ &\geq 1 - \tfrac{k}{nk/2} \\ &= 1 - \tfrac{2}{n}\end{aligned}$$

Observation: min-degree $\geq k$

(holds for all G_i due to 1st observation)

$$m = \frac{1}{2}\sum_{v \in V} \deg(v) \geq \frac{1}{2}\sum_{v \in V} k \geq \frac{1}{2}nk$$

Karger's Algorithm

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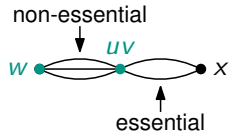

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- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i =$ "C in G_i" **Observation**: min-degree $\geq k$

$\Pr[\mathcal{E}_1] \geq 1 - \frac{2}{n}$ (holds for all G_i due to 1st observation)

$\Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \geq 1 - \frac{2}{n-1} \longrightarrow \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}] \geq 1 - \frac{2}{n-i+1}$

$$\Pr[\mathcal{E}_{n-2}] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_{n-2} \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{n-3}]$$

$$\geq \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\cdots\left(\frac{2}{\cancel{4}}\right)\left(\frac{1}{\cancel{3}}\right)$$

$$= \frac{2}{n(n-1)} \geq \frac{2}{n^2}$$

Karger's Algorithm Amplified

Theorem: On a graph with n nodes, Karger's algorithm runs in $O(n^2)$ time and returns a minimum cut with probability at least $\frac{2}{n^2}$.

$$\Pr[\text{“min-cut found”}] \geq 1 - \exp(-\tfrac{2}{n^2} \cdot t) = 1 - \frac{1}{n} \quad \text{for } t = \tfrac{n^2}{2} \ln(n)$$

Success probability $\geq p$
Number of repetitions t
Amplified prob. $\geq 1 - e^{-pt}$

Corollary: On a graph with n nodes, $O(n^2 \log(n))$ Karger repetitions run in $O(n^4 \log(n))$ total time and return a min-cut with high probability. Much better than exp. time of Randomized Cut!

Sidenote: Number of minimum cuts

- Let $C_1, \ldots, C_\ell$ be all the min-cuts in G and $\underbrace{\mathcal{E}^i_{n-2}}_{\text{disjoint, since the algorithm returns only one cut}}$ for $i \in [\ell]$ be the event that C_i is returned by Karger's algorithm
- Just seen: $\Pr[\mathcal{E}^i_{n-2}] \geq \frac{2}{n^2}$

$$1 \geq \Pr\left[\bigcup_{i \in [\ell]} \mathcal{E}^i_{n-2}\right] = \sum_{i \in [\ell]} \Pr[\mathcal{E}^i_{n-2}] \geq \tfrac{2 \cdot \ell}{n^2}$$

Observation: $\ell \leq \frac{n^2}{2}$.

More Amplification: Karger-Stein

Motivation

- Probability that a min-cut survives i contractions

$$\Pr[\mathcal{E}_i] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}]$$
$$\geq \left(1-\frac{2}{n}\right)\left(1-\frac{2}{n-1}\right)\left(1-\frac{2}{n-2}\right)\cdots\left(1-\frac{2}{n-i+2}\right)\left(1-\frac{2}{n-i+1}\right)$$
$$= \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\cdots\left(\frac{n-i}{\cancel{n-i+2}}\right)\left(\frac{n-i-1}{\cancel{n-i+1}}\right)$$
$$= \frac{(n-i)(n-i-1)}{n(n-1)} \geq \frac{(n-i-1)(n-i-1)}{n \cdot n} = \left(1-\frac{i+1}{n}\right)^2.$$

- Probability becomes very small only towards the very end.

KargerStein$(G_0 = (V_0, E_0))$
  if $|V_0| = 2$ **then** return unique cut-set
  for $i = 1$ to $s = |V_0| - \frac{|V_0|}{\sqrt{2}} - 1$ **do**
    sample $e \sim \mathcal{U}(E_{i-1})$
    $G_i \leftarrow G_{i-1}$.**contract**(e)
$C_1 \leftarrow$ **KargerStein**(G_s) // independent runs
$C_2 \leftarrow$ **KargerStein**(G_s) // runs
return smaller of C_1, C_2

- Idea: stop when a min-cut is still likely to exist and recurse
- After $s = n - n/\sqrt{2} - 1$ steps we have

$$\Pr[\mathcal{E}_s] \geq \left(1 - \frac{n - n/\sqrt{2}}{n}\right) = \left(1 - (1 - 1/\sqrt{2})\right)^2 = (1/\sqrt{2})^2 = \frac{1}{2}$$

Karger-Stein: Running Time

Recursion

- After $t = n - n/\sqrt{2} - 1$ steps the number of nodes is $n/\sqrt{2} + 1$

$$T(n) = 2T\left(\frac{n}{\sqrt{2}} + 1\right) + O(n^2)$$

Solution (essentially by Master Theorem)

$$T(n) = O(n^2 \log n)$$

```
KargerStein(G_0 = (V_0, E_0))
  if |V_0| = 2 then return unique cut-set                    // O(1)
  for i = 1 to s = |V_0| - |V_0|/√2 - 1 do                   // O(n)
    sample e ~ U(E_{i-1})                                    // O(1)
    G_i ← G_{i-1}.contract(e)                                // O(n)
C_1 ← KargerStein(G_s)   // independent runs
C_2 ← KargerStein(G_s)
return smaller of C_1, C_2
```

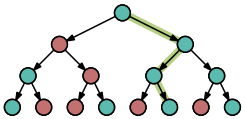
Karger-Stein: Success Probability

Know: Each call to Karger-Stein breaks the min-cut with probability at most $\frac{1}{2}$.

↑ before calling itself recursively

Auxiliary Problem

Given complete binary tree of height d where each node is randomly coloured red or green (with probability $\frac{1}{2}$ each).
What is the probability p_d that a green root-to-leaf path exists?



$p_0 = 1/2$ // root green $\quad p_d = \frac{1}{2}(1-(1-p_{d-1})^2)$ // root green, **not** no path in both left and right subtree

Claim: $p_d \geq \frac{1}{d+2}$. Proof by induction.

$$p_0 = \tfrac{1}{2} = \tfrac{1}{0+2} \quad \checkmark$$

$$p_d = \tfrac{1}{2}\left(1-(1-p_{d-1})^2\right) \geq \tfrac{1}{2}\left(1-(1-\tfrac{1}{d+1})^2\right) = \tfrac{1}{2}\left(\tfrac{2}{d+1} - \tfrac{1}{(d+1)^2}\right)$$

$$= \tfrac{1}{2}\cdot\tfrac{2d+2-1}{(d+1)^2} = \tfrac{1}{2}\cdot\tfrac{2d+1}{d^2+2d+1} \geq \tfrac{1}{2}\cdot\tfrac{2d}{d^2+2d} = \tfrac{1}{d+2} \quad \text{// for } 1 \leq a \leq b \text{ we have } \tfrac{a}{b} \geq \tfrac{a-1}{b-1}$$

Corollary: Karger-Stein succeeds with probability at least $p_{\log_{\sqrt{2}}(n)} = \frac{1}{O(\log n)}$.

Karger-Stein Amplified

Theorem: On a graph with n nodes, Karger-Stein runs in $O(n^2 \log(n))$ time and returns a minimum cut with probability at least $1/O(\log(n))$.

Amplification

$$\Pr[\text{“min-cut found”}] \geq 1 - \exp\left(-\frac{t}{O(\log(n))}\right) = 1 - O\left(\frac{1}{n}\right) \quad \text{for } t = \log^2(n)$$

Success probability $\geq p$
Number of repetitions t
Amplified prob. $\geq 1 - e^{-pt}$

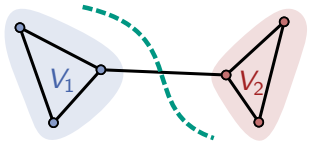
Corollary: On a graph with n nodes, $O(\log^2(n))$ repetitions of Karger-Stein run in $O(n^2 \log^3(n))$ total time and return a minimum cut with high probability.

- Compared to $O(n^4 \log(n))$ for Karger
- Compared to $\Omega(n^3)$ for deterministic approaches

Conclusion

Minimum Cut

- Fundamental graph problem
- Many deterministic flow-based algorithms ...
- ... with worst-case running times in $\Omega(n^3)$



Randomized Algorithms

- Karger's edge-contraction algorithm

Probability Amplification

- Monte Carlo algorithms with and without biases
- Repetitions amplify success probability
- Karger-Stein: Amplify before failure probability gets large

Algo Output \ Correct Answer	✗	✓
✗	true neg	false neg
✓	false pos	true pos

Outlook

"Minimum cuts in near-linear time", Karger, J.Acm. '00

Success w.h.p. in time $O(m \log^3(n))$

"Faster algorithms for edge connectivity via random 2-out contractions", Ghaffari & Nowicki & Thorup, SODA'20

Success w.h.p. in time $O(m \log(n))$ and $O(m + n \log^3(n))$

Possible Exam Questions

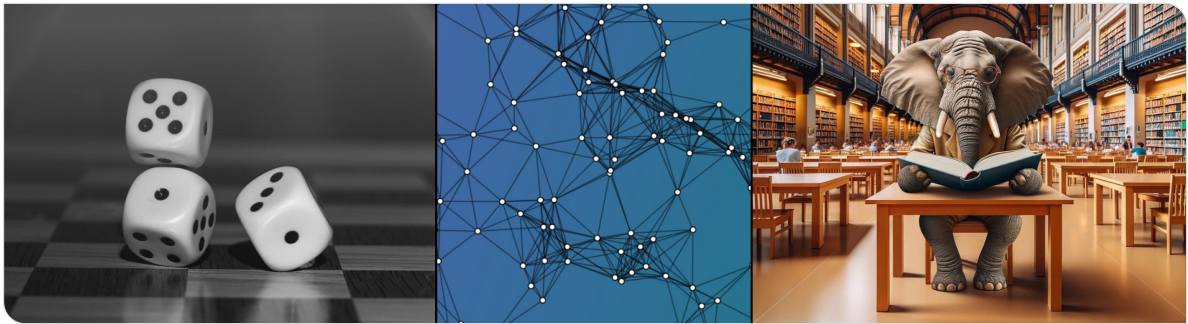
- What is a Monte Carlo algorithm?
  - Which variants exist?
- What is meant by probability amplification?
- How does probability amplification work...
  - ... in the case of one-sided error?
  - ... in the case of two-sided error?
  - ... for optimization problems?
  - How does the error probability relate to the number of repetitions?
- What is the Minimum Cut problem?
  - What do the best known deterministic algorithms achieve?
  - What are success probability and running time of the trivial random cut algorithm?
  - How does Karger's algorithm work?
    - What does $\Pr[\mathcal{E}_t]$ mean, and how did we estimate this probability?
    - What follows for the running time and success probability?
  - How is the algorithm by Karger and Stein obtained from Karger's algorithm?
    - How did we estimate the success probability and running time?
    - How do we achieve a success probability of $1 - \frac{1}{n}$?

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Concentration Bounds

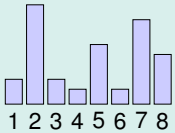
Stefan Walzer | WS 2025/2026



Content

How to describe the shape of a real-valued distribution?

In general: Cannot be summarise in a few numbers



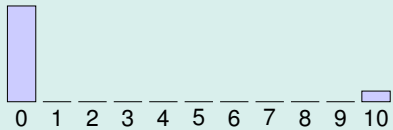
distribution on $[n]$ is given by $\vec{p} \in \mathbb{R}^n_{\geq 0}$ with $\sum_{i=1}^{n} p_i = 1$

- $n - 1$ degrees of freedom
- a few numbers convey little information

Expectation is not always meaningful

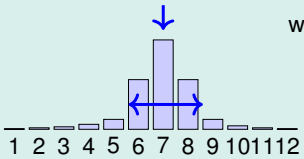
sniper "expected" to hit the target

I "expect" one hair in my soup



How to describe the shape of a real-valued distribution?

Many distributions: Unimodal, i.e. one “bump”



well summarised by

- where is the bump?
- how wide is the bump?
- how much is outside the bump?

Concentration Bound

Statement of the form:

$$\Pr[X \notin [a, b]] \leq \varepsilon.$$

Bound is strong if $b - a$ and ε are small.

Why is this relevant for us?

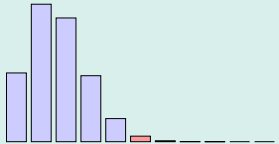
Randomised algorithm might work only if $X \in [a, b]$. Want to bound the failure probability by ε.

Special Cases

Concentration around Expectation:	$\Pr[\lvert X - \mathbb{E}[X]\rvert > c] \leq \varepsilon$	$([a, b] = [\mathbb{E}[X] - c, \mathbb{E}[X] + c])$
Tail bound:	$\Pr[X > b] \leq \varepsilon$	$(a = -\infty)$

Benchmark for What Follows

Running Example



Let $X \sim \text{Bin}(10, \frac{1}{6})$ the number of sixes when rolling a fair die 10 times.

$\Pr[X \geq 5] \approx 0.016.$

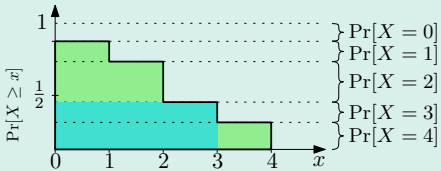
- 0.016 is calculated explicitly // only feasible for tiny examples
- later: we use general methods for obtaining bounds on $\Pr[X \geq 5]$

Markov: Tail bound for non-negative random variables

Markov Inequality

Let X be a non-negative random variable and $b \geq 0$. Then $\Pr[X \geq b] \leq \mathbb{E}[X]/b$.

Proof by Picture



From the picture ($b = 3$):

$$b \cdot \Pr[X \geq b] \leq \sum_{i \geq 1} \Pr[X \geq i] \overset{\text{TSF}}{=} \mathbb{E}[X]$$

Rearranging gives:
$\Pr[X \geq b] \leq \mathbb{E}[X]/b$.

Note: Tight if and only if $\Pr[X \in \{0, b\}] = 1$.

Actual Proof

$$\mathbb{E}[X] \overset{\text{LTE}}{=} \underbrace{\mathbb{E}[X \mid X \geq b]}_{\geq b} \cdot \Pr[X \geq b] + \underbrace{\mathbb{E}[X \mid X < b]}_{\geq 0} \cdot \Pr[X < b] \geq b \cdot \Pr[X \geq b]. \quad \square$$

Markov's Inequality: Benchmark

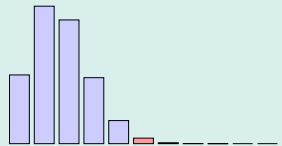
Markov Inequality

$\Pr[X \geq b] \leq \mathbb{E}[X]/b.$

Bound from Markov Inequality

$$\Pr[X \geq 5] \leq \frac{\mathbb{E}[X]}{5} = \frac{10/6}{5} \approx 0.333.$$

Running Example



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Chebyshev's inequality ~~Corollaries to Markov's Inequality~~

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Markov Inequality

Let X be a non-negative random variable and $b \geq 0$. Then $\Pr[X \geq b] \leq \mathbb{E}[X]/b$.

Corollary for non-negative X and strictly increasing $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$\Pr[X \geq b] = \Pr[f(X) \geq f(b)] \leq \mathbb{E}[f(X)]/f(b)$.

Corollary for any real-valued X

$\Pr[|X - \mathbb{E}X| \geq b] \leq \mathbb{E}[|X - \mathbb{E}[X]|]/b$. // but absolute values can be a pain to work with...

Chebyshev's Inequality

$\Pr[|X - \mathbb{E}X| \geq b] = \Pr[|X - \mathbb{E}X|^2 \geq b^2] \leq \mathbb{E}[|X - \mathbb{E}X|^2]/b^2 = \mathrm{Var}(X)/b^2$.

Moments[1]

Definitions

Let X be real-valued random variable and $n \in \mathbb{N}$. Then

$\mathbb{E}[X^n]$ is the nth **(raw) moment**, $\mathbb{E}[(X - \mathbb{E}[X])^n]$ is the nth **central moment**,
$\mathbb{E}[|X|^n]$ is the nth absolute[a] moment, $\mathbb{E}[|X - \mathbb{E}[X]|^n]$ is the nth absolute central moment.

[a] Note: $|\cdot|$ makes a difference only for odd n.

What's so special about the second central moment $\mathrm{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$?

Intuitive Meaning

It's the mean squared distance.

Exercise: Nice mathematical properties

If X, Y are independent, then
$\mathrm{Var}(aX + bY) = a^2 \cdot \mathrm{Var}(X) + b^2 \cdot \mathrm{Var}(Y)$.

[1] deutsch: *das* Moment

Chebyshev's Inequality: Benchmark

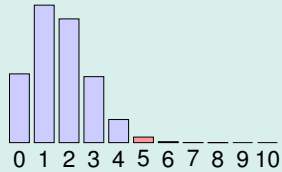
Chebyshev's Inequality

$\Pr[|X - \mathbb{E}X| \geq b] \leq \mathrm{Var}(X)/b^2$.

Variance Calculation

- let $X_i = [i\text{th roll is a six}]$, $X = \sum_{i=1}^{10} X_i$
- $\mathbb{E}[X_i] = \frac{1}{6}$, $\mathbb{E}[X] = \frac{10}{6} = \frac{5}{3}$
- $\mathrm{Var}(X_i) = \frac{1}{6} \cdot (\frac{5}{6})^2 + \frac{5}{6} \cdot (\frac{1}{6})^2 = \frac{5}{36}$.
- $\mathrm{Var}(X) = 10 \cdot \mathrm{Var}(X_1) = \frac{50}{36}$.

Running Example



Let $X \sim \mathrm{Bin}(10, \frac{1}{6})$ the number of sixes when rolling a fair die 10 times.

$\Pr[X \geq 5] \approx 0.016$.

Bound from Chebyshev's Inequality

$\Pr[X \geq 5] = \Pr[X - \mathbb{E}[X] \geq 5 - \frac{5}{3} = \frac{10}{3}] \leq \Pr[|X - \mathbb{E}[X]| \geq \frac{10}{3}] \leq \mathrm{Var}(X)/(\frac{10}{3})^2 = \frac{50 \cdot 9}{36 \cdot 100} = \frac{1}{8} = 0.125$.

Moment Generating Function

Definition

For real-valued random variable X call $M_X(t) = \mathbb{E}[e^{tX}]$ its *moment generating function*.

Remark: Why it's called "moment generating function".

$$M_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}\Big[\sum_{i=0}^{\infty} \frac{(tX)^i}{i!}\Big] = \sum_{i=0}^{\infty} \frac{t^i}{i!} \cdot \mathbb{E}[X^i].$$ // generating function[a] for $(\mathbb{E}[X^0], \mathbb{E}[X^1], \mathbb{E}[X^2], \dots)$

[a] https://en.wikipedia.org/wiki/Generating_function#Exponential_generating_function_(EGF)

Chernoff Bound

Let X be a real-valued RV and $b \geq 0$. Then

- $\Pr[X \geq b] \leq \inf_{t>0} \mathbb{E}[e^{tX}]/e^{tb}$ and
- $\Pr[X \leq b] \leq \inf_{t<0} \mathbb{E}[e^{tX}]/e^{tb}$.

Proof of first variant (second works simlarly)

Let $t > 0$ be arbitrary. Then $x \mapsto e^{tx}$ is increasing.

$$\Pr[X \geq b] = \Pr[e^{tX} \geq e^{tb}] \overset{\text{Markov}}{\leq} \mathbb{E}[e^{tX}]/e^{tb}.$$

Done, since $t > 0$ was arbitrary.

Generic Chernoff Bound: Benchmark

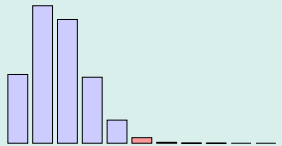
Chernoff Bound

$\Pr[X \geq b] \leq \inf_{t>0} \mathbb{E}[e^{tX}]/e^{tb}$

Calculations

- let $X_i = [i\text{th roll is a six}]$, $X = \sum_{i=1}^{10} X_i$
- $\mathbb{E}[e^{t \cdot X_i}] = \frac{1}{6} \cdot e^t + \frac{5}{6} = \frac{e^t+5}{6}$
- $\mathbb{E}[e^{t \cdot X}] = \mathbb{E}[e^{t \cdot X_1}]^{10} = (\frac{e^t+5}{6})^{10}$

Running Example



Let $X \sim \text{Bin}(10, \frac{1}{6})$ the number of sixes when rolling a fair die 10 times.

$\Pr[X \geq 5] \approx 0.016.$

Note: For independent $X_1, \ldots, X_n$

$\mathbb{E}[e^{t(X_1+\cdots+X_n)}] = \mathbb{E}[e^{tX_1} \cdot \ldots \cdot e^{tX_n}] = \mathbb{E}[e^{tX_1}] \cdot \ldots \cdot \mathbb{E}[e^{tX_n}].$

Resulting Chernoff Bound

$$\Pr[X \geq 5] \leq \inf_{t>0} \mathbb{E}[e^{tX}]/e^{5t} = \inf_{t>0} \left(\frac{e^t+5}{6}\right)^{10}/e^{5t} \overset{\text{Wolfram Alpha}}{\approx} 0.053. \quad // \text{ with } t = \ln(5)$$

What just happened?

Let $X \in \mathbb{N}$ be random variable and $p_i = \Pr[X = i]$ for $i \in \mathbb{N}$.
Consider bounds for $\Pr[X \geq b]$.

Markov

The bound is $\frac{\mathbb{E}[X]}{b}$. The contribution of "$X = i$" to is $\frac{p_i \cdot i}{b}$.

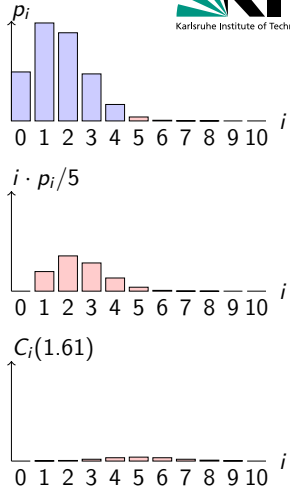
Chernoff

The bound is $\frac{\mathbb{E}[e^{tX}]}{e^{tb}}$. The contribution of "$X = i$" is $C_i(t) := \frac{p_i \cdot e^{ti}}{e^{tb}} = p_i \cdot e^{t(i-b)}$.

- if $i < b$ then $C_i(t)$ is decreasing in t
  $\hookrightarrow$ If $\Pr[X \geq b] = 0$ then Chernoff can prove this with $t \to \infty$.
- if $i > b$ then $C_i(t)$ is increasing in t
  $\hookrightarrow$ that's okay for a while as long as p_i are very small for $i > b$.

Remark: Finding the best t numerically is easy

$C_i''(t) > 0$ for all t, so $t \mapsto \mathbb{E}[e^{tX}]/e^{tb}$ is convex.



The Chernoff Bound for *your* Favourite Random Variable

Maybe someone already did the work...?

- The general Chernoff bound is hard to use
  - How to compute $\mathbb{E}[e^{tX}]$?
  - How to choose t?
- Many variants for special cases exist.

Multiplicative Form for Sums of Bernoulli RV

Theorem

Let $X = X_1 + \cdots + X_n$ with independent $X_i \sim \mathrm{Ber}(p_i)$ and $\mu := \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then for any $\delta > 0$:

$$\Pr[X \geq (1+\delta)\mu] \leq \Bigg(\underbrace{\frac{e^{\delta}}{(1+\delta)^{1+\delta}}}_{f(\delta)} \Bigg)^{\mu}.$$

Note: $f(0) = 1$ and f is decreasing on $(0, \infty)$.
Hence for constant $\delta > 0$ the bound is exponentially small in μ.

Proof.

- $\mathbb{E}[e^{tX_i}] = p_i \cdot e^t + (1 - p_i) = 1 + p_i(e^t - 1) \leq e^{p_i(e^t - 1)}$.
- $\mathbb{E}[e^{tX}] = \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] \leq \prod_{i=1}^{n} e^{p_i(e^t-1)} = e^{\sum_{i=1}^{n} p_i(e^t-1)} = e^{\mu(e^t-1)}$.
- $\Pr[X \geq (1+\delta)\mu] \overset{\text{Chernoff}}{\leq} \frac{\mathbb{E}[e^{tX}]}{e^{(1+\delta)\mu t}} \leq \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}} = \left(\frac{e^{e^t-1}}{e^{(1+\delta)t}}\right)^{\mu} \overset{t=\ln(1+\delta)}{=} \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$.

Consider $\ln(f(\delta)) = \delta - (1+\delta)\ln(1+\delta) =: g(\delta)$. Note that $g'(\delta) = 1 - \frac{1+\delta}{1+\delta} - \ln(1+\delta) = -\ln(1+\delta) < 0$ for $\delta > 0$.
Hence $g(\delta)$ is decreasing on $\mathbb{R}_{\geq 0}$ and so $f(\delta)$ is also decreasing on $\mathbb{R}_{\geq 0}$. □

Multiplicative Form for Sums of Bernoulli RV (ii)

The function $f(\delta) = \frac{e^\delta}{(1+\delta)^{1+\delta}}$ is still inconvenient to work with...

Theorem (from last slide, slightly generalised)

Let $X = X_1 + \cdots + X_n$ with ind. $X_i \sim \text{Ber}(p_i)$ and $\mu := \mathbb{E}[X]$.

Then $\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$ for any $\delta \geq 0$

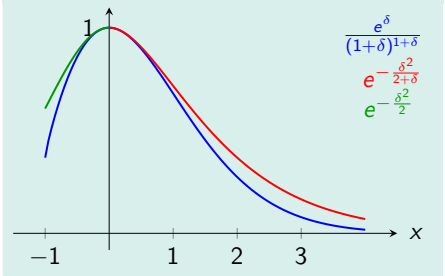
and $\Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^\mu$ for any $\delta \in [0,1)$.

Corollary (in the same setting)

- $\Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\delta^2}{2+\delta}\mu}$ for $\delta \geq 0$,
- $\Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\delta^2}{2}\mu}$ for $\delta \in [0,1)$, and
- $\Pr[|X-\mu| \geq \delta\mu] \leq 2e^{-\frac{\delta^2}{3}\mu}$ for $0 \leq \delta \leq 1$

See also https://en.wikipedia.org/wiki/Chernoff_bound

"Proof by Plot"



Simplified Chernoff Bounds: Benchmark

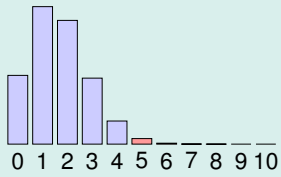
A Simplified Chernoff Bound

$\Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\delta^2}{2+\delta}\mu}$ for $0 \leq \delta$.

Calculations

- $\mu = \mathbb{E}[X] = \frac{10}{6}$
- $5 = 3 \cdot \frac{10}{6} = \mu + 2\mu$ // $\delta = 2$

Running Example



Let $X \sim \text{Bin}(10, \frac{1}{6})$ the number of sixes when rolling a fair die 10 times.

$\Pr[X \geq 5] \approx 0.016$.

Resulting Chernoff Bound

$\Pr[X \geq 5] = \Pr[X \geq (1+2)\mu] \overset{\text{Chernoff}}{\leq} e^{-\frac{2^2}{2+2}\mu} = e^{-\frac{10}{6}} \approx 0.189$.

How do the methods compare?

Exercise

When throwing a fair die n times, bound the probability for seeing twice as many sixes as expected...

- ... using Markov,
- ... using Chebyshev,
- ... using Chernoff (simplified).

Compare the results.

Conclusion

Summary: Methods for Obtaining Concentration Bounds for real-valued X

Method	Assumptions	Formula	Challenges	$\Pr[X \geq 5]$ in Benchmark
—	—	$\Pr[X \geq b] = \sum_{i=b}^{\infty} \Pr[X = i]$	tedious calculation	0.016
Markov	$X \geq 0$	$\Pr[X \geq b] \leq \frac{\mathbb{E}[X]}{b}$	compute $\mathbb{E}[X]$	0.333
Chebyshev	—	$\Pr[\lvert X - \mathbb{E}[X]\rvert \geq b] \leq \frac{\mathrm{Var}(X)}{b^2}$	compute $\mathrm{Var}(X)$	0.125
Chernoff	—	$\Pr[X \geq b] \leq \inf_{t>0} \frac{\mathbb{E}[e^{tX}]}{e^{tb}}$	compute $\mathbb{E}[e^{tX}]$, choose t	0.053
simpl. Ch.	X sum of ind. Ber-RVs	$\Pr[X \geq (1+\delta)\mathbb{E}[X]] \leq e^{-\frac{\delta^2}{2+\delta}\mathbb{E}[X]}$	compute $\mathbb{E}[X]$	0.189

Further Chernoff-flavoured concentration bounds (X aggregates independent $(X_i)_{i\in\mathbb{N}}$)

Hoeffding's inequality, McDiarmid's inequality, Bernstein inequalities.

Appendix: Possible Exam Questions I

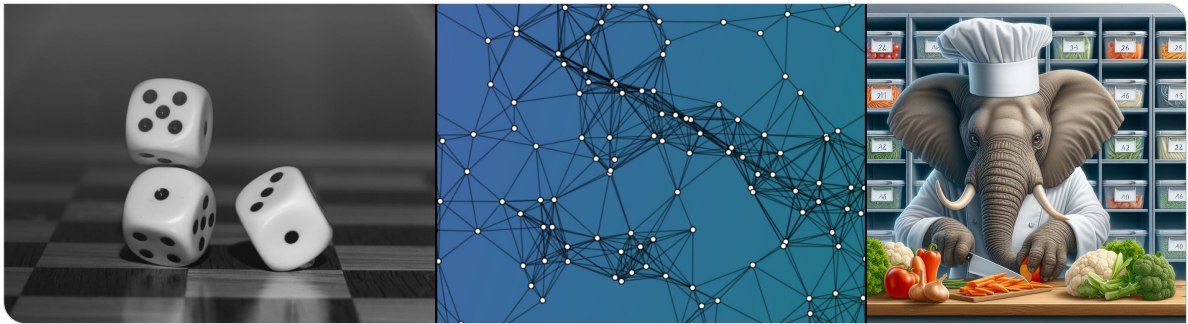
- What is meant by a concentration bound?
- What do the Markov inequality / Chebyshev inequality / Chernoff inequality state?
  - What are the respective assumptions?
  - How difficult are the bounds to apply?
  - How good or bad are the resulting concentration bounds in comparison?
- Detailed questions:
  - Prove the Markov inequality.
  - Prove the Chebyshev inequality (from the Markov inequality).
    - Instead of the second moment, are higher moments suitable for deriving an inequality?
  - Prove the Chernoff inequality (from the Markov inequality).
  - What is the moment-generating function, and why is it called that?
  - How can we bound $\mathbb{E}[e^{tX}]$ when X is a sum of Bernoulli random variables?

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Classic Hash Tables

Stefan Walzer | WS 2025/2026



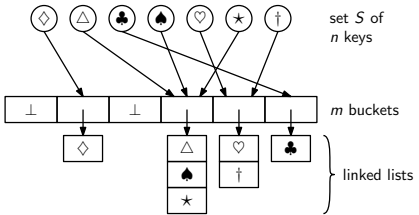
Content

Hash Table with Chaining

e.g. `std::unordered_set`, `java.util.HashMap`

Terminology

D:	*Universe* (or domain) of *keys* (strings, integers, game states in chess)
$S \subseteq D$:	set of n keys (possibly with associated data)
$h : D \to R$:	hash function, range usually $R = [m]$
$\alpha = \frac{n}{m}$:	*load factor*, $\alpha = \mathcal{O}(1)$



Goal

Operations in time t with $\mathbb{E}[t] = \mathcal{O}(1)$.
Randomness comes from the hash function.

Ideal Hash Functions

Every function from D to R is equally likely to be h.

Ideal Hash Functions are Impractical

Naive Idea

- Let R^D denote all functions from D to R. We pick $h \sim \mathcal{U}(R^D)$.
- There are $|R|$ options for the hash of each $x \in D$
- Hence: $|R^D| = |R|^{|D|}$

$x \in D$	x_1	x_2	x_3	...	$x_{\lvert D\rvert}$
$h(x) \in R$	?	?	?	...	?

Why $h \sim \mathcal{U}(R^D)$ is desirable

- $h \sim \mathcal{U}(R^D) \Leftrightarrow \forall x_1, \ldots, x_n \in D : h(x_1), h(x_2), \ldots, h(x_n)$ are *independent* and uniformly random in R.
  $\hookrightarrow$ independence is very useful in an analysis
- In particular: $\forall x_1, \ldots, x_n \in D, \forall i_1, \ldots, i_n : \Pr_{h \sim \mathcal{U}(R^D)}[h(x_1) = i_1 \wedge \ldots \wedge h(x_n) = i_n] = |R|^{-n}$.

Why $h \sim \mathcal{U}(R^D)$ is unwieldy

$\log_2(|R|^{|D|}) = |D| \cdot \log_2(|R|)$ bits to store $h \sim \mathcal{U}(R^D)$ $\quad\rightsquigarrow\quad$ for $D = \{0,1\}^{64}$: more than 2^{64} bits.

Content

What is a Hash Function?

(it depends on who you ask)

Cryptographic Hash Function

A **collision resistant** function such as $h =$ `sha256sum`

```
$ sha256sum myfile.txt
018a7eaee8a...3e79043e21ab4  myfile.txt
```

Range $R = \{0,1\}^{256}$. It is hard to find x, y with $h(x) = h(y)$.
$\hookrightarrow$ Files with equal hashes are likely the same.

Cryptographic Pseudorandom Function

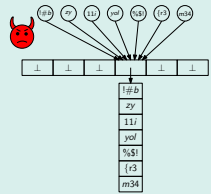
A function f : Seeds $\times D \to R$ where $\log_2 |\text{Seeds}|$ is small and no efficient algorithm can distinguish

- $f(s, \cdot)$ for $s \sim \mathcal{U}(\text{Seeds})$ and
- $h(\cdot)$ for $h \sim \mathcal{U}(R^D)$,

except with negligible probability.

Hash Function in Algorithm Engineering

- typically small range $|R| = \mathcal{O}(n)$
  $\hookrightarrow$ cannot be collision resistant
- should **behave like** $h \sim \mathcal{U}(R^D)$ **in my application**
- should be **fast** to evaluate
- adversarial settings rarely considered, although:



⚠ HashDoS is a thing.
However: Hash function and hash values need not be public.

High-Speed Hashing in Practical Data Structures

Black Magic, do not touch!

Karlsruhe Institute of Technology

MurmurHash

Bitshifts, Magic Constants, ...

```
uint32_t murmur3_32(const uint8_t* key,
          size_t len, uint32_t seed) {
    uint32_t h = seed;
    uint32_t k;
    for (size_t i = len >> 2; i; i--) {
        memcpy(&k, key, sizeof(uint32_t));
        key += sizeof(uint32_t);
        h ^= murmur_32_scramble(k);
        h = (h << 13) | (h >> 19);
        h = h * 5 + 0xe6546b64;
    }
    [...]
    return h;
}
static inline uint32_t murmur_32_scramble(uint32_t k) {
    k *= 0xcc9e2d51;
    k = (k << 15) | (k >> 17);
    k *= 0x1b873593;
    return k;
}
```

Usage

For $R = [m]$, pick seed $\sim \mathcal{U}(\{0,1\}^{32})$ and use

$$h(x) = \texttt{murmur3\_32}(x, \text{seed}) \bmod m \quad \text{// mod slow in practice}$$

$$h'(x) = \lfloor \texttt{murmur3\_32}(x, \text{seed}) \cdot m/2^{32} \rfloor \quad \text{// “fast range”}$$

(see https://github.com/lemire/fastrange)

Does *h* behave like a random function?

- **YES,** with respect to many statistical tests.
  see https://github.com/aappleby/smhasher
- **NO,** HashDoS attacks are known.
  see https://en.wikipedia.org/wiki/MurmurHash#Vulnerabilities
- **MAYBE,** for your favourite application.

Content

What should a Theorist do?

Approach 1: Ignore the Problem

Simple Uniform Hashing Assumption (SUHA)

- We have access to $h \sim \mathcal{U}(R^D)$ for any R and D.
- h takes $\mathcal{O}(1)$ time to evaluate.
- h takes no space to store.

How to Analyse your Algorithm

1. *Assume* SUHA holds.
2. *Analyse* algorithm under SUHA.
3. *Hope* that algorithm still works with real hash functions.

SUHA is “wrong” but adequate

- *Modelling* assumption.
  $\hookrightarrow$ like e.g. ideal gas law in physics
- Excellent track record in non-adversarial settings.

What should a Theorist do?

Approach 2: Bring your own Hash Functions

Analyse Algorithm using Universal Hashing

1. Define family $\mathcal{H} \subseteq R^D$ of hash functions with $\log(|\mathcal{H}|)$ not too large.
   $\hookrightarrow$ sampling and storing $h \in \mathcal{H}$ is cheap
2. Proof that algorithm with $h \sim \mathcal{U}(\mathcal{H})$ has good expected behaviour.

Remarks

- Mathematical structure of $\mathcal{H}$ must be amenable to analysis.
- *Rigorously* covers non-adversarial settings.
- Proofs often difficult.
  $\hookrightarrow$ Wider theory practice gap than with SUHA.

What should a Theorist do?

Approach 3: Let the Cryptographers do the Work

How to Analyse your Algorithm using Cryptographic Assumptions

1. Analyse algorithm under *SUHA*.
2. Actually use *cryptographic pseudorandom function f*.
   - **Case 1:** Everything still works. Great! :-)
   - **Case 2:** Something fails.
     $\Rightarrow$ Your use case can tell the difference between f and true randomness.
     $\hookrightarrow$ The cryptographers said this is impossible. $\lightning$

Should we use cryptographic pseudorandom functions?

- **YES.** Algorithms become robust even in some adversarial settings.
  $\hookrightarrow$ e.g. Python, Haskell, Ruby, Rust use **SipHash** by default
  `https://en.wikipedia.org/wiki/SipHash`
- **NO.** Too slow in high-performance settings.

Hash Function	MiB / sec
SipHash	944
Murmur3F	7623
xxHash64	12109

(source: `https://github.com/rurban/smhasher`)

Content

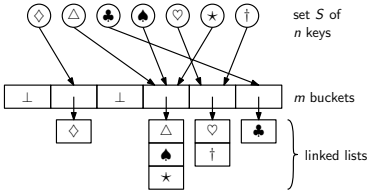
Hash Table with Chaining

Search Time under Chaining

For $n, m \in \mathbb{N}$ and a family $\mathcal{H} \subseteq [m]^D$ of hash functions the *maximum expected search time* is at most

$$T_{\text{chaining}}(n, m, \mathcal{H}) = \max_{\substack{S \subseteq D \\ |S|=n}} \max_{x \in D} \mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})} \Big[1 + |\{y \in S \mid h(y) = h(x)\}| \Big]$$

⚠ Key set is *worst case*. Only $h \in \mathcal{H}$ is random. Key set is fixed *before* h is chosen.



Theorem: Hash Table with Chaining under SUHA

If $\mathcal{H} = [m]^D$ then $T_{\text{chaining}}(n, m, \mathcal{H}) \leq 2 + \alpha = \mathcal{O}(1)$ if $\alpha \in \mathcal{O}(1)$.

Analysis of Hash Table with Chaining under SUHA

Theorem: Hash Table with Chaining under SUHA

Let $\mathcal{H} = [m]^D$, $S \subseteq D$ with $|S| = n$ and $x \in D$ then

$$\mathbb{E}_{h\sim\mathcal{U}(\mathcal{H})}\Big[1 + |\{y \in S \mid h(y) = h(x)\}|\Big] \leq 2 + \alpha$$

Proof.

$$\begin{aligned}
&\mathbb{E}_{h\sim\mathcal{U}(\mathcal{H})}\Big[1 + |\{y \in S \mid h(y) = h(x)\}|\Big] \\
&= \mathbb{E}_{h\sim\mathcal{U}(\mathcal{H})}\Big[1 + \sum_{y\in S}[h(y) = h(x)]\Big] \\
&= 1 + \sum_{y\in S}\mathbb{E}_{h\sim\mathcal{U}(\mathcal{H})}\Big[[h(y) = h(x)]\Big] \\
&= 1 + \sum_{y\in S}\Pr_{h\sim\mathcal{U}(\mathcal{H})}[h(y) = h(x)] \\
&\leq 1 + 1 + \sum_{y\in S\setminus\{x\}}\Pr_{h\sim\mathcal{U}(\mathcal{H})}[h(y) = h(x)] \\
&= 2 + \sum_{y\in S\setminus\{x\}}\frac{1}{m} \leq 2 + \frac{n}{m} = 2 + \alpha. \quad \square
\end{aligned}$$

Content

A Universal Hash Family

Definition: c-universal hash family

A class $\mathcal{H} \subseteq [m]^D$ is called *c-universal* if: $\forall x \neq y \in D : \Pr_{h \sim \mathcal{U}(\mathcal{H})}[h(x) = h(y)] \leq \frac{c}{m}$.

Note: $\mathcal{H} = [m]^D$ is 1-universal.

Reminder (?): Finite Fields

Let $\mathbb{F}_p = \{0, \ldots, p-1\}$ for a prime number p. Then $(\mathbb{F}_p, \times, \oplus)$ is a field where

$$a \times b := (a \cdot b) \bmod p \qquad \textit{and} \qquad a \oplus b := (a + b) \bmod p.$$

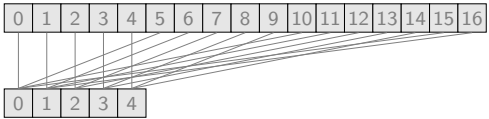
In particular $(\mathbb{F}_p^* := \mathbb{F}_p \setminus \{0\}, \times)$ is a group.

The class of Linear Hash Functions

Assume $D \subseteq \mathbb{F}_p$ for prime p. Then the following class is 1-universal:

$$\mathcal{H}_{p,m}^{\text{lin}} := \{x \mapsto ((a \times x) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}.$$

Can’t we use a simpler class?



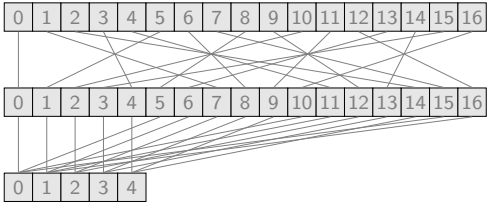
$\bmod m$

$p = 17$
$m = 5$

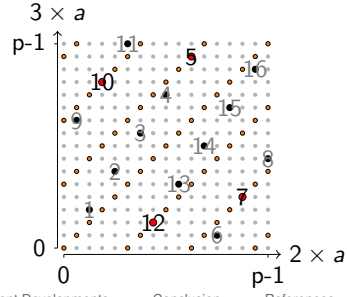
What about just $x \mapsto x \bmod m$?

Nothing is random. We have $h(0) = h(5)$ but this should hold only with probability $1/5$.

Can't we use a simpler class?



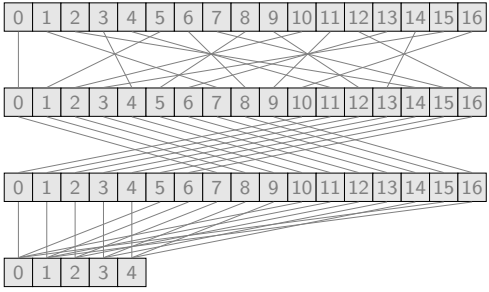
$p = 17$
$m = 5$
$a = 7$ // $\sim \mathcal{U}(\mathbb{F}_p^*)$



What about just $\{x \mapsto (a \times x) \bmod m \mid a \in \mathbb{F}_p^*\}$?

- Example: Do 2 and 3 collide? Picture $\{(a \times 2, a \times 3) \mid a \in \mathbb{F}_p^*\}$.
  $\Pr_{a \sim \mathcal{U}(\mathbb{F}_p^*)}[h(2) = h(3)] = \Pr[a \in \{5, 7, 10, 12\}] = \frac{4}{16} > \frac{3}{15} = \frac{1}{5}$.
  $\Rightarrow$ not 1-universal (but reportedly 2-universal)
- Also note: $h(0) = 0$ is not random.

Can’t we use a simpler class?



$p = 17$
$m = 5$
$a = 7 \;//\; \sim \mathcal{U}(\mathbb{F}_p^*)$
$b = 7 \;//\; \sim \mathcal{U}(\mathbb{F}_p)$

Back to $\{x \mapsto ((a \times x) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}$

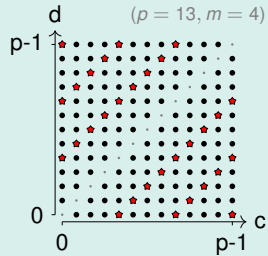
Mathematically “cleaner”. Proof of 1-universality on next slide.

Remark: “$\ldots \bmod m$” can be replaced with “$\lfloor \ldots \cdot \frac{m}{p} \rfloor$” with no downsides.

Proof that $\mathcal{H}^{\text{lin}}_{p,m} := \{x \mapsto ((a \times x) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}$ is 1-universal.

Let $x \neq y \in \mathbb{F}_p$. (To show: $\Pr_{h \sim \mathcal{H}^{\text{lin}}_{p,m}}[h(x) = h(y)] \leq 1/m$.)

- Define $\begin{matrix} c = (a \times x) \oplus b \\ d = (a \times y) \oplus b \end{matrix} \Leftrightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \underbrace{\begin{pmatrix} x & 1 \\ y & 1 \end{pmatrix}}_{\text{regular!}} \begin{pmatrix} a \\ b \end{pmatrix}$.
- The mapping $(a, b) \mapsto (c, d)$ is a bijection (for every $x \neq y$) from $\mathbb{F}_p^* \times \mathbb{F}_p \to P$. // $(a, b) \sim \mathcal{U}(\mathbb{F}_p^* \times \mathbb{F}_p) \Rightarrow (c, d) \sim \mathcal{U}(P)$
- Define *bad set* $B := \{(c, d) \in P \mid c \bmod m = d \bmod m\}$.
  $\hookrightarrow$ from picture: $\frac{|B|}{|P|} \leq \frac{1}{m}$.



$P := \mathbb{F}_p \times \mathbb{F}_p \setminus \{(b, b) \mid b \in \mathbb{F}_p\}$

$$\Pr_{(a,b) \sim \mathcal{U}(\mathbb{F}_p^* \times \mathbb{F}_p)}[h(x) = h(y)] = \Pr_{(a,b)}[((a \times x) \oplus b) \bmod m = ((a \times y) \oplus b) \bmod m]$$

$$= \Pr_{(a,b)}[c \bmod m = d \bmod m] = \Pr_{a,b}[(c, d) \in B] = \Pr_{(c,d) \sim \mathcal{U}(P)}[(c, d) \in B] = \frac{|B|}{|P|} \leq \frac{1}{m}. \quad \square$$

Analysis of Hash Table with Chaining

. . . using a Universal Hash Family

Theorem

If $\mathcal{H} \subseteq [m]^D$ is a c-universal hash family then $T_{\text{chaining}}(n, m, \mathcal{H}) \leq 2 + c\alpha = \mathcal{O}(1)$ if $\alpha \in \mathcal{O}(1)$ and $c \in \mathcal{O}(1)$.

Proof: Mostly the same.

$$\forall S \subseteq [D], \forall x \in D: \quad \mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})}\left[1 + |\{y \in S \mid h(y) = h(x)\}|\right]$$
$$\leq \ldots \leq 2 + \sum_{y \in S \setminus \{x\}} \Pr_{h \sim \mathcal{U}(\mathcal{H})}[h(y) = h(x)]$$
$$= 2 + \sum_{y \in S \setminus \{x\}} \frac{c}{m} \leq 2 + \frac{cn}{m} = 2 + c\alpha. \quad \square$$

Examples for Universal Hash Families

- "$((\boldsymbol{ax} + \boldsymbol{b}) \bmod \boldsymbol{p}) \bmod \boldsymbol{m}$" is 1-universal

  as discussed: $D = \mathbb{F}_p, \qquad R = [m],$

  $$\mathcal{H}^{\text{lin}}_{p,m} := \{x \mapsto ((a \times b) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}$$

- "$(\boldsymbol{ax} \bmod \boldsymbol{p}) \bmod \boldsymbol{m}$" is only 2-universal[a]:

  $$D = \mathbb{F}_p, \qquad R = [m],$$
  $$\mathcal{H} = \{x \mapsto (a \times x) \bmod m \mid a \in \mathbb{F}_p^*\}$$

- **Multiply-Shift** is 1-universal:

  $$D = \{0, \ldots, 2^w - 1\}, \qquad R = \{0, \ldots, 2^\ell - 1\}$$
  $$\mathcal{H} = \{x \mapsto \lfloor((a \cdot x + b) \bmod 2^{2w})/2^{2w-\ell}\rfloor \mid a, b \in \{0, \ldots, 2^{2w} - 1\}\}.$$

[a] I failed to find an exact proof for this, though...

Selling point of multiply shift:

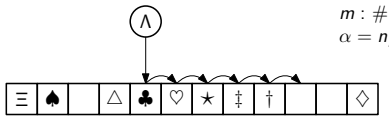
- "$x \bmod 2^{2w}$" drops some higher order bits
- "$\lfloor x/2^{2w-\ell}\rfloor$ drops some lower order bits
- No division or modulo operation needed!

For $w = 32$ (taken from Thorup 2015):

```
uint32_t hash(uint32_t x, uint32_t l,
              uint64_t a, uint64_t b) {
        return (a * x + b) >> (64-l);
}
```

Content

Hash Table with Linear Probing



Operations

For key x *probe* buckets $h(x), h(x)+1, h(x)+2, \ldots \pmod m$.

- **Insert.** Put x into first empty bucket.
- **Lookup.** Look for x, abort when encountering empty bucket.
- **Delete.** Lookup and remove x and ⚠ check if a key to the right wants to move into the hole.[a]

↪ For details see https://en.wikipedia.org/wiki/Linear_probing

[a] Alternative implementations leaves special *tombstones* markers.

We Focus on Insertion Times

- Lookup($x \in S$): At most x's insertion time.
- Lookup($x \notin S$): At most the time it *would take* to insert x now.
- Delete($x \in S$): At most the time it *would take* to insert $y \notin S$ with $h(y) = h(x)$.

Theorem: Linear Probing under SUHA

Let $T_{n,m}$ be the random insertion time into a linear probing hash table. If $\frac{1}{2} \leq \alpha < 1$ then under SUHA we have

$$\mathbb{E}[T_{n,m}] = \mathcal{O}\left(\tfrac{1}{(1-\alpha)^2}\right) = \mathcal{O}(1). \quad \text{(not here)}$$

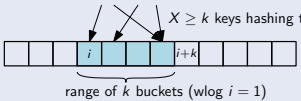
Content

Preparation: A concentration bound

Chernoff

For $X \sim \text{Bin}(n,p)$ and $\delta \in [0,1]$ we have $\Pr[X \geq (1+\delta)\mathbb{E}[X]] \leq \exp(-\delta^2\mathbb{E}[X]/3)$.

Lemma: $\Pr[\geq k$ hits in segment of length $k]$



Let $k \in \mathbb{N}$ and $X = |\{y \in S \mid h(y) \in \{1,\dots,k\}\}|$.

Then $\Pr_{h \sim \mathcal{U}(R^D)}[X \geq k] \leq \exp(-(1-\alpha)^2 k/3)$.

Proof

Let $S = \{x_1,\dots,x_n\}$ and $X_i = [h(x_i) \in \{1,\dots,k\}] \sim \text{Ber}(\frac{k}{m})$.
Then $X = \sum_{i\in[n]} X_i \sim \text{Bin}(n, \frac{k}{m})$ with $\mathbb{E}[X] = \frac{kn}{m} = \alpha k$.

$$\begin{aligned}\Pr[X \geq k] &= \Pr[X \geq \tfrac{1}{\alpha}\mathbb{E}[X]] \\ &= \Pr[X \geq (1+\tfrac{1-\alpha}{\alpha})\mathbb{E}[X]] \\ &\leq \exp(-(\tfrac{1-\alpha}{\alpha})^2 \alpha k/3) \\ &\leq \exp(-(1-\alpha)^2 k/3).\end{aligned}$$

Proof: Expected LP-Insertion Time under SUHA is $\mathcal{O}(1)$

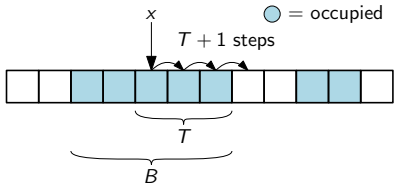
$$\mathbb{E}[T] \leq \mathbb{E}[B] = \sum_{k\geq 1} k \cdot \Pr[B = k] = \sum_{k\geq 1} k \cdot \Pr\left[\bigcup_{d=0}^{k-1} A_{h(x)-d,h(x)-d+k-1}\right]$$

$$\overset{(1)}{\leq} \sum_{k\geq 1} k \cdot \sum_{d=0}^{k-1} \Pr\left[A_{h(x)-d,h(x)-d+k-1}\right] \overset{(2)}{=} \sum_{k\geq 1} k \cdot k \cdot \Pr[A_{1,k}]$$

$$\overset{(3)}{\leq} \sum_{k\geq 1} k^2 \cdot \Pr[|\{y \in S \mid h(y) \in \{1,\ldots,k\}| \geq k]$$

$$\overset{(4)}{\leq} \sum_{k\geq 1} k^2 \cdot \exp(-(1-\alpha)^2 k/3) = \mathcal{O}(1).$$

Wolfram Alpha gives: $\int_0^\infty k^2 \exp(-(1-\alpha)^2 k/3) = \frac{54}{(1-\alpha)^6}$.



$A_{u,v}$: $\{u, v\}$ is maximal occupied block:

Reasoning:

(1) Union Bound. (actually "=")

(2) $h(x)$ is independent of keys in the table and hash distribution is invariant under cyclic shifts.

(3) Note: Keys stored in block cannot come in from the left.

(4) Lemma from previous slide.

Content

Degrees of Independence

(Mutual / Collective) Independence

A family $\mathcal{E}$ of **events** is **independent** if $\forall k \in \mathbb{N}$ and distinct $E_1, \ldots, E_k \in \mathcal{E}$ we have

$$\Pr\left[\bigcap_{i=1}^{k} E_i\right] = \prod_{i=1}^{k} \Pr[E_i].$$

A family $\mathcal{X}$ of discrete **random variables** is **independent** if $\forall k \in \mathbb{N}$, distinct $X_1, \ldots, X_k \in \mathcal{X}$ and all $x_1, \ldots, x_k$ we have

$$\Pr\left[\bigwedge_{i=1}^{k} X_i = x_i\right] = \prod_{i=1}^{k} \Pr[X_i = x_i].$$

Pairwise Independence

A family of **events** is **pairwise independent** if any subfamily of size 2 is independent.

A family of **random variables** is **pairwise independent** if any subfamily of size 2 is independent.

d-wise Independence

A family of **events** is **d-wise independent** if any subfamily of size at most d is independent.

A family of **random variables** is **d-wise independent** if any subfamily of size at most d is independent.

d-Independent Hash Family

Definition: d-Independent Hash Family

A family $\mathcal{H} \subseteq R^D$ of hash functions is *d-independent* if for distinct $x_1, \ldots, x_d \in D$ and any $i_1, \ldots, i_d \in R$: (grey is implied by black)

$$\Pr_{h \sim \mathcal{U}(\mathcal{H})}[h(x_1) = i_1 \wedge \ldots \wedge h(x_d) = i_d] = \prod_{j=1}^{d} \Pr_{h \sim \mathcal{U}(\mathcal{H})}[h(x_j) = i_j] = |R|^{-d}.$$

Alternative Definition

$\mathcal{H}$ is d-independent if for $h \sim \mathcal{U}(\mathcal{H})$

- the family $(h(x))_{x \in D}$ of random variables is d-independent *and*
- $h(x) \sim \mathcal{U}(R)$ for each $x \in D$.

Theorem

Let $D = R = \mathbb{F}$ be a finite field. Then

$$\mathcal{H} := \{x \mapsto \sum_{i=0}^{d-1} a_i x^i \mid a_0, \ldots, a_{d-1} \in \mathbb{F}\}$$

is a d-independent family.

Note: $\mathcal{H} \subseteq \mathbb{F}^{\mathbb{F}} \rightsquigarrow$ not yet useful.

Corollary: Smaller Ranges (proof omitted)

- If m divides $|\mathbb{F}|$, then adding "$\bmod m$" gives a d-independent family $\mathcal{H}' \subseteq [m]^{\mathbb{F}}$.
- If m does not divide $|\mathbb{F}|$, then adding "$\bmod m$" gives a family $\mathcal{H}' \subseteq [m]^{\mathbb{F}}$ such that for $h \sim \mathcal{U}(\mathcal{H}')$ the family $(h(x))_{x \in \mathbb{F}}$ is d-independent but only *approximately* uniformly distributed in $[m]$.

Proof: $\mathcal{H} := \{x \mapsto \sum_{i=0}^{d-1} a_i x^i \mid a_0, \ldots, a_{d-1} \in \mathbb{F}\}$ is d-independent

Let $x_1, \ldots, x_d \in \mathbb{F}$ be distinct keys and $i_1, \ldots i_d \in \mathbb{F}$ arbitrary.

$\hookrightarrow$ to show : $\Pr_{h \sim \mathcal{U}(\mathcal{H})}[\forall j \in [d] : h(x_j) = i_j] = |\mathbb{F}|^{-d}$.

For $h \in \mathcal{H}$ (given via $a_0, \ldots, a_{d-1}$) the following is equivalent:

$$\begin{array}{l} h(x_1) = i_1 \\ h(x_2) = i_2 \\ \vdots \\ h(x_d) = i_d \end{array} \iff \begin{array}{l} a_0 + a_1 x_1 + \cdots + a_{d-1} x_1^{d-1} = i_1 \\ a_0 + a_1 x_2 + \cdots + a_{d-1} x_2^{d-1} = i_2 \\ \vdots \\ a_0 + a_1 x_d + \cdots + a_{d-1} x_d^{d-1} = i_d \end{array} \iff \underbrace{\begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{d-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{d-1} \\ \vdots & & & \ddots & \vdots \\ 1 & x_d & x_d^2 & \ldots & x_d^{d-1} \end{pmatrix}}_{\text{Vandermonde matrix } M \Rightarrow \text{ regular}} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix} = \begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_d \end{pmatrix}$$

Exactly one vector $\vec{a} = M^{-1} \cdot \vec{i}$ solves the equation.

$$\Rightarrow \Pr_{h \sim \mathcal{U}(\mathcal{H})}[\forall j : h(x_j) = i_j] = \Pr_{a_0, \ldots, a_{d-1} \sim \mathcal{U}(\mathbb{F})}[\vec{a} = M^{-1} \cdot \vec{i}] = |\mathbb{F}|^{-d}. \quad \square$$

Concentration Bound for d-Independent Variables

(Tricky) Exercise

Let d be even and $X_1, \dots, X_n \sim \mathrm{Ber}(p)$ a d-independent family of random variables with $p = \Omega(1/n)$.
Let $X = \sum_{i=1}^{n} X_i$. Then for any $\delta > 0$ we have

$$\Pr[X - \mathbb{E}[X] \geq \delta \mathbb{E}[X]] = \mathcal{O}(\delta^{-d} \mathbb{E}[X]^{-d/2}).$$

Remark: Weaker than Chernoff, stronger than Chebyshev

- Chebycheff gives $\Pr[X - \mathbb{E}[X] \geq \delta \mathbb{E}[X]] \leq \frac{1-p}{\delta^2 \mathbb{E}[X]}$. (requires $d = 2$)
  $\hookrightarrow$ uses that $\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$ for pairwise independent $X_1, \dots, X_n$.
- Chernoff gave $\Pr[X - \mathbb{E}[X] \geq \delta \mathbb{E}[X]] \leq \exp(-\delta^2 \mathbb{E}[X]/3)$. (requires $d = n$).

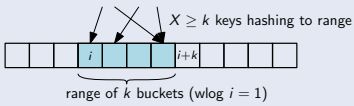
Preparation: A Concentration Bound

again for d-independence

Lemma (last slide)

For d-independent $X_1, \dots, X_n \sim \text{Ber}(p)$ and $X = \sum_{i \in [n]} X_i$ we have $\Pr[X \geq (1+\delta)\mathbb{E}[X]] = \mathcal{O}(\delta^{-d}\mathbb{E}[X]^{-d/2})$.

Lemma: $\geq k$ hits in segment of length k



Let $\mathcal{H}$ be a d-independent hash family and $h \sim \mathcal{U}(\mathcal{H})$.
Let $k \in \mathbb{N}$ and $X = |\{y \in S \mid h(y) \in \{1, \dots, k\}\}|$.

Then $\Pr[X \geq k] \leq \mathcal{O}((1-\alpha)^{-d}k^{-d/2})$.

Proof

Let $S = \{x_1, \dots, x_n\}$ and
$X_i = [h(x_i) \in \{1, \dots, k\}] \sim \text{Ber}(\frac{k}{m})$.
Then $X = \sum_{i \in [n]} X_i$ fits the Lemma with $\mathbb{E}[X] = \frac{kn}{m} = \alpha k$.

$$\begin{aligned}
\Pr[X \geq k] &= \Pr[X \geq \tfrac{1}{\alpha}\mathbb{E}[X]] \\
&= \Pr[X \geq (1 + \tfrac{1-\alpha}{\alpha})\mathbb{E}[X]] \\
&= \mathcal{O}\Big(\Big(\tfrac{1-\alpha}{\alpha}\Big)^{-d}(\alpha k)^{-d/2}\Big) \\
&\leq \mathcal{O}((1-\alpha)^{-d}k^{-d/2}). \quad \text{(using } \alpha \leq 1\text{)}
\end{aligned}$$

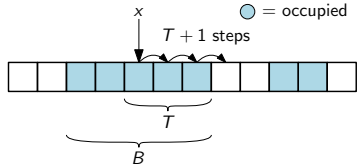
Theorem: Linear Probing with d-independence

Under the same conditions as before, except with 9-independent hash functions, the insertion time $T_{n,m}$ for linear probing satisfies:

$$\mathbb{E}[T_{n,m}] = \mathcal{O}(1)$$

Proof Sketch

$$\begin{aligned}
\mathbb{E}[T] &\leq \mathbb{E}[B] \leq \dots \\
&\overset{(1)}{\leq} \sum_{k \geq 1} k^2 \cdot \Pr[|\{y \in S \mid h(y) \in \{1, \dots, k\}\}| \geq k] \\
&\overset{(2)}{\leq} \sum_{k \geq 1} k^2 \cdot \mathcal{O}((1-\alpha)^{-8} k^{-8/2}) \\
&\leq \sum_{k \geq 1} k^{-2} \cdot \mathcal{O}((1-\alpha)^{-8}) \\
&\overset{(3)}{=} \tfrac{\pi^2}{6} \mathcal{O}((1-\alpha)^{-8}) = \mathcal{O}(1). \quad \square
\end{aligned}$$



$A_{u,v}$: $\{u, v\}$ is a maximal occupied block:

Reasoning:

(1) Same as before, except we have to condition on $h(x)$ and may only use 8-independence in the following. (this is the hand wavy part!)

(2) Concentration bound from previous slide for $d = 8$.

(3) If interested, see 3Blue1Brown video:
`https://www.youtube.com/watch?v=d-o3eB9sfls`

Final Remarks on Linear Probing + Universal Hashing

Much more is known about insertion times of linear probing:

- Any 5-independent family gives $\mathcal{O}(\frac{1}{(1-\alpha)^2})$.
  ↪ A. Pagh, R. Pagh, and Ruzic 2011
- An (artificially bad) 4-independent family gives $\Omega(\log n)$.
  ↪ Pătraşcu and Thorup 2016
- A (well-designed) 3-independent family gives $\mathcal{O}(\frac{1}{(1-\alpha)^2})$.
  ↪ Aamand et al. 2020

Content

Theoretically Good[1] Hash Tables

Remark: Storing $x \in D$ using $\log_2(|D|)$ Bits Wastes $\Theta(\log_2 n)$ bits per key (Cleary 1984)

Example: If $D = [2n]$ and $|S| = n$, then bitvector of $2n$ bits suffices.
↪ Much smaller than Array with $\mathcal{O}(n)$ entries of $\log_2 |D|$ bits!
In General: Can aim for OPT $= \log_2 \binom{|D|}{n} = n \log_2(|D|) - \Theta(n \log_2(n))$ bits rather than $n \cdot \log_2 |D|$.
↪ better than load factor 1

Theoreticians can "have it all" (Arbitman, Naor, and Segev 2010)

- worst case $\mathcal{O}(1)$ operations with high probability (e.g. probability $1 - n^{-100}$)
- *succinctness*, i.e. memory usage $(1 + o(1))$OPT
- explicit families of hash functions

[1] and practically bad

Universal Hashing Beyond k-independence

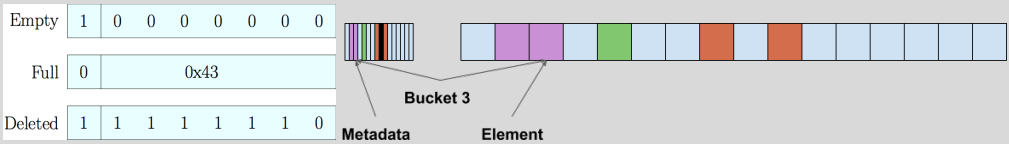
Fast hashing with strong concentration bounds (Aamand et al. 2020)

Variants of “tabulation hashing”:

- – are only 3-independent
- \+ offer Chernoff-style concentration bounds
- \+ are much faster than highly independent hash functions
- – are still slower than what algorithm engineers would use

Linear Probing holds up well

Used at Google: “SwissTable” / `absl::flat_hash_set` = Linear Probing + SIMD



https://abseil.io/about/design/swisstables

Linear Probing: Is “Primary Clustering” a Myth?

Theorem: Primary Clustering

Assume we insert $n = (1 - \varepsilon)m$ elements into a linear probing hash table of size m.
Then the last insertion takes $\mathcal{O}(1/\varepsilon^2)$ time in expectation.

Linear Probing Revisited[2]

- Linear probing tables with tombstones give *amortised* times of $\tilde{\mathcal{O}}(1/\varepsilon^{3/2})$.
- Graveyard hashing improves this to $\mathcal{O}(1/\varepsilon)$.

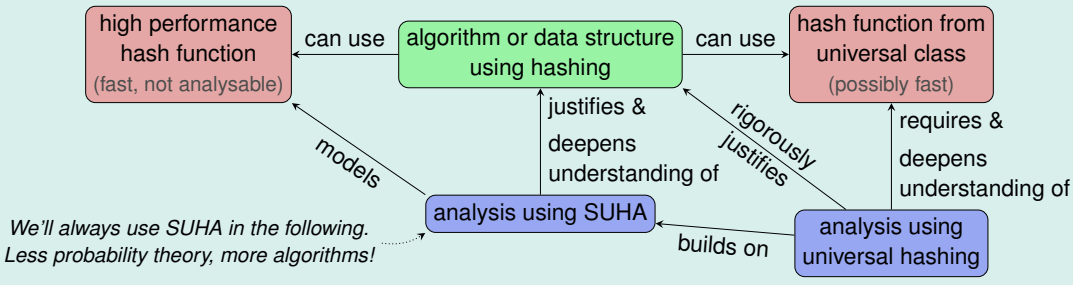
[2] Linear Probing Revisited: Tombstones Mark the Demise of Primary Clustering Bender, B. C. Kuszmaul, and W. Kuszmaul 2022

Conclusion

Technical Takeaway: Performance of Hash Tables

For both an **ideal hash function** (SUHA) and a random hash function from a suitable **universal class**, a hash table using **linear probing** or **chaining** provably has an expected running time of $\mathcal{O}(\mathbf{1})$ per operation.

Non-Technical Takeaway: Approaches to analyse hashing based algorithms



Appendix: Possible Exam Questions I

- What would be an ideal notion of a hash function? In what way would an ideal hash function be useful? What is the problem with this notion?
- What is the Simple Uniform Hashing Assumption (SUHA)? What speaks in favor of making this assumption? What alternatives exist?
- In what way is a pseudorandom function with cryptographic indistinguishability guarantees useful for us? What is the connection to SUHA?*
- Universal hashing:
  - How is c-universality defined?
  - Which c-universal hash class did we study? How did we prove its c-universality?
  - How is d-independence defined for a hash class?
  - Which d-universal hash class did we study?
  - What is the relationship between d-independence and c-universality? (exercise)
  - Chernoff bounds are intended for sums of independent random variables. What can one do when the random variables are only d-independent?*

Appendix: Possible Exam Questions II

- Consider hashing with chaining:
  - What bound on the expected insertion time did we prove? How?
  - At what point does the distribution of the hash function matter?
  - Name a sufficient property that a universal hash class should have so that the proof works.
- Consider hashing with linear probing:
  - What bound on the expected insertion time did we prove? How?
  - At which point does the distribution of the hash function matter?
  - Name a sufficient property that a universal hash class should have so that the proof works.
  - How did we use this property?*

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[2] Yuriy Arbitman, Moni Naor, and Gil Segev. “Backyard Cuckoo Hashing: Constant Worst-Case Operations with a Succinct Representation”. In: *Proc. 51th FOCS*. 2010, pp. 787–796. DOI: 10.1109/FOCS.2010.80.

[3] Michael A. Bender, Bradley C. Kuszmaul, and William Kuszmaul. “Linear Probing Revisited: Tombstones Mark the Demise of Primary Clustering”. In: *FOCS*. 2022, pp. 1171–1182. DOI: 10.1109/FOCS52979.2021.00115.

[4] John G. Cleary. “Compact Hash Tables Using Bidirectional Linear Probing”. In: *IEEE Trans. Computers* 33.9 (1984), pp. 828–834. DOI: 10.1109/TC.1984.1676499.

[5] Anna Pagh, Rasmus Pagh, and Milan Ruzic. “Linear Probing with 5-wise Independence”. In: *SIAM Rev.* 53.3 (2011), pp. 547–558. DOI: 10.1137/110827831. URL: https://doi.org/10.1137/110827831.

References II

[6] Mihai Pătraşcu and Mikkel Thorup. “On the *k*-Independence Required by Linear Probing and Minwise Independence”. In: *ACM Trans. Algorithms* 12.1 (2016), 8:1–8:27. DOI: 10.1145/2716317. URL: https://doi.org/10.1145/2716317.

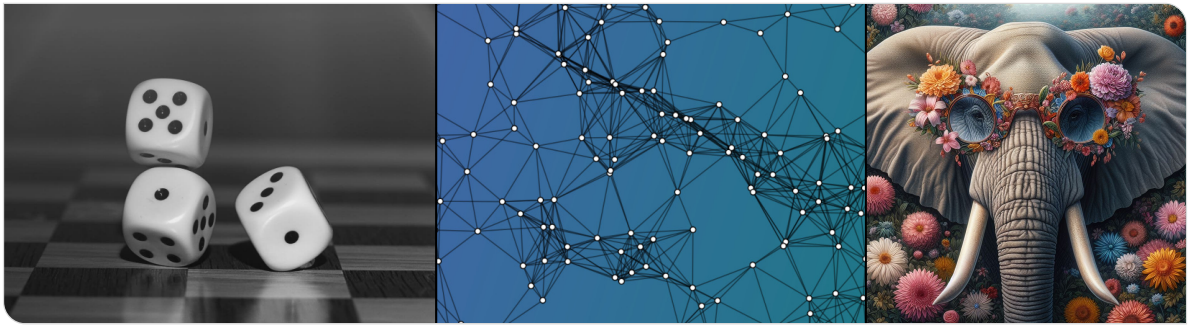
[7] Mikkel Thorup. “High Speed Hashing for Integers and Strings”. In: *CoRR* abs/1504.06804 (2015). arXiv: 1504.06804. URL: http://arxiv.org/abs/1504.06804.

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Bounded Differences and Bloom Filters

Stefan Walzer | WS 2025/2026



Content

More Concentration Bounds

Hoeffding: Let $X = X_1 + \cdots + X_n$ where $a_i \leq X_i \leq b_i$, and $X_1, \ldots, X_n$ independent

$\Pr[X - \mathbb{E}[X] \geq t] \leq \exp(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2})$. // proof combines Chernoff with a different lemma

Special Case: $X_1, \ldots, X_n$ are Bernoulli random variables

$\Pr[X - \mathbb{E}[X] \geq t] \leq \exp(-2t^2/n)$. // often a bit weaker than Chernoff: $\Pr[X \geq (1+\delta)\mathbb{E}[X]] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$

Generalisation: McDiarmid / Method of Bounded Differences

- Assume $X_1, \ldots, X_n$ are independent and $X = f(X_1, \ldots, X_n)$.
- Assume for each i, a change in X_i changes $f(X_1, \ldots, X_n)$ by at most c_i.

$\Pr[X - \mathbb{E}[X] \geq t] \leq \exp(-\frac{2t^2}{\sum_{i=1}^{n} c_i^2})$ // same bound holds for $\Pr[\mathbb{E}[X] - X \geq t]$.

Proof uses Martingales... not here.

A Simple Use Case for the Method of Bounded Differences

Setting

- fixed graph $G = (V, E)$ and $s, t \in V$
- independent edge lengths $X_1, \ldots, X_m \sim \mathcal{U}([0, 1])$
- let P be the shortest $s - t$ path. // unique with probability 1

McDiarmid / Bounded Differences

- $X_1, \ldots, X_n$ independent and $X = f(X_1, \ldots, X_n)$.
- changing X_i changes $f(X_1, \ldots, X_n)$ by at most c_i.

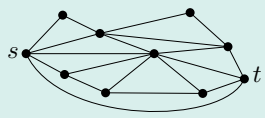
$\Rightarrow \Pr[X - \mathbb{E}[X] \geq t] \leq \exp(-\frac{2t^2}{\sum_{i=1}^{n} c_i^2})$.

Example where McDiarmid might be useful

Let $L :=$ length of $P = \min_{P \text{ is } (s,t) \text{ path}} \sum_{e \text{ on } P} X_e$.

- $L = f(X_1, \ldots, X_m)$ for independent $X_1, \ldots, X_m$. ✓
- Changing X_i changes L by at most 1. $\rightsquigarrow c_i = 1$. ✓

$\Pr[L - \mathbb{E}[L] \geq t] \leq \exp(-\frac{2t^2}{m})$ // might be useful



A Simple Use Case for the Method of Bounded Differences

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McDiarmid / Bounded Differences

- $X_1, \ldots, X_n$ independent and $X = f(X_1, \ldots, X_n)$.
- changing X_i changes $f(X_1, \ldots, X_n)$ by at most c_i.

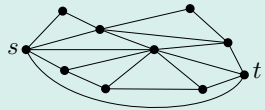
$\Rightarrow \Pr[X - \mathbb{E}[X] \geq t] \leq \exp(-\frac{2t^2}{\sum_{i=1}^{n} c_i^2})$.

Example where McDiarmid seems useless

Let $H :=$ number of edges in P. // "hops"

- $H = f(X_1, \ldots, X_m)$ for independent $X_1, \ldots, X_m$. ✓
- Changing X_i might change H by $n - 2$. $\rightsquigarrow c_i = O(n)$.

$\Pr[H - \mathbb{E}[H] \geq t] \leq \exp(-\frac{2t^2}{\mathcal{O}(mn^2)})$ // too weak



Application of the Method of Bounded Differences

Exercise: “Balls, Bins and Bounded Differences”

Prove that querying all n keys in a hash table with linear chaining takes time $\mathcal{O}(n)$, except with probability n^{-100}.

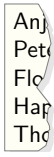
Content

Filter = Approximate Membership Query Data Structure

a set S:

Anja Blancani
Peter Sanders
Florian Kurpicz
Hape Lehmann
Thomas Worsch

a filter for S:



query(Peter Sanders) = **YES**
query(Petra Mutzel) = **YES**
query(Donald Knuth) = **NO**

The **YES** answers are **unreliable**.
The **NO** answers are **reliable**.

Setting

- universe D of possible keys
- a set $S \subseteq D$ of $n = |S|$
- a false positive probability ε

Want: Data structure representing S.

Space Requirement

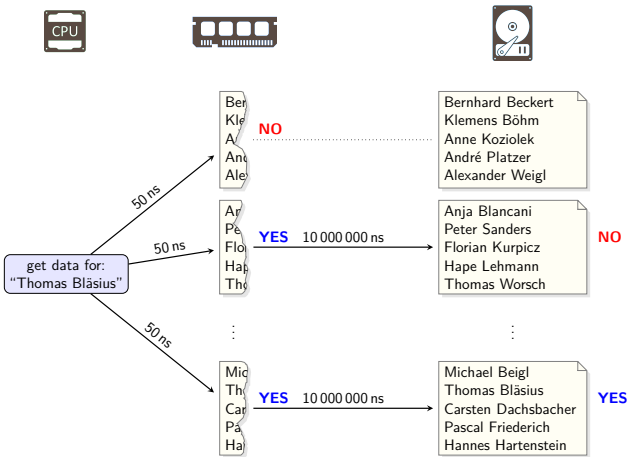
- want $\mathcal{O}(n \log_2(1/\varepsilon))$ bits
- *much* smaller than $\mathcal{O}(n \log_2 |D|)$ bits needed for hash table

Operations

- **insert** elements to S and **delete** elements from S (optional)
- **query**: given $x \in D$ answer "is $x \in S$?" *approximately*:

$$\textbf{query}(x) = \textbf{YES} \text{ for } x \in S$$
$$\Pr[\textbf{query}(x) = \textbf{NO}] \geq 1 - \varepsilon \text{ for } x \notin S$$

Simplified Use Case for Filters



General Idea

If the reliable **NO** answers are frequent, a filter access can replace a (costly) access to a reliable data structure.

Further Example

Filter stores set of malicious URLs.

- Most accessed URLs will be non-malicious.
- Only false positives and true positives have to access reliable data structure (e.g. web-service).

Content

Reminder: SUHA

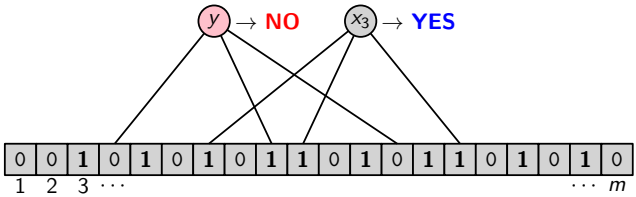
Simple Uniform Hashing Assumption (SUHA)

- We have access to $h \sim \mathcal{U}(R^D)$ for any R and D.
- h takes $\mathcal{O}(1)$ time to evaluate.
- h takes no space to store.

The Bloom Filter Data Structure

Parameters

m	length of a bit array $A[1..m]$ that we use
$k \in \mathcal{O}(1)$	number of hash functions $h_1, \ldots, h_k \sim \mathcal{U}([m]^D)$
n	number of keys in $S \subseteq D$ (dynamic)
$\alpha \in \mathcal{O}(1)$	load n/m (dynamic)



insert(x):

for $i \in [k]$ **do**
  $A[h_i(x)] = 1$

query(x):

for $i \in [k]$ **do**
  if $A[h_i(x)] = 0$ **then**
    return NO
return YES

delete(x):

⟨ not supported ⟩

Content

Preparation

Exercise: Some approximations of e

$$\forall n \in \mathbb{N} : (1 + \tfrac{1}{n})^n \leq e \leq (1 + \tfrac{1}{n})^{n+1}$$
$$\text{and} \quad (1 - \tfrac{1}{n})^n \leq e^{-1} \leq (1 - \tfrac{1}{n})^{n-1}.$$

Corollaries

$$\forall n \in \mathbb{N} : (1 + \tfrac{1}{n})^n = e - \mathcal{O}(1/n)$$
$$\text{and} \quad (1 - \tfrac{1}{n})^n = e^{-1} - \mathcal{O}(1/n).$$

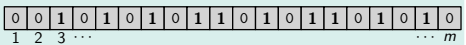
Bloom Filter Analysis (i)

Lemma

Assume $S = \{x_1, \ldots, x_n\}$ is inserted into the Bloom filter. Let $(A_1, \ldots, A_m) \in \{0,1\}^m$ be the random filter state and $Z := \sum_{i=1}^{m}(1 - A_i)$ the number of zeroes. Then

i $\mathbb{E}[\frac{Z}{m}] = (1 - \frac{1}{m})^{m\alpha k} = e^{-\alpha k} - o(1)$

ii For $y \notin S$: $\Pr[\textbf{query}(y) = \textbf{YES} \mid Z = z] = (1 - \frac{z}{m})^k$



Proof of (i).

$$\mathbb{E}[\tfrac{Z}{m}] = \tfrac{1}{m}\mathbb{E}[\sum_{i=1}^{m}(1 - A_i)] = \tfrac{1}{m}\sum_{i=1}^{m}\Pr[A_i = 0] = \tfrac{1}{m}\sum_{i=1}^{m}\Pr[A_1 = 0] = \Pr[A_1 = 0]$$

$$= \Pr[\forall x \in S : \forall i \in [k] : h_i(x) \neq 1] \overset{\text{SUHA}}{=} \prod_{x \in S}\prod_{i \in [k]}\Pr[h_i(x) \neq 1] \overset{\text{SUHA}}{=} \prod_{x \in S}\prod_{i \in [k]}(1 - \tfrac{1}{m})$$

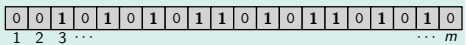
$$= (1 - \tfrac{1}{m})^{nk} = (1 - \tfrac{1}{m})^{m\alpha k} = (e^{-1} - o(1))^{\alpha k} = e^{-\alpha k} - o(1).$$

Bloom Filter Analysis (i)

Lemma

Assume $S = \{x_1, \ldots, x_n\}$ is inserted into the Bloom filter. Let $(A_1, \ldots, A_m) \in \{0,1\}^m$ be the random filter state and $Z := \sum_{i=1}^{m}(1 - A_i)$ the number of zeroes. Then

i. $\mathbb{E}[\frac{Z}{m}] = (1 - \frac{1}{m})^{m\alpha k} = e^{-\alpha k} - o(1)$
ii. For $y \notin S$: $\Pr[\textbf{query}(y) = \textbf{YES} \mid Z = z] = (1 - \frac{z}{m})^k$



Proof of (ii).

$$\Pr[\textbf{query}(y) = \textbf{YES} \mid Z = z] = \Pr[\forall i \in [k] : A_{h_i(y)} = 1 \mid Z = z] \overset{\text{SUHA}}{=} \prod_{i \in [k]} \left(\frac{m - z}{m}\right) = (1 - \tfrac{z}{m})^k.$$

How should a Bloom filter be configured?

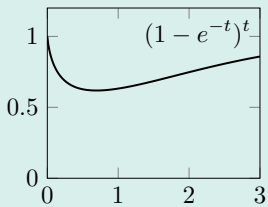
Approximate false positive rate

From the previous Lemma we get for $y \notin S$:

$$\varepsilon = \Pr[\textbf{query}(y) = \textbf{YES}] \overset{?}{\approx} \Pr[\textbf{query}(y) = \textbf{YES} \mid Z = \mathbb{E}[Z]]$$
$$\overset{\text{ii}}{=} \left(1 - \frac{\mathbb{E}[Z]}{m}\right)^k \overset{\text{i}}{=} (1 - e^{-\alpha k} + o(1))^k \approx (1 - e^{-\alpha k})^k.$$

Which k minimises ε? (when α is fixed)

$$\begin{aligned} & \arg\min_{k \in \mathbb{N}} (1 - e^{-\alpha k})^k \\ = \; & \arg\min_{k \in \mathbb{N}} (1 - e^{-\alpha k})^{\alpha k} \\ \approx \; & \tfrac{1}{\alpha} \arg\min_{t \in \mathbb{R}_+} (1 - e^{-t})^t \\ = \; & \tfrac{1}{\alpha} \arg\min_{t \in \mathbb{R}_+} t \ln(1 - e^{-t}) \end{aligned}$$



- plot $(1 - e^{-t})^t \rightsquigarrow$ one global minimum.
- deriving $t \ln(1 - e^{-t})$ gives $\ln(1 - e^{-t}) + \frac{te^{-t}}{1 - e^{-t}}$
- $t = \ln(2)$ is root of the derivative.

$\hookrightarrow k = \ln(2)/\alpha$ is optimal for fixed α.
$\hookrightarrow$ choose α and k such that $\alpha k = \ln(2)$

False Positive Probability and Space

Intuition for optimality of $\alpha k = \ln(2)$

- gives $\mathbb{E}[\frac{Z}{m}] \approx e^{-\alpha k} = \frac{1}{2}$
- maximises *entropy* of the filter bits

Theorem

A Bloom filter with $k \in \mathbb{N}$ hash functions and load factor $\alpha = \ln(2)/k$ has

i. a **space requirement** of $m = n/\alpha = \frac{kn}{\ln 2} \approx 1.44kn$ bits
ii. and a **false positive probability** of $\varepsilon = 2^{-k} + o(1)$.

(i) Space Requirement

Nothing to proof.

(ii) False Positive Probability

- **Know:** $\mathbb{E}[\frac{Z}{m}] \approx e^{-\alpha k} = \frac{1}{2}$ by choice of α
- **Know:** $\frac{Z}{m} = \frac{1}{2} \Rightarrow \varepsilon = 2^{-k}$
- **Need:** $\frac{Z}{m} \approx \mathbb{E}[\frac{Z}{m}]$, i.e. good concentration

Concentration bound for Z

Lemma

i. $\Pr[Z \leq \mathbb{E}[Z] - t \quad] \leq \exp(-\Theta(t^2/m))$ for any $t > 0$,

ii. $\Pr[Z \leq \mathbb{E}[Z] - m^{2/3}] \leq \exp(-\Theta(m^{1/3}))$ by setting $t = m^{2/3}$.

McDiarmid / Bounded Differences

- $X_1, \ldots, X_n$ independent and $X = f(X_1, \ldots, X_n)$.
- changing X_i changes $f(X_1, \ldots, X_n)$ by at most c_i.

$\Rightarrow \Pr[\mathbb{E}[X] - X \geq t] \leq \exp(-\frac{2t^2}{\sum_{i=1}^{n} c_i^2})$.

Difficulty: Filter bits $A_1, \ldots, A_m$ are not independent

- note e.g. that $A_1 = \ldots = A_m = 0$ is impossible (if $n > 0$)
- Generally: $A_1, \ldots, A_m$ are *negatively associated*:
  - Knowing that some bits are zero makes it more likely that others are 1 and vice versa.
- Standard Chernoff bounds don't apply.

Concentration bound for Z

Lemma

i. $\Pr[Z \leq \mathbb{E}[Z] - t \quad] \leq \exp(-\Theta(t^2/m))$ for any $t > 0$,

ii. $\Pr[Z \leq \mathbb{E}[Z] - m^{2/3}] \leq \exp(-\Theta(m^{1/3}))$ by setting $t = m^{2/3}$.

McDiarmid / Bounded Differences

- $X_1, \ldots, X_n$ independent and $X = f(X_1, \ldots, X_n)$.
- changing X_i changes $f(X_1, \ldots, X_n)$ by at most c_i.

$\Rightarrow \Pr[\mathbb{E}[X] - X \geq t] \leq \exp(-\frac{2t^2}{\sum_{i=1}^{n} c_i^2})$.

Proof of (i) using the method of bounded differences.

- Z is a function of kn independent hash values // SUHA
- each hash value can change Z by at most 1

$$\Rightarrow \Pr[Z \leq \mathbb{E}[Z] - t] \leq \Pr[\mathbb{E}[Z] - Z \geq t] = \exp\left(\frac{-2t^2}{nk}\right) = \exp\left(\frac{-2t^2}{m\alpha k}\right) = \exp\left(\frac{-2t^2}{m\ln(2)}\right). \quad \square$$

False Positive Probability of Bloom filters

Proof of Theorem, part (ii): $\varepsilon = 2^{-k} + o(1)$

By choice of k and α we have $\mathbb{E}[\frac{Z}{m}] = e^{-\alpha k} - o(1) = \frac{1}{2} - o(1)$.
Let $y \notin S$ and $B = \lfloor \mathbb{E}[Z] - m^{2/3} \rfloor$.

$$\varepsilon = \Pr[\textbf{query}(y) = \textbf{YES}] \overset{\text{LTP}}{=} \sum_{z=1}^{m} \Pr[Z = z] \cdot \Pr[\textbf{query}(y) = \textbf{YES} \mid Z = z] = \sum_{z=1}^{m} \Pr[Z = z] \cdot (1 - \tfrac{z}{m})^k$$

$$\leq \sum_{z=1}^{B} \Pr[Z = z] + \sum_{z=B+1}^{m} \Pr[Z = z](1 - \tfrac{B+1}{m})^k \leq \Pr[Z \leq B] + (1 - \tfrac{B+1}{m})^k$$

$$\leq \Pr[Z \leq \mathbb{E}[Z] - m^{2/3}] + \big(1 - \tfrac{\mathbb{E}[Z] - m^{2/3}}{m}\big)^k \overset{\text{ii}}{\leq} \exp(-\Theta(m^{1/3})) + (1 - \tfrac{1}{2} + o(1))^k = 2^{-k} + o(1). \quad \square$$

How to Configure Your Bloom Filter

Theorem

A Bloom filter with $k \in \mathbb{N}$ hash functions and load factor $\alpha = \ln(2)/k$ has

i. a **space requirement** of $m = n/\alpha = \frac{kn}{\ln 2} \approx 1.44kn$ bits
ii. and a **false positive probability** of $\varepsilon = 2^{-k} + o(1)$.

How to determine m and k (the parameters you actually need)

1. n: determined by input
2. ε: choose a trade-off between space usage and reliability
   - If utility comes from negative answers "$x \notin S$, definitely" and running time is negligible, then possibly:
     - want to maximise utility − disutility, where: ($\propto$ means "proportional to")
     - utility $\propto$ negative answers = queries $\cdot \Pr_{x \sim \text{QueryDistribution}}[x \notin S] \cdot (1 - \varepsilon)$
     - disutility $\propto$ space consumption $= 1.44 \log_2(1/\varepsilon) n$ bits of RAM or cache
3. compute $k = \lceil \log_2(1/\varepsilon) \rceil$ // effectively restricts ε to powers of 2
4. compute $\alpha = \ln(2)/k$ and $m = \lceil n/\alpha \rceil$

Remarks: Much More is Known

Exercise: Bloom Filters that can Delete

Discuss the possibilities and limitations of *Counting Bloom Filters*.

Blocked Bloom Filters

All positions of a key are in the same cache line.

- better query and insertion times
- slightly worse trade-off between space and ε

Cuckoo Filters

Require only $n \log_2(1/\varepsilon) + \mathcal{O}(n)$ bits (rather than $\approx 1.44 n \log_2(1/\varepsilon)$)

Static Filters

Require only $n \log_2(1/\varepsilon) + o(n)$ bits.

Conclusion

Method of Bounded Differences

- requires $X = f(X_1, \ldots, X_n)$ for independent $X_1, \ldots, X_n$
- requires that changing X_i changes X only a little
- gives Chernoff-like concentration bounds

Approximate Membership Data Structure

- decides "is $x \in S$?" with *false positive probability* ε
- much smaller than exact membership data structure
  - $\approx 1.44 \log_2(1/\varepsilon)$ bits per element for Bloom filters
  - often fits into cache or RAM
- used to avoid costly access to exact data structure
  - reliable on **NO** answers
  - useful if **NO** answers are frequent

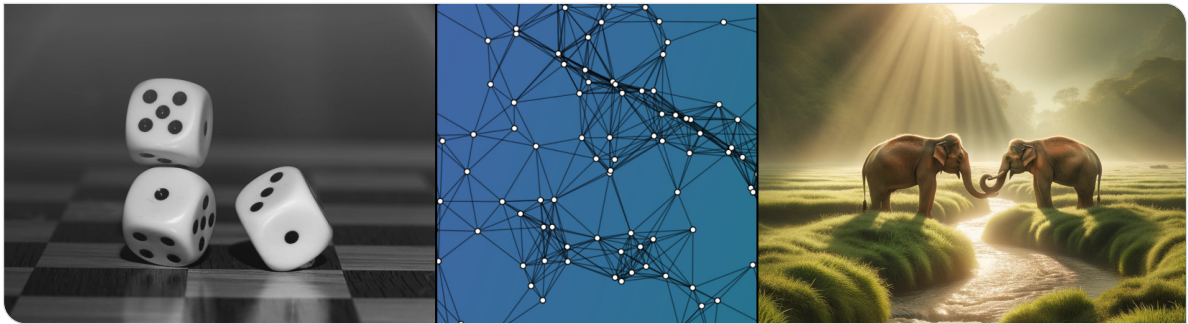
Appendix: Possible Exam Questions I

- Method of Bounded Differences
  - What are the assumptions and guarantees of McDiarmid's inequality?
  - What is an example where it provides good or poor guarantees?
- Approximate-Membership-Query data structures in general
  - What is the task of an AMQ data structure?
  - What is the advantage compared to an exact data structure?
  - What is an application in which an AMQ data structure is useful?
- Bloom filters
  - How is a Bloom filter constructed, and which operations does it support?
  - Which parameters exist, and how are they related?
  - What does our analysis say about the clever choice of parameters? How are the remaining parameters chosen? What memory usage results?
  - Questions regarding the analysis
    - How many zeros or ones do we expect?
    - How is the false-positive probability related to the number of zeros or ones?
    - How can we argue that the number of zeros or ones in the Bloom filter is close to its expected value?

Karlsruhe Institute of Technology

Probability and Computing
Coupling, Balls into Bins, Poissonisation and the Poisson Point Process

Stefan Walzer | WS 2025/2026



Content

An easy choice?

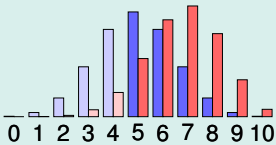
A Simple Game

You win if you get ≥ 5 heads in 10 coin tosses. Choose:

i. a fair coin with Pr[“heads”] $= \frac{1}{2}$

ii. a biased coin with Pr[“heads”] $= \frac{2}{3}$

How to prove that (ii) is the better choice?



fair coin

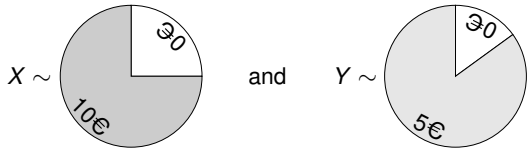
biased coin

$$\sum_{i=5}^{10} \binom{10}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{10-i} \stackrel{?}{<} \sum_{i=5}^{10} \binom{10}{i} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{10-i}$$

Shouldn’t there be an answer that needs no calculation?

Which Lottery do you prefer?

Consider two “wheels of fortune”:

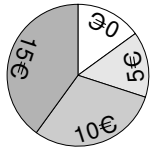


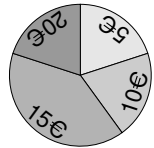
Both can be rationally preferred

- $\mathbb{E}[X] > \mathbb{E}[Y]$ // maximises expected reward
- $\Pr[Y \geq 5€] > \Pr[X \geq 5€]$ // maximises probability that you can afford ice cream

See `https://en.wikipedia.org/wiki/Von_Neumann%E2%80%93Morgenstern_utility_theorem` to get started on rational choice theory.

Which Lottery do you prefer?

$X \sim$ 

and $Y \sim$ 

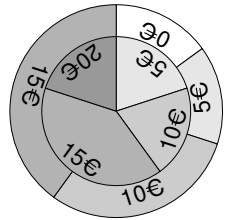
Formal Reason you should prefer Y

For every c we have:

$$\Pr[X \geq c] \leq \Pr[Y \geq c].$$

Intuitive Reason you should prefer Y

Glueing the wheels together guarantees $X < Y$.

$(X, Y) \sim$ 

Content

Equality in Distribution

Notation

We write $X \overset{\mathrm{d}}{=} X'$ for two random variables if X and X' have the same distribution.

Equivalent Definitions

$$X \overset{\mathrm{d}}{=} X' \Leftrightarrow \forall x : \Pr[X = x] = \Pr[X' = x] \qquad \text{(for discrete R.V. } X \text{ and } X')$$

$$\Leftrightarrow \forall x : \Pr[X \leq x] = \Pr[X' \leq x] \qquad \text{(for real-valued R.V. } X \text{ and } X')$$

To Clarify:

If $X, Y \sim \mathcal{U}([0,1])$ are independent then

- $X \overset{\mathrm{d}}{=} Y$
- $\Pr[X = Y] = 0$

Definition: Coupling of X and Y

A random variable X ↓ **A Coupling of X and Y**

A random variable (X', Y') with

- $X' \stackrel{\mathrm{d}}{=} X$
- $Y' \stackrel{\mathrm{d}}{=} Y$

$X \sim$  ↓ **A Coupling of X and Y**

$(X', Y') \sim$ 

- $X' \stackrel{\mathrm{d}}{=} X$ ✓
- $Y' \stackrel{\mathrm{d}}{=} Y$ ✓

$Y \sim$  ↓

Remarks

- No assumption on joint distribution of X and Y. Might be independent, correlated or undefined.
- X' and Y' should be correlated in an interesting/useful way.
- Example coupling shows:

$$\Pr[X \geq c] \overset{X \stackrel{\mathrm{d}}{=} X'}{=} \Pr[X' \geq c] \overset{X' \leq Y'}{\leq} \Pr[Y' \geq c] \overset{Y \stackrel{\mathrm{d}}{=} Y'}{=} \Pr[Y \geq c]$$

An easy choice!

Karlsruhe Institute of Technology

A Simple Game (Generalised)

You win if your random variable exceeds $c \in \mathbb{N}$. Choose:

i. $X \sim \mathrm{Bin}(n, \frac{1}{2})$ // number of heads of fair coin

ii. $Y \sim \mathrm{Bin}(n, \frac{2}{3})$ // number of heads of biased coin

Prove that Y is better than X using a Coupling

Let $R_1, \ldots, R_n \sim \mathcal{U}([6])$ be n fair dice rolls.

- $X' = |\{i \in [n] \mid R_i \in \{1,2,3\}\}|$
- $Y' = |\{i \in [n] \mid R_i \in \{1,2,3,4\}\}|$

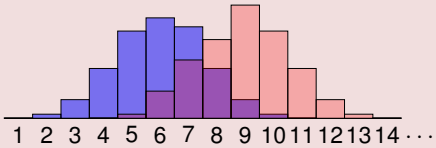
Observe:

- $X' \overset{\mathrm{d}}{=} X$
- $Y' \overset{\mathrm{d}}{=} Y$
- $X' \leq Y'$ guaranteed

Hence: $\Pr[X \geq c] = \Pr[X' \geq c] \leq \Pr[Y' \geq c] = \Pr[Y \geq c]$.

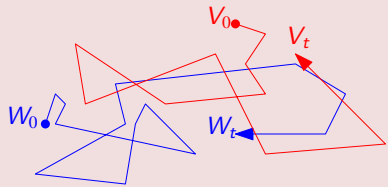
Exercises

Coupling and Total Variation Distance



The histograms of X and Y intersect in an area of A if and only if there exists a coupling (X', Y') of X and Y such that $\Pr[X' = Y'] = A$.

Coupling of Random Walks



For any pair of start points W_0 and V_0, intuitively:

- The distributions of W_t and V_t are asymptotically indinstinguishable for large t.
- Random walks "forget" where they started.
- There is a coupling $(W'_t, V'_t)_{t\geq 0}$ of $(W_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ such that $\lim_{t\to\infty} \Pr[W'_t = V'_t] = 1$.

Content

Balls Into Bins

General Terminology

- m balls are randomly distributed among n bins
- the *load* of a bin is the number of balls in it

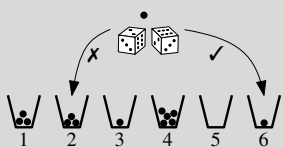
$m = 13, \quad n = 6$

Fully Random Allocation

- $X_1, \ldots, X_m \sim \mathcal{U}([n])$ independent
- $L_i := |\{j \in [m] \mid X_j = i\}|$
  is the load of bin $i \in [n]$
- $(L_1, \ldots, L_n)$ follows a (specific) *multinomial distribution*

Example for Partially Random Allocation (not in this lecture)

- balls are placed sequentially
- each ball chooses the *least loaded* among two randomly chosen bins (ties broken randomly)



Balls into Bins: Many Interesting Questions

- What is the distribution/expectation/concentration of

random variable	interpretation
$\max_{i\in[n]} L_i$	most loaded bin
$\max_{i\in[n]} L_i - \frac{m}{n}$	most loaded bin relative to mean
$\lvert\{i \in [n] \mid L_i = 0\}$	number of empty bins

- Can we make the allocation more balanced by intervening in some way?
  - e.g. with partially random allocation (last slide) $\lim_{m\to\infty} \mathbb{E}[\max_{i\in[n]} L_i - \frac{m}{n}] = \Theta(\log\log n)$ for any fixed n.

Countless variants exist...

“Balls into Bins” is everywhere

Hashing with Chaining $\longleftrightarrow$ n Balls into m Bins

length of the list in bucket i $\longleftrightarrow$ number of balls in bin i

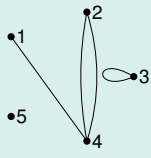
Bloom Filter with k Hash Functions $\longleftrightarrow$ kn Balls into m Bins

a filter bit is set to 0 $\longleftrightarrow$ ith bin is empty

Degree Sequence of Random (Multi-)Graph $\longleftrightarrow$ $2m$ Balls into n bins

Given independent $v_1, \ldots, v_{2m} \sim \mathcal{U}([n])$ let
$G = (V = [n], E = \{\{v_1, v_2\}, \ldots, \{v_{2m-1}, v_{2m}\}\})$
(we allow multi-edges and loops in G)

degree of vertex i $\longleftrightarrow$ load of bin i



$n = 5, \quad m = 4$

$v_1 = 1, \quad v_2 = 4$
$v_3 = 4, \quad v_4 = 2$
$v_5 = 3, \quad v_6 = 3$
$v_7 = 2, \quad v_8 = 4$

“Balls into Bins” is the standard language for discussing underlying mathematical questions.

Content

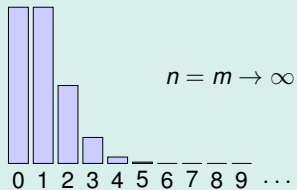
Load of a Single Bin

Setting: Expected Constant Load per Bin

- fully random allocation
- $m = \lambda n$ balls n bins for large n
- λ fixed constant

Load of the First Bin

Consider $L^{(n)} \sim \text{Bin}(\lambda n, \frac{1}{n})$. For $\lambda = 1$:



Poisson Distribution

For $\lambda \in \mathbb{R}_{\geq 0}$, $X \sim \text{Pois}(\lambda)$ is a random variable with

$$\Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$$ // note: probabilities sum to 1

Theorem (proof on blackboard)

$\lim_{n \to \infty} \Pr[L^{(n)} = i] = \Pr[X = i]$.

Remarks

- we say "$L^{(n)}$ converges in distribution to X"
- we write $L^{(n)} \xrightarrow{d} X$
- this formally refers to convergence of CDFs

Properties of the Poisson Distribution

Exercise: $X \sim \text{Pois}(\lambda)$ has Nice Properties

i. $\mathbb{E}[X] = \lambda$.

ii. $\text{Var}(X) = \lambda$.

iii. Let $Y \sim \text{Pois}(\rho)$ be independent of X. Then $X + Y \sim \text{Pois}(\lambda + \rho)$.

iv. Let $X' \sim \text{Bin}(X, p)$. Then $X' \sim \text{Pois}(\lambda p)$.

Poissonised Balls into Bins

λn Balls into n Bins Model

- $X_1, \dots, X_{\lambda n} \sim \mathcal{U}([n])$
- $L_i := |\{j \in [m] \mid X_j = i\}| \sim \mathrm{Bin}(\lambda n, \frac{1}{n})$
- $(L_i)_{i \in [n]}$ *not independent*
  - e.g. large L_1 is (weak) evidence for small L_2
  - annoying in analysis
- number λn of balls *fixed*

"Poissonised" Model

- $L_1, \dots, L_n \sim \mathrm{Pois}(\lambda)$ independent
  - extremely convenient for analysis
- $\mathbb{E}[L_1 + \dots + L_n] = \lambda n$
- number of balls *random* $\sim \mathrm{Pois}(\lambda n)$
  - unusual setting in practice

Wouldn't it be nice...

... if we could switch between the models whenever convenient?

Connection: Poissonised and Regular Balls into Bins

Lemma 1

Let $n \in \mathbb{N}$ and $\lambda > 0$. Consider two variants of Poissonised balls into bins:

Regular Variant:

- sample $L_1, \dots, L_n \sim \text{Pois}(\lambda)$

Sum-First-Variant:

- sample $M \sim \text{Pois}(\lambda n)$
- perform a regular M balls into n bins experiment
  - sample $X_1, \dots, X_M \sim \mathcal{U}([n])$
  - let $L'_i := |\{j \in [M] \mid X_j = i\}|$

Both variants are equivalent, i.e. $(L_1, \dots, L_n) \overset{\text{d}}{=} (L'_1, \dots, L'_n)$.

What we need to show (calculation on blackboard):

For arbitrary $(\ell_1, \dots, \ell_n) \in \mathbb{N}^n$: $\Pr[(L_1, \dots, L_n) = (\ell_1, \dots, \ell_n)] = \Pr[(L'_1, \dots, L'_n) = (\ell_1, \dots, \ell_n)]$.

Some Concentration Bounds

Lemma 2

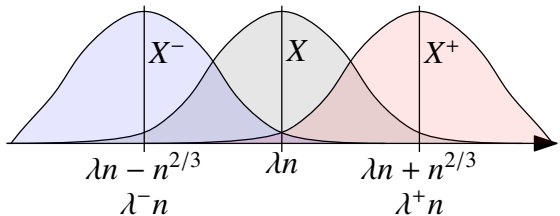
i. Let $\Lambda > 0$ and $X \sim \text{Pois}(\Lambda)$. Then $\Pr[|X - \Lambda| > t] \leq \frac{\Lambda}{t^2}$ for any $t > 0$. // Chebyschev

ii. Let $\lambda = \Theta(1)$, and $X \sim \text{Pois}(\lambda n)$ then $\Pr[X = \lambda n \pm \mathcal{O}(n^{2/3})] = 1 - o(1)$.

iii. Let $\lambda = \Theta(1)$, $\lambda^+ := \lambda + n^{-1/3}$ and $X^+ \sim \text{Pois}(\lambda^+ n)$ then $\Pr[X^+ \geq \lambda n] = 1 - o(1)$.

iv. Let $\lambda = \Theta(1)$, $\lambda^- := \lambda - n^{-1/3}$ and $X^- \sim \text{Pois}(\lambda^- n)$ then $\Pr[X^- \leq \lambda n] = 1 - o(1)$.

v. In particular: $\Pr[X^- \leq \lambda n \leq X^+] = 1 - o(1)$.



Coupling of Poissonised and Regular Balls into Bins

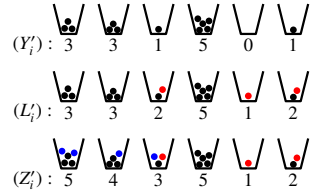
Theorem

Let $n, \lambda, \lambda^+, \lambda^-$ be as before. Consider three "balls into bins" models:

1. $Y_1, \dots, Y_n \sim \text{Pois}(\lambda^-)$ // poissonised with slightly reduced λ
2. $L_1, \dots, L_n$ arising from regular $m = \lambda n$ balls into n bins
3. $Z_1, \dots, Z_n \sim \text{Pois}(\lambda^+)$ // poissonised with slightly increased λ

There is a coupling $(Y_i', L_i', Z_i')_{i\in[n]}$ of $(Y_i)_{i\in[n]}$, $(L_i)_{i\in[n]}$, $(Z_i)_{i\in[n]}$ such that

with probability $1 - o(1)$: $Y_i' \le L_i' \le Z_i'$ for all $i \in [n]$.



Proof.

Let $X_1, X_2, \dots \sim \mathcal{U}([n])$, $M^- \sim \text{Pois}(\lambda^- n)$, $M^+ \sim \text{Pois}(\lambda^+ n)$.

- $Y_i' := |\{j \in [M^-] \mid X_j = i\}|$ for $i \in [n]$.
- $L_i' := |\{j \in [m\ \ ] \mid X_j = i\}|$ for $i \in [n]$.
- $Z_i' := |\{j \in [M^+] \mid X_j = i\}|$ for $i \in [n]$.

This is indeed a coupling as claimed:

- $(Y_i')_{i\in[n]} \overset{\text{d}}{=} (Y_i)_{i\in[n]}$ by Lemma 1.
- $(L_i')_{i\in[n]} \overset{\text{d}}{=} (L_i)_{i\in[n]}$ by construction.
- $(Z_i')_{i\in[n]} \overset{\text{d}}{=} (Z_i)_{i\in[n]}$ by Lemma 1.

By Lemma 2 (v) we have $M^- \le m \le M^+$ with probability $1 - o(1)$. In that case clearly $Y_i' \le L_i' \le Z_i'$ for all $i \in [n]$. □

Coupling of Poissonised and Regular Balls into Bins

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There is a coupling $(Y_i', L_i', Z_i')_{i\in[n]}$ of $(Y_i)_{i\in[n]}$, $(L_i)_{i\in[n]}$, $(Z_i)_{i\in[n]}$ such that

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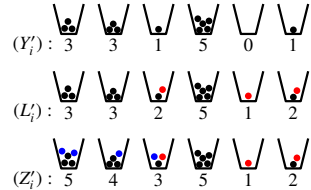


Application involving Monotonous Functions

Let $f : \mathbb{N}_0^n \to \mathbb{R}$ be non-decreasing in each argument.
Examples:

- maximum load of a bin
- longest run of non-empty bins
- collision number // numbers of pairs of co-located balls

For some bound $B \in \mathbb{R}$ let

- $p^- := \Pr[f((Y_i)_{i\in[n]}) \ge B]$ // easier to compute
- $p \;\; := \Pr[f((L_i)_{i\in[n]}) \ge B]$ // what we want
- $p^+ := \Pr[f((Z_i)_{i\in[n]}) \ge B]$ // easier to compute

Then $p \in [p^- - o(1), p^+ + o(1)]$.

Back to Bloom Filters

Exercise:

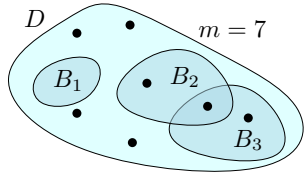
Analyse Bloom filters in a “Poissonised” model and discuss how the results can be transferred to the exact model.

Content

What about “Balls into Continuous Domain”?

Setting

- D is space of finite measure
- $m \in \mathbb{N}$ // number of balls
- $X_1, \dots, X_m \sim \mathcal{U}(D)$ // randomly thrown into D



Note: If $D = \{1, \dots, n\}$ we have discrete balls into bins.

Same annoying issue

If $B_1, B_2 \subseteq D$ with $B_1 \cap B_2 = \varnothing$ are two “bins” then the numbers L_1 and L_2 of “balls” in B_1 and B_2 are correlated.

Similar elegant solution

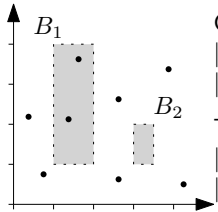
- We can “Poissonise” the setting.
- But we drop “balls into bins” terminology:
  - we allow infinite domains D
  - we allow infinite number of balls

Definitions of the Poisson Point Process

General Definition

Let D be a measurable space with measure μ. // e.g. $D = \mathbb{R}^2$ and $\mu =$"area"
The Poisson point process with parameter $\lambda \in \mathbb{R}_{\geq 0}$ is a random set $P \subseteq D$ such that

1. $|P \cap B| \sim \text{Pois}(\lambda\mu(B))$ for any $B \subseteq D$ with $\mu(B) < \infty$
2. $|P \cap B_1|$ and $|P \cap B_2|$ are independent whenever $B_1 \cap B_2 = \varnothing$



Generally:
$|B_1 \cap P| \sim \text{Po}(3\lambda)$
$|B_2 \cap P| \sim \text{Po}(\frac{1}{2}\lambda)$

This outcome:
$|B_1 \cap P| = 2$
$|B_2 \cap P| = 0$

Equivalent "Sum First" variant if $\mu(D) < \infty$

- sample $M \sim \text{Pois}(\lambda\mu(D))$
- sample $X_1, \ldots, X_M \sim \mathcal{U}(D)$

Then $P \overset{\text{d}}{=} \{X_1, X_2, \ldots, X_M\}$.

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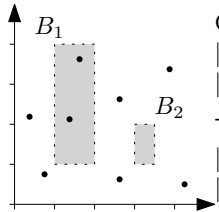

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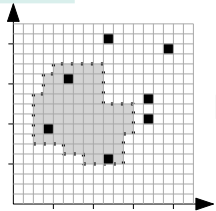
$|B_2 \cap P| = 0$

Construction as a limit

- subdivide D into pieces of measure ε
- let each piece contain a point with probability $\varepsilon\lambda$
- consider the limit for $\varepsilon \to 0$



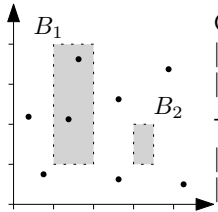
$$|B \cap P| \sim \text{Bin}(\mu(B)/\epsilon, \lambda\epsilon) \xrightarrow{\epsilon \to 0} \text{Po}(\lambda\mu(B))$$

Definitions of the Poisson Point Process

General Definition

Let D be a measurable space with measure μ. // e.g. $D = \mathbb{R}^2$ and $\mu =$"area"
The Poisson point process with parameter $\lambda \in \mathbb{R}_{\geq 0}$ is a random set $P \subseteq D$ such that

1. $|P \cap B| \sim \text{Pois}(\lambda\mu(B))$ for any $B \subseteq D$ with $\mu(B) < \infty$
2. $|P \cap B_1|$ and $|P \cap B_2|$ are independent whenever $B_1 \cap B_2 = \varnothing$

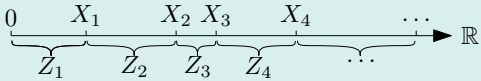


Generally:
$|B_1 \cap P| \sim \text{Po}(3\lambda)$
$|B_2 \cap P| \sim \text{Po}(\frac{1}{2}\lambda)$

This outcome:
$|B_1 \cap P| = 2$
$|B_2 \cap P| = 0$

Equivalent Definition if $D = \mathbb{R}_{\geq 0}$ (where μ is the Borel measure)

- sample $Z_1, Z_2, \ldots \sim \text{Exp}(\lambda)$
- define $X_i = \sum_{j=1}^{i} Z_j$



Then $P \overset{\text{d}}{=} \{X_1, X_2, \ldots\}$.

Proof idea: $\Pr[\min P > t] = \Pr[|P \cap [0,t]| = 0] = \Pr_{X \sim \text{Pois}(\lambda t)}[X = 0] = e^{-\lambda t} \overset{\text{def}}{=} \Pr[Z_1 > t] = \Pr[X_1 > t]$.

Conclusion

Coupling

- embedding of two random variables X and Y into a common probability space
- relationships between distributions of X and Y become visible as relationships between outcomes of X' and Y'

Balls into Bins

- standard language when m objects are randomly assigned to n other objects

Poissonisation

- the act of replacing multinomially distributed $(L_1, \ldots, L_n)$ with independent Poisson random variables $(L'_1, \ldots, L'_n)$
- often much easier to think about
- often formally justifiable

Poisson Point Process

- important model where points from a continuous space occur independently from each other with fixed density λ

Appendix: Possible Exam Questions I

- What is a coupling?
  - Give examples in which a coupling can be useful.
  - What does equality in distribution mean?
- Where in the lecture have we (implicitly or explicitly) considered balls-into-bins processes?
- Poissonisation:
  - Which annoying property does the load distribution in balls-into-bins processes have? What is different in a Poissonised model?
  - How do Poisson distributions arise in a balls-into-bins setting?
  - How did we relate the Poissonised and the regular balls-into-bins models? How can switching between the models be formally justified?
- Poisson point processes
  - How are Poisson point processes defined?

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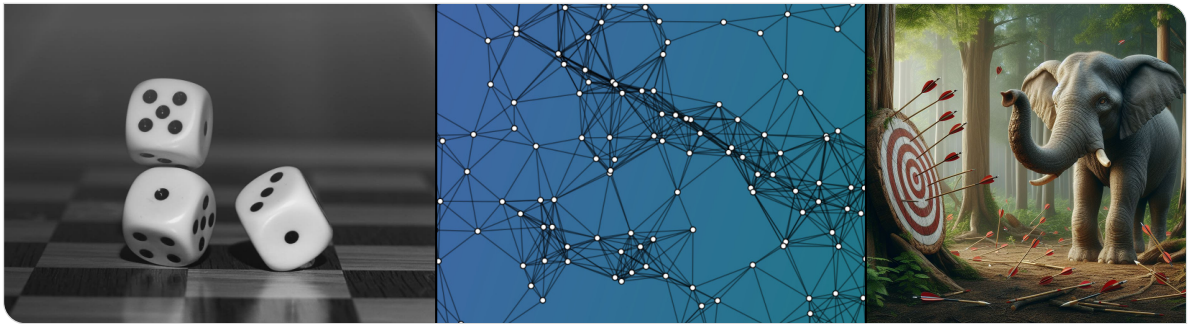
Algorithm Engineering

Probability and Computing – Approximation Algorithms

Stefan Walzer | WS 2025/2026



Lecture Notes by Worsch

This lecture's content is covered in Thomas Worsch's notes from 2019.

Content

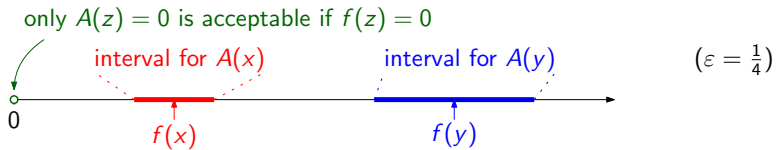
Randomised Approximate Counting

Definition

- let f : Input $\to \mathbb{R}_{\geq 0}$
- let A be randomised algorithm
- let $\varepsilon, \delta > 0$

Algorithm A approximates f with *relative error* ε and failure probability δ if

$$\forall x \in \text{Inputs} : \Pr[|A(x) - f(x)| \leq \varepsilon \cdot f(x)] \geq 1 - \delta.$$



Remark: Related Complexity Classes

PRAS. Problems admitting A with running time polynomial in $|x|$, but not necessarily in $\frac{1}{\varepsilon}$ (for $\delta = 1/4$).

FPRAS. Problems admitting A with running time polynomial in $|x|$ and $\frac{1}{\varepsilon}$ (for $\delta = 1/4$).

Note: Also defined where $f(x)$ is not a *number*. For instance: Want to compute a *vertex cover* with a size close to optimal.

Counting Satisfiable Assignments of Boolean Formulas

A counting problem

For Boolean formula $B(x_1, \ldots, x_n)$ let $\#B$ be the number of satisfying assignments:

$$\#B = |\{(x_1, \ldots, x_n) \in \{0,1\}^n \mid B(x_1, \ldots, x_n) = 1\}|.$$

Example

$$B = (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_3)$$
$$\#B = |\{(0,0,0), (0,0,1), (1,0,1), (1,1,1)\}| = 4$$

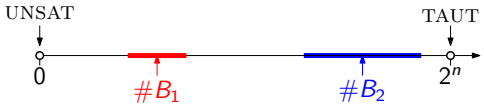
Approximation algorithm for $\#B$ in general? Unlikely.

Assume A satisfies $\Pr[|A(B) - \#B| \leq \varepsilon(\#B)] \leq 1 - \delta$ for $\varepsilon = \frac{1}{2}$ and $\delta = \frac{1}{4}$. Then

$$B \text{ is UNSAT} \Leftrightarrow \#B = 0 \Leftrightarrow \Pr[A(B) = 0] \geq \tfrac{3}{4}$$
$$B \text{ is SAT} \Leftrightarrow \#B > 0 \Leftrightarrow \Pr[A(B) > 0] \geq \tfrac{3}{4}$$

If A is polynomial time then A is BPP algorithm for SAT.
Then SAT $\in$ BPP and NP $\subseteq$ BPP. Hard to believe...

What could be a tractable special case?



We must distinguish: B is UNSAT $\Leftrightarrow \#B = 0$ from B is SAT $\Leftrightarrow \#B \geq 1$.
We need not distinguish: B is TAUT $\Leftrightarrow \#B = 2^n$ from B is NON-TAUT $\Leftrightarrow \#B < 2^n$.

CNF is hard on the wrong end

$$(x_1 \vee \bar{x}_2 \vee x_{42}) \wedge \ldots \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_{37})$$

Deciding UNSAT is NP-hard.
Deciding TAUT is easy.
$\hookrightarrow$ only empty CNF-formula is TAUT.

DNF looks good

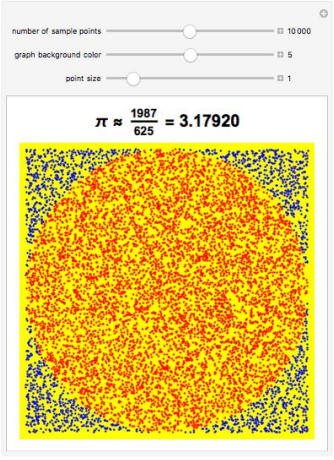
$$(\bar{x}_1 \wedge x_2 \wedge x_{42}) \vee \ldots \vee (x_1 \wedge \bar{x}_3 \wedge x_{37})$$

Deciding UNSAT is easy.
$\hookrightarrow$ only empty DNF-formula is UNSAT.
Deciding TAUT is hard.

Goal: Approximate $\#B$ for DNF formula B.

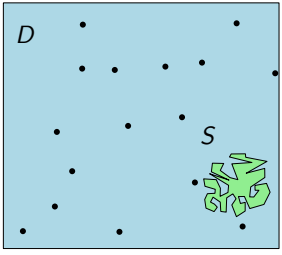
(Equivalently: Approximate $2^n - \#B$ for CNF formula B.)

Intuition: Approximating π



https://demonstrations.wolfram.com/ApproximatingPiByTheMonteCarloMethod/

Intuition: Approximate $|S|$ for $S \subseteq D$ by sampling from D



$|S| \approx |D| \cdot \frac{0}{16}$

Requirements

For this to work we must be able to

1. compute the size of D
2. sample uniformly from D
3. decide for $x \in D$ whether $x \in S$

soft requirement:

4. $\frac{|S|}{|D|}$ should not be too small

Approximate $|S|$ for $S \subseteq D$ by sampling from D

Algorithm approxSetSize**:**

hits $\leftarrow 0$
for $i = 1$ *to* N **do**
  sample $x \sim \mathcal{U}(D)$
  hits $\leftarrow$ hits $+ \mathbb{1}_{x \in S}$
return $\frac{\text{hits}}{N} \cdot |D|$

Chernoff

For $\varepsilon \in (0,1)$ and $X \sim \text{Bin}(N,p)$:

$$\Pr[|X - \mathbb{E}[X]| > \varepsilon \mathbb{E}[X]] < 2\exp(-\varepsilon^2 \mathbb{E}[X]/3).$$

Simple Theorem

Let D be a finite set and $S \subseteq D$ such that we can efficiently

1. compute $|D|$
2. sample uniformly from D
3. decide for $x \in D$ whether $x \in S$

Let $p = |S|/|D|$. Then approxSetSize with $N = \frac{3\log(2/\delta)}{\varepsilon^2 p}$ approximates $|S|$ with relative error ε and failure probability δ.

$\hookrightarrow$ Special Case $\varepsilon, \delta = \Theta(1)$: Need $N = \Omega(1/p)$ samples.

Proof: Apply Chernoff to hits $\sim \text{Bin}(N,p)$.

$$\begin{aligned}\Pr[\text{fail}] &= \Pr[|\text{result} - |S|| > \varepsilon|S|] = \Pr[|\tfrac{\text{hits}}{N} \cdot |D| - |S|| > \varepsilon|S|] = \Pr[|\text{hits} - \tfrac{|S|}{|D|}N| > \varepsilon\tfrac{|S|}{|D|}N] \\ &= \Pr[|\text{hits} - pN| > \varepsilon pN] = \Pr[|\text{hits} - \mathbb{E}[\text{hits}]| > \varepsilon\mathbb{E}[\text{hits}]] \leq 2\exp(-\varepsilon^2\mathbb{E}[\text{hits}]/3) = 2\exp(-\varepsilon^2 pN/3) = \delta.\end{aligned}$$

Approximate $|S|$ for $S \subseteq D$ by sampling from D

Algorithm approxSetSize**:**

hits $\leftarrow 0$
for $i = 1$ *to* N **do**
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$\hookrightarrow$ Special Case $\varepsilon, \delta = \Theta(1)$: Need $N = \Omega(1/p)$ samples.

Application to $\#B$

- $S =$ satisfying assignments of B
- $D = \{0,1\}^n$
- $p = \frac{|S|}{|D|} = \frac{\#B}{2^n}$
- We may have $p = 1/2^n$
- $N = \Omega(2^n)$ required
- :-(

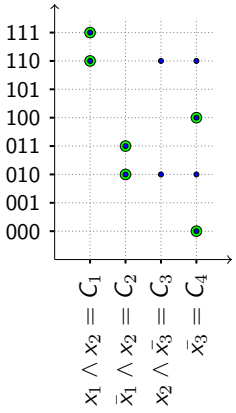
No Surprise

Of course this didn't work

Did not exploit that B is in DNF.

Approximating $\#B$ for B in DNF

Assume $B = C_1 \vee \ldots \vee C_m$ where C_i contains ℓ_i literals.



- $D_i := \{x \in \{0,1\}^n \mid C_i(x) = 1\}$ (satisfying assignments of C_i)
- $D := \{(i, x) \mid i \in [m], x \in D_i\}$ $(= D_1 \dot\cup \ldots \dot\cup D_m)$
- $S := \{(i, x) \mid i \in [m], x \in D_i, x \notin D_1 \cup \ldots D_{i-1}\}$

Claim: We can approximate $|S| = \#B$ by sampling from D

1. We can compute $|D_i| = 2^{n-\ell_i}$ and $|D| = \sum_{i=1}^{m} |D_i|$. ✓
2. We can efficiently sample $(I, X) \sim \mathcal{U}(D)$ ✓
   - sample I such that $\Pr[I = i] = \frac{|D_i|}{|D|}$ // time $\mathcal{O}(m)$ or better
   - sample $X \sim \mathcal{U}(D_i)$
     ↪ set variables appearing in C_i as required, sample others from $\mathrm{Ber}(1/2)$.
   - Yields $\Pr[(I, X) = (i, x)] = \frac{|D_i|}{|D|} \cdot \frac{1}{|D_i|} = \frac{1}{|D|}$ for all $(i, x) \in D$.
3. We can efficiently decide "is $(i, x) \in S$?" ✓
   ↪ just plug x into all clauses $C_1, \ldots, C_i$ // time $\mathcal{O}(mn)$
4. $p = \frac{|S|}{|D|}$ satisfies $p \geq \frac{1}{m}$. ✓

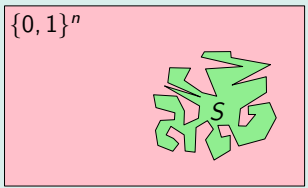
Takeaway

Theorem

If B is in DNF, then we can approximate $\#B$ in polynomial time (using $N = m \cdot \frac{3\log(2/\delta)}{\varepsilon^2}$ samples) with relative error ε and failure probability δ.

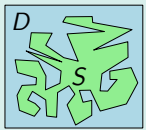
Intuition: Why did this work?

Naive strategy:



Problem: $|S|/|\{0,1\}^n|$ may be exponentially small

Improved strategy:



Advantage: $|S|/|D|$ is $\Omega(1/m)$.

Conclusion

Randomised Approximation is Powerful

For B in DNF:

- Computing $\#B$ *exactly* is $\#\boldsymbol{P}$-complete.
- no *deterministic approximation* algorithm for such problems is known
- we analysed an efficient *randomised approximation* algorithm

Appendix: Possible Exam Questions

- What is a randomized approximation algorithm (for a counting problem)?
- We considered the counting problem $\#B$ for Boolean formulas. Did we succeed in the general case? Why not?
- Which special case did we focus on? Why does the problem of general case not apply?
- We studied an algorithm that estimates the size of $|S|$ for two sets $S \subseteq D$.
  - Under which assumptions is this applicable?
  - How did the algorithm work?
  - How does the number of required samples depend on $|S|$ and $|D|$?
- To estimate $\#B$ for a DNF formula B, we learned a more clever approach.
  - How did this approach work?
  - How does it avoid the problem of the naive approach?

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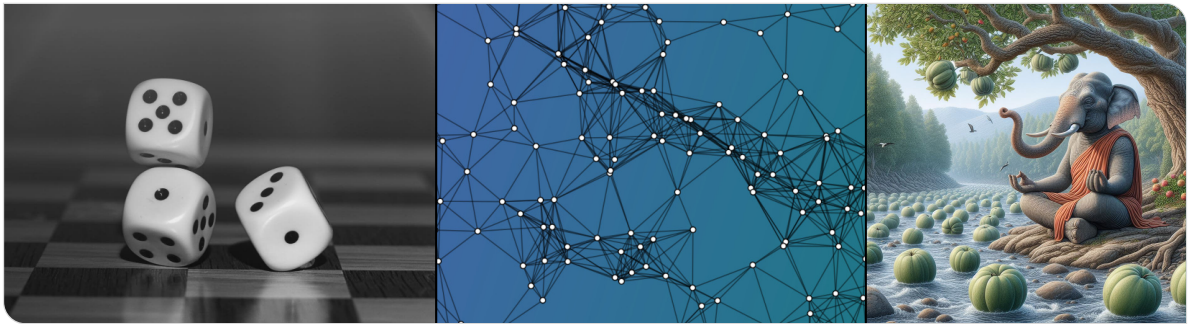
Probability and Computing – Streaming

Stefan Walzer | WS 2025/2026



Content

What is a Streaming Algorithm?

- looong input data stream $(a_1, \ldots, a_m) \in [n]^m$ can only be read *once* from left to right
- goal: approximate some value $F = F(a_1, \ldots, a_m)$ with small relative error ε and failure probability δ.
  $\hookrightarrow$ streaming algorithms are approximation algorithms
- challenge: use less *space* than exact algorithm (in particular: cannot store $(a_1, \ldots, a_m)$).
  $\hookrightarrow$ don't care about running time

Formally, a streaming algorithm is given by three algorithms init, update and result used as follows:

$Z \leftarrow \text{init}()$
for $i = 1$ **to** m **do**
  $Z \leftarrow \text{update}(Z, a_i)$
return $\text{result}(Z)$

Its space complexity is the space required for Z.

Today's Motivating Examples

A Router approximately counts traffic over each connection.
  $\hookrightarrow$ maybe: detect anomalies related to DDoS

B Website approximately counts number of unique users visiting a resource.

Today's Formal Results

A Approximate $F_1(a_1, \ldots, a_m) = m$ in expected space $\frac{1}{\varepsilon^2 \delta} \log \log m$.

B Approximate $F_0(a_1, \ldots, a_m) = |\{a_1, \ldots, a_m\}|$ in expected space $\frac{1}{\varepsilon^2} \log(n) \cdot \log(m/\delta)$.

Content

Attempt I: Naive Counting

Approximate Counting
- stream $(a_1, \ldots, a_m)$
- want $F_1 = m$

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Naive Counting

Algorithm init**:**
  $Z \leftarrow 0$
  return Z

Algorithm update(Z, a)**:**
  $Z \leftarrow Z + 1$
  return Z

Algorithm result(Z)**:**
  return Z

Observations on Naive counting

- No errors ($\varepsilon = \delta = 0$).
- Requires $\lceil \log(m+1) \rceil$ bits of memory.
- No *deterministic* algorithm can use less space
  - Would have to "reuse" a state Z.
  - Is then trapped in an infinite loop.
  - Result arbitrarily far off if m large enough.

Attempt II: Lossy Counting

Approximate Counting
- stream $(a_1, \ldots, a_m)$
- want $F_1 = m$

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Lossy Counting, parameter p

Algorithm init**:**
  $Z \leftarrow 0$
  return Z

Algorithm update(Z, a)**:**
  with probability p **do**
    $Z \leftarrow Z + 1$
  return Z

Algorithm result(Z)**:**
  return Z/p

Analysis (Exercise)

For any $p \in (0, 1]$ we have
- $\mathbb{E}[\text{result}] = m$
- $\Pr[|\text{result} - m| \leq \varepsilon m] \geq 1 - 2\exp(-\varepsilon^2 pm/3)$.
- $\mathbb{E}[\text{space}] \leq \log_2(1 + mp) + 1$.

Corollary

By choosing $p = \frac{3}{\varepsilon^2 m} \ln(2/\delta)$ we get

$$\Pr[\text{fail}] \leq \delta \text{ and } \mathbb{E}[\text{space}] \leq \mathcal{O}(\log(\tfrac{1}{\varepsilon}) + \log\log(1/\delta))).$$

Serious Objection

Correctly choosing p requires already knowing m.

(or at least the order of magnitude of m)

Attempt III: Morris' Algorithm

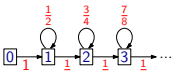
Approximate Counting
- stream $(a_1, \ldots, a_m)$
- want $F_1 = m$

Morris' Algorithm

Algorithm init**:**
  $Z \leftarrow 0$
  return Z

Algorithm update(Z, a)**:**
  with probability 2^{-Z} **do**
    $Z \leftarrow Z + 1$
  return Z

Algorithm result(Z)**:**
  return $2^Z - 1$



states of Z
transition probabilities ($\rho = 2$)

Lemma: Morris' Algorithm is an *Unbiased Estimator*

$\mathbb{E}[\text{result}] = m$.

Proof

Let Z_i for $i \in [m]$ denote the value of Z after i updates.
Consider the expected change to 2^Z in one step...

- ... conditioned on a current value $j \in \mathbb{N}$:

$$\mathbb{E}[2^{Z_{i+1}} - 2^{Z_i} \mid Z_i = j] = 2^{-j} \cdot (2^{j+1} - 2^j) + (1 - 2^{-j}) \cdot \underbrace{(2^j - 2^j)}_{=0} = 2 - 1 = 1.$$

- ... unconditionally:

$$\mathbb{E}[2^{Z_{i+1}} - 2^{Z_i}] \overset{\text{LTE}}{=} \sum_{j \geq 0} \Pr[Z_i = j] \cdot \underbrace{\mathbb{E}[2^{Z_{i+1}} - 2^{Z_i} \mid Z_i = j]}_{=1} = \sum_{j \geq 0} \Pr[Z_i = j] = 1.$$

Hence:

$$\mathbb{E}[\text{result}] = \mathbb{E}[2^{Z_m} - 1] = \mathbb{E}[2^{Z_m} - 2^{Z_0}] = \mathbb{E}[\sum_{i=1}^{m} 2^{Z_{i+1}} - 2^{Z_i}] = \sum_{i=1}^{m} \underbrace{\mathbb{E}[2^{Z_{i+1}} - 2^{Z_i}]}_{=1} = m.$$

Attempt III: Morris' Algorithm

Approximate Counting
- stream $(a_1, \ldots, a_m)$
- want $F_1 = m$

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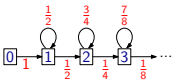
Morris' Algorithm

Algorithm init:
  $Z \leftarrow 0$
  return Z

Algorithm update(Z, a):
  with probability 2^{-Z} **do**
    $Z \leftarrow Z + 1$
  return Z

Algorithm result(Z):
  return $2^Z - 1$



Lemma 1: Worryingly large Variance

$\mathrm{Var}(2^{Z_i}) = \frac{i^2 - i}{2} = \Theta(i^2)$.

Lemma 2

$\mathbb{E}[2^{2Z_i}] = \frac{3i(i+1)}{2} + 1$.

Proof of Lemma 1 using Lemma 2.

$$\mathrm{Var}(2^{Z_i}) = \mathbb{E}[2^{2Z_i}] - \mathbb{E}[2^{Z_i}]^2 \overset{\text{Lem. 2}}{=} \frac{3i(i+1)}{2} + 1 - (i+1)^2 = \frac{3}{2}i^2 - i^2 \pm \mathcal{O}(i) = \Theta(i^2)$$

Attempt III: Morris' Algorithm

Approximate Counting
- stream $(a_1, \ldots, a_m)$
- want $F_1 = m$

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Morris' Algorithm

Algorithm init:
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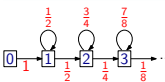

states of Z
transition probabilities ($\rho = 2$)

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$\mathrm{Var}(2^{Z_i}) = \frac{i^2 - i}{2} = \Theta(i^2)$.

Lemma 2

$\mathbb{E}[2^{2Z_i}] = \frac{3i(i+1)}{2} + 1$.

Proof of Lemma 2.

For $i \in \{0, 1\}$ ✓. Let now $i \geq 1$. Note $\Pr[Z_{i+1} = 0] = \Pr[Z_i = 0] = 0$.

$$\begin{aligned}
\mathbb{E}[2^{2Z_{i+1}}] &= \sum_{j \geq 1} 2^{2j} \Pr[Z_{i+1} = j] = \sum_{j \geq 1} 2^{2j} (\Pr[Z_i = j-1] \cdot 2^{-j+1} + \Pr[Z_i = j] \cdot (1 - 2^{-j})) \\
&= \sum_{j \geq 1} 2^{j+1} \Pr[Z_i = j-1] + \sum_{j \geq 1} 2^{2j} \Pr[Z_i = j] - \sum_{j \geq 1} 2^j \Pr[Z_i = j] \\
&= 4 \sum_{j \geq 0} 2^j \Pr[Z_i = j] + \sum_{j \geq 0} 2^{2j} \Pr[Z_i = j] - \sum_{j \geq 0} 2^j \Pr[Z_i = j] \\
&= 4\mathbb{E}[2^{Z_i}] + \mathbb{E}[2^{2Z_i}] - \mathbb{E}[2^{Z_i}] = 3\mathbb{E}[2^{Z_i}] + \mathbb{E}[2^{2Z_i}] = 3(i+1) + \mathbb{E}[2^{2Z_i}] \\
&\overset{\text{Ind.}}{=} 3(i+1) + \frac{3i(i+1)}{2} + 1 = \frac{3(i+2)(i+1)}{2} + 1. \quad \square
\end{aligned}$$

Space

Expected Space

$$\mathbb{E}[\text{space}] \leq \mathbb{E}[\lceil \log_2(1+Z_m) \rceil] \leq 1 + \mathbb{E}[\log_2(1+Z_m)] = 1 + \mathbb{E}[\log_2(1+\log_2(2^{Z_m}))]$$
$$\overset{(\star)}{\leq} 1 + \log_2(1+\log_2(\mathbb{E}[2^{Z_m}])) = 1 + \log_2(1+\log_2(m+1)) = \Theta(\log\log m).$$

($\star$) uses Jensen's inequality that you'll prove as an exercise.

Interim Conclusion: Morris is not good enough *yet*

- $\mathbb{E}[\text{result}] = m$ ✓ unbiased estimator
- $\mathbb{E}[\text{space}] = \mathcal{O}(\log\log m)$ ✓ highly space efficient
- $\text{Var}(\text{result}) = \Theta(m^2)$ ✗
  - Standarddeviation $\Theta(m)$
    $\rightsquigarrow$ typically right order of magnitude, but not better.

Morris$^+$: Use many copies of Morris' Algorithm

Theorem

Consider a streaming algorithm that maintains a sequence $Z = (Z_1, \dots, Z_s)$ of independent Morris-counters and returns $\text{result}(Z) := \frac{\text{result}(Z_1)+\cdots+\text{result}(Z_s)}{s}$. For $s = \frac{1}{\varepsilon^2\delta}$ we obtain

- $\mathbb{E}[\text{result}(Z)] = m$ and $\mathbb{E}[\text{space}] = \mathcal{O}(\frac{1}{\varepsilon^2\delta} \log\log m)$
- $\Pr[|\text{result}(Z) - m| \leq \varepsilon m] = 1 - \mathcal{O}(\delta)$.

Reminder on Variance

If X, Y are independent random variables and $s > 0$ then

- $\text{Var}(sX) = s^2 \text{Var}(X)$
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

Proof of Concentration using Chebyshev

$$\text{Var}(\text{result}(Z)) = \text{Var}(\tfrac{1}{s}\sum_{i=1}^{s} \text{result}(Z_i)) = \tfrac{1}{s^2}\text{Var}(\sum_{i=1}^{s} \text{result}(Z_i))$$
$$= \tfrac{1}{s^2}\sum_{i=1}^{s} \text{Var}(\text{result}(Z_i)) = \tfrac{s}{s^2}\text{Var}(\text{result}(Z_1)) = \tfrac{1}{s}\Theta(m^2) = \Theta(m^2/s).$$

Chebyschev:

$\Pr[X - \mathbb{E}[X] > c] \leq \frac{\text{Var}(X)}{c^2}$.

$$\Pr[\text{fail}] = \Pr[|\text{result}(Z) - m| > \varepsilon m] = \Pr[|\text{result}(Z) - \mathbb{E}[\text{result}(Z)]| > \varepsilon m] \leq \frac{\text{Var}(\text{result}(Z))}{\varepsilon^2 m^2} = \Theta(1/(\varepsilon^2 s)) = \Theta(\delta). \quad \square$$

Morris*: Use a different base in Morris' Algorithm

Morris with base $1 < \rho < 2$

- In every update: increment Z with probability ρ^{-Z}.
- In the end: return $\frac{\rho^Z - 1}{\rho - 1}$.

Modified Analysis

Show similarly to before: // calculations much more technical...

- $\mathbb{E}[\text{result}] = m$
- $\text{Var}(\text{result}) = \Theta((\rho - 1)m^2)$

Choosing $\rho = 1 + \varepsilon^2\delta$ gives:

- $\Pr[|\text{result} - m| > \varepsilon m] = \mathcal{O}(\delta)$.
- $\mathbb{E}[\text{space}] = \mathcal{O}(\log\log m + \log 1/\delta + \log 1/\varepsilon)$.

Content

History

Counting Distinct Elements

- stream $(a_1, \ldots, a_m) \in [n]^m$
- want $F_0 = |\{a_1, \ldots, a_m\}|$

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Remark: CVM is not well-known

Popular line of algorithms for F_0 by Philippe Flajolet et al:

- ~~1984: Flajolet-Martin~~ (deprecated)
  ↪ https://en.wikipedia.org/wiki/Flajolet-Martin_algorithm
- ~~2003: LogLog~~ (deprecated)
- 2007: HyperLogLog
  ↪ `https://en.wikipedia.org/wiki/HyperLogLog`

The CVM-Algorithm

- 2022: European Symposium on *Simplicity* in Algorithms 2022
  ↪ „Distinct Elements in Streams: An Algorithm for the (Text) Book“
- is a bit worse than HyperLogLog
- is easier to analyse than HyperLogLog

Next: We develop CVM in three steps.

Attempt I: Naively storing the set

Counting Distinct Elements

- stream $(a_1, \ldots, a_m) \in [n]^m$
- want $F_0 = |\{a_1, \ldots, a_m\}|$

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Naive Storing

Algorithm init**:**
  $Z \leftarrow \varnothing$
  return Z

Algorithm update(Z, a)**:**
  $Z \leftarrow Z \cup \{a\}$
  return Z

Algorithm result(Z)**:**
  return $|Z|$

Observation

Naively storing the set requires $\Omega(F_0 \cdot \log n)$ bits.

Attempt II: Storing the set lossily

Counting Distinct Elements
- stream $(a_1, \ldots, a_m) \in [n]^m$
- want $F_0 = |\{a_1, \ldots, a_m\}|$

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LossyStore, parameter p

Algorithm init**:**
  $Z \leftarrow \varnothing$
  return Z

Algorithm update(Z, a)**:**
  $Z \leftarrow Z \setminus \{a\}$
  with probability p **do**
    $Z \leftarrow Z \cup \{a\}$
  return Z

Algorithm result(Z)**:**
  return $|Z|$/p;

Chernoff for $X \sim \mathrm{Bin}(n, p)$

$\Pr[|X - \mathbb{E}[X]| > \varepsilon\mathbb{E}[X]] \leq 2\exp(-\varepsilon^2\mathbb{E}[X]/3)$.

Analysis

Let $Z_0, \ldots, Z_m$ be the states of Z over time. Invariant: Each $a \in \{a_1, \ldots, a_i\}$ is in Z_i independently with probability p. Hence $|Z_m| \sim \mathrm{Bin}(F_0, p)$.

- $\mathbb{E}[\text{result}] = \mathbb{E}[|Z_m|/p] = \mathbb{E}[|Z_m|]/p = F_0 p/p = F_0$.
  $\hookrightarrow$ result is *unbiased estimator* of F_0.
- $\Pr[\text{fail}] = \Pr[|\text{result} - F_0| > \varepsilon F_0] = \Pr\big[\big||Z_m|/p - F_0\big| > \varepsilon F_0\big]$
  $= \Pr\big[\big||Z_m| - pF_0\big| > \varepsilon pF_0\big] = \Pr\big[\big||Z_m| - \mathbb{E}[|Z_m|]\big| > \varepsilon\mathbb{E}[|Z_m|]\big]$
  $\overset{\text{Chern.}}{\leq} 2\exp(-\varepsilon^2\mathbb{E}[|Z_m|]/3) = 2\exp(-\varepsilon^2 pF_0/3)$.
  $\hookrightarrow$ choose $p = p_\delta := \frac{3\log(2/\delta)}{\varepsilon^2 F_0}$ for $\Pr[\text{fail}] \leq \delta$.
- Expected space *in the end* for $p = p_\delta$ (⚠ $\neq$ peak space consumption)
  $\mathbb{E}[|Z_m| \cdot \mathcal{O}(\log n)] = F_0 p_\delta \cdot \mathcal{O}(\log n) = \mathcal{O}(\frac{\log(1/\delta)}{\varepsilon^2}\log n)$.

Serious Objection: Need to know F_0 to choose p

- for $p \gg p_\delta$: space is wasted
- for $p \ll p_\delta$: failure becomes likely

Attempt III: Adjust lossiness dynamically

Counting Distinct Elements
- stream $(a_1, \dots, a_m) \in [n]^m$
- want $F_0 = |\{a_1, \dots, a_m\}|$

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CVM, parameter T

Algorithm init**:**
- $Z \leftarrow \varnothing$
- $P \leftarrow 1$
- **return** (P, Z)

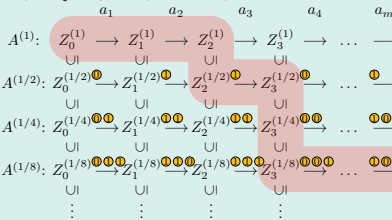
Algorithm update$((P, Z), a)$**:**
- $Z \leftarrow Z \setminus \{a\}$
- **with probability** P **do**
  - $Z \leftarrow Z \cup \{a\}$
- **while** $|Z| \geq T$ **do** // shrink
  - $Z' \leftarrow \varnothing$
  - **for** $a \in Z$ **do**
    - **with probability** $1/2$ **do**
      - $Z' \leftarrow Z' \cup \{a\}$
  - $(Z, P) \leftarrow (Z', P/2)$
- **return** (P, Z)

Algorithm result$((P, Z))$**:**
- **return** $|Z|/P$

CVM behaves like LossyStore with dynamic p

Consider $A^{(p)} := \text{LossyStore}(p)$ with states $Z_0^{(p)}, \dots, Z_m^{(p)}$ for $p \in \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$.

Let $(P_0, Z_0^{(\text{CVM})}), \dots, (P_m, Z_m^{(\text{CVM})})$ be the state of CVM.



Intuition: The path of CVM:

$(x, y) \leftarrow (0, 0)$ // top left
for $i = 1$ **to** m **do** // m updates
- $x \leftarrow x + 1$ // go right
- **while** $|Z_x^{(2^{-y})}| \geq T$ **do**
  - $y \leftarrow y + 1$ // go down

final state is $Z_m^{(2^{-y})}$

Coupling between executions of $A^{(p)}$ and CVM:
- $A^{(p/2)}$ uses coin tosses of $A^{(p)}$ and one more.
  "$A^{(p/2)}$ keeps half of what $A^{(p)}$ keeps."
- CVM uses coin tosses of $A^{(P)}$ to process elements.
- When shrinking, CVM inspects past coin tosses done by $A^{(P/2)}$. (the next unused coin for all $a \in Z$)

Effects of the coupling:
- $Z_j^{(\text{CVM})} = Z_j^{(P_j)}$ for $j \in [m]$
- $\text{result}^{(\text{CVM})} = \text{result}^{(P_m)}$
- $\text{fail}^{(\text{CVM})} = \text{fail}^{(P_m)}$

Attempt III: Adjust lossiness dynamically

Counting Distinct Elements
- stream $(a_1, \ldots, a_m) \in [n]^m$
- want $F_0 = |\{a_1, \ldots, a_m\}|$

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CVM, parameter T

Algorithm init:
- $Z \leftarrow \varnothing$
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- **return** (P, Z)

Algorithm update($(P, Z), a$):
- $Z \leftarrow Z \setminus \{a\}$
- **with probability** P **do**
  - $Z \leftarrow Z \cup \{a\}$
- **while** $|Z| \geq T$ **do** // shrink
  - $Z' \leftarrow \varnothing$
  - **for** $a \in Z$ **do**
    - **with probability** $1/2$ **do**
      - $Z' \leftarrow Z' \cup \{a\}$
  - $(Z, P) \leftarrow (Z', P/2)$
- **return** (P, Z)

Algorithm result((P, Z)):
- **return** $|Z|/P$

Lemma: Failure Probability and Space

With $T = \frac{18 \log_2(2m/\delta)}{\varepsilon^2}$ we get $\Pr[\text{fail}^{\text{CVM}}] = \mathcal{O}(\delta)$ and $\text{space}^{\text{CVM}} = \mathcal{O}(\frac{\log(m/\delta)}{\varepsilon^2} \log n) + \lceil \log_2(\log_2(1/P_m)) \rceil$.

Analysis of CVM's failure probability (a bit sketchy)

- Recall: $\text{LossyStore}(p_\delta = \frac{3 \log(2/\delta)}{\varepsilon^2 F_0})$ has failure probability $\leq \delta$. Assume p_δ is power of 2.
- Then $\Pr[\text{fail}^{(p_\delta)}] \leq \delta$, $\Pr[\text{fail}^{(2p_\delta)}] \leq \delta^2$, $\Pr[\text{fail}^{(4p_\delta)}] \leq \delta^4$, ...
- Therefore $\Pr[\text{fail}^{(p_\delta)}] + \Pr[\text{fail}^{(2p_\delta)}] + \ldots + \Pr[\text{fail}^{(1)}] \leq \delta + \delta^2 + \delta^4 + \ldots = \mathcal{O}(\delta)$.
-
$$\Pr[P_m < p_\delta] = \Pr[|Z_j^{(p_\delta)}| \geq T \text{ for some } j \in [m]] \leq m \cdot \Pr[|Z_m^{(p_\delta)}| \geq T]$$
$$= m \cdot \Pr_{Z \sim \text{Bin}(F_0, p_\delta)}[Z \geq T] \overset{\Delta}{\leq} m \cdot 2^{-T} \leq m \cdot 2^{-\log(m/\delta)} = \delta.$$
where Δ uses a Chernoff bound and $6\mathbb{E}[Z] = 6F_0 p_\delta = \frac{18 \log_2(2/\delta)}{\varepsilon^2} \leq T$.
- $\text{fail}^{\text{CVM}} \Leftrightarrow \text{fail}^{(P_m)} \Rightarrow \left(P_m < p_\delta \vee \text{fail}^{(1)} \vee \text{fail}^{(1/2)} \vee \ldots \vee \text{fail}^{(p_\delta)}\right)$

$$\text{Finally: } \Pr[\text{fail}^{\text{CVM}}] \leq \Pr[P_m < p_\delta \vee \text{fail}^{(1)} \vee \text{fail}^{(1/2)} \vee \ldots \vee \text{fail}^{(p_\delta)}] \overset{\text{UB}}{\leq} \delta + \mathcal{O}(\delta) = \mathcal{O}(\delta).$$

Conclusion

Streaming Algorithms

- Input read only once, from left to right.
- Goal: Use little space. (less than what is needed to store input stream)
- Motivation: Network actor wants to maintain statistic on traffic.

Morris$^+$ Algorithm for Counting the Stream Length

- approximation in space $\mathcal{O}(\frac{1}{\varepsilon^2\delta}\log\log m)$ // or $\mathcal{O}(\log\log m + \log\frac{1}{\delta\varepsilon})$ using Morris*?
  (ε = relative error, δ = failure probability)
- deterministic algorithms need space $\lceil\log(1+m)\rceil$

CVM Algorithm for Counting *Distinct* Elements

- approximation in space $\mathcal{O}(\frac{1}{\varepsilon^2}\log(n)\log(m/\delta))$

Appendix: Possible Exam Questions I

- Definition of streaming algorithms:
  - What is the task of a streaming algorithm (with respect to a quantity $F = F(a_1, \ldots, a_m)$)?
  - What is the specific challenge for streaming algorithms?
- Streaming algorithms for $F_1 = m$:
  - What could be an application in which one would like to estimate F_1?
  - How much memory is needed if one simply counts? Can a deterministic algorithm do something smarter?
  - How does the LossyCounting algorithm work? Why does it not help us here?
  - How does Morris' algorithm work?
  - Prove that Morris' algorithm is unbiased.*
  - Prove that the memory usage of Morris is doubly logarithmic in m.
  - What other weakness did Morris' algorithm have, and how did we fix it?

Appendix: Possible Exam Questions II

- Streaming algorithms for $F_0 = \{a_1, \ldots, a_m\}$:
  - What could be an application in which one would like to estimate F_0?
  - How much memory does the naive deterministic algorithm require? What can we achieve with CVM?
  - As an intermediate step, we formulated the LossyStore algorithm. How does it work?
  - How does the CVM algorithm work? How is it related to the LossyStore algorithm?
  - In the analysis of the error probability of CVM, we distinguished two types of problems. Which ones?*

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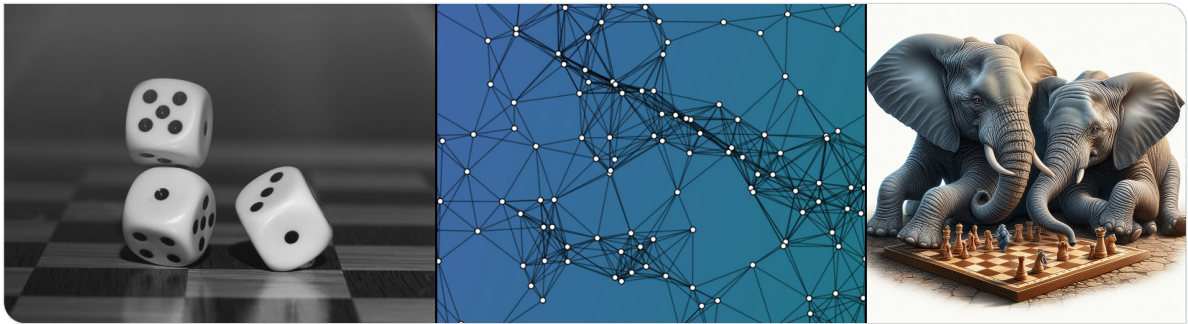
Probability and Computing – Game Theory & Yao's Principle: Proving Lower Bounds for Randomised Algorithms

Stefan Walzer | WS 2025/2026



Lecture Notes by Worsch

Some of this lecture's content is covered in Thomas Worsch's notes from 2019.

Content

Content

Content

Prisoner's Dilemma

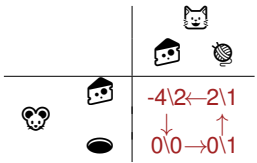
Setting

- *strategies* and available to both *players*
- table shows *payoffs* for players depending on chosen strategies
- here: always better to choose
  ↪ pair (,) is unique *equilibrium*

Definition: Equilibrium

Combination of strategies such that no one can profit by unilaterally switching his or her own strategy.

A cat and mouse game



Someone always regrets their decision

		reaction
		should have played
		should have played
		should have played
		should have played

↪ No combination of *pure* strategies is an *equilibrium*.

Equilibrium

Combination of strategies such that no one can profit by unilaterally switching his or her own strategy.

Nash Equilibria

What a Game *is*

- Finite sets S_1, S_2 of *pure strategies*.
- Utility functions $u_1, u_2 : S_1 \times S_2 \to \mathbb{R}$.

How a Game is played

- Players pick a strategy simultaneously
  $\hookrightarrow$ gives pair $(s_1, s_2) \in S_1 \times S_2$.
- player 1 gets payoff $u_1(s_1, s_2)$ and
  player 2 gets payoff $u_2(s_1, s_2)$.

Existence of Mixed-Strategy Nash Equilibria

There exist distributions $\mathcal{S}_1^*$ on S_1 and $\mathcal{S}_2^*$ on S_2, called *mixed strategies* such that $(\mathcal{S}_1^*, \mathcal{S}_2^*)$ is an equilibrium:

player 1 cannot increase expected payoff: $\mathbb{E}_{s_1 \sim \mathcal{S}_1^*, s_2 \sim \mathcal{S}_2^*}[u_1(s_1, s_2)] = \max_{s_1 \in S_1} \mathbb{E}_{s_2 \sim \mathcal{S}_2^*}[u_1(s_1, s_2)]$.

player 2 cannot increase expected payoff: $\mathbb{E}_{s_1 \sim \mathcal{S}_1^*, s_2 \sim \mathcal{S}_2^*}[u_2(s_1, s_2)] = \max_{s_2 \in S_2} \mathbb{E}_{s_1 \sim \mathcal{S}_1^*}[u_2(s_1, s_2)]$.

Remark: Theorem holds for $n \geq 3$ players as well.

Nash Equilibrium in Cat & Mouse Game

		-4\2	2\1
		0\0	0\1

Equilibrium

$\mathcal{S}_{\text{mouse}} = \{$: $\frac{1}{2}$, : $\frac{1}{2}\}$

$\mathcal{S}_{\text{cat}} = \{$: $\frac{1}{3}$, : $\frac{2}{3}\}$

Verification of Equilibrium Property: Calculating Expected Payoffs

for :

- playing gives expected payoff $\frac{1}{3} \cdot (-4) + \frac{2}{3} \cdot 2 = 0$
- playing gives expected payoff $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 0 = 0$
- playing $\mathcal{S}_{\text{mouse}}$ is a mix of both
  $\hookrightarrow$ also expected payoff 0.

for :

- playing gives expected payoff $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$
- playing gives expected payoff $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$
- playing $\mathcal{S}_{\text{cat}}$ is a mix of both
  $\hookrightarrow$ also expected payoff 1.

Content

Two Player Zero Sum Games and their Matrix Formulation

- Finite sets of pure strategies
  - S_1 for player 1
  - S_2 for player 2
- utility function $u : S_1 \times S_2 \to \mathbb{R}$
  - player 1 gets $u(s_1, s_2)$
  - player 2 gets $-u(s_1, s_2)$

- Implicit sets of pure strategies
  - $S_1 = [n]$ for the *row player*
  - $S_2 = [m]$ for the *column players*
- matrix $M \in \mathbb{R}^{n \times m}$
  - row player gets M_{s_1, s_2}
  - column player gets $-M_{s_1, s_2}$

		0	-1	1
		1	0	-1
		-1	1	0

Unique equilibrium of

$\mathcal{S}_1 = \mathcal{S}_2 = \{ : \frac{1}{3}, : \frac{1}{3}, : \frac{1}{3}\}$

Two Player Zero Sum Games and their Matrix Formulation

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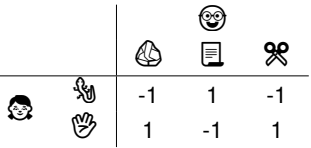

Equilibria of 🪨 📄 ✂ 🦎 🖖

Work it out yourself!

Nash Equilibria for Two-Player Zero-Sum Games

Nash's Theorem (1950), Special Case

For any $M \in \mathbb{R}^{n \times m}$ there exist distributions $\mathcal{S}_1^*$ on $[n]$ and $\mathcal{S}_2^*$ on $[m]$ such that

$$\mathbb{E}_{s_1 \sim \mathcal{S}_1^*, s_2 \sim \mathcal{S}_2^*}[M_{s_1,s_2}] = \max_{s_1 \in [n]} \mathbb{E}_{s_2 \sim \mathcal{S}_2^*}[M_{s_1,s_2}] = \min_{s_2 \in [m]} \mathbb{E}_{s_1 \sim \mathcal{S}_1^*}[M_{s_1,s_2}].$$

Intuition

When the players play according to $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$, then no player can benefit by deviating from his strategy.

Corollary: Loomis (1946) Von Neumann (1928)

For any $M \in \mathbb{R}^{n \times m}$ we have

$$\max_{\mathcal{S}_1} \min_{s_2 \in [m]} \mathbb{E}_{s_1 \sim \mathcal{S}_1}[M_{s_1,s_2}] = \min_{\mathcal{S}_2} \max_{s_1 \in [n]} \mathbb{E}_{s_2 \sim \mathcal{S}_2}[M_{s_1,s_2}]$$

Intuition

No first-mover disadvantage if

- first player choses mixed strategy
- second player answers with pure strategy

Proof of Corollary ("$\geq$")

$$\max_{\mathcal{S}_1} \min_{s_2 \in [m]} \mathbb{E}_{s_1 \sim \mathcal{S}_1}[M_{s_1,s_2}] \geq \min_{s_2 \in [m]} \mathbb{E}_{s_1 \sim \mathcal{S}_1^*}[M_{s_1,s_2}] \overset{\text{Nash}}{=} \max_{s_1 \in [n]} \mathbb{E}_{s_2 \sim \mathcal{S}_2^*}[M_{s_1,s_2}] \geq \min_{\mathcal{S}_2} \max_{s_1 \in [n]} \mathbb{E}_{s_2 \sim \mathcal{S}_2}[M_{s_1,s_2}]$$

Nash Equilibria for Two-Player Zero-Sum Games

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Intuition

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- second player answers with pure strategy

Proof of Corollary ("$\leq$")

$$\max_{\mathcal{S}_1} \min_{s_2 \in [m]} \mathbb{E}_{s_1 \sim \mathcal{S}_1}[M_{s_1,s_2}] = \max_{\mathcal{S}_1} \min_{\mathcal{S}_2} \mathbb{E}_{s_1 \sim \mathcal{S}_1, s_2 \sim \mathcal{S}_2}[M_{s_1,s_2}] \leq \min_{\mathcal{S}_2} \max_{\mathcal{S}_1} \mathbb{E}_{s_1 \sim \mathcal{S}_1, s_2 \sim \mathcal{S}_2}[M_{s_1,s_2}] = \min_{\mathcal{S}_2} \max_{s_1 \in [n]} \mathbb{E}_{s_2 \sim \mathcal{S}_2}[M_{s_1,s_2}]$$

Content

Algorithm Design as a 2-Player Zero-Sum Game

Setting

- P: a computational problem
- **Inputs**: *finite* set of inputs
- **Algos**: *finite* set of deterministic algorithms
- $C(A, I) \in \mathbb{R}$ cost of $A \in$ **Algos** on $I \in$ **Inputs**.

A Two-Player Zero-Sum Game

- Designer chooses (randomised) algorithm, i.e. a distribution on **Algos**.
  $\hookrightarrow$ Goal: Minimise (expected) cost.
- Adversary chooses (randomised) input, i.e. a distribution on **Inputs**.
  $\hookrightarrow$ Goal: Maximise (expected) cost.

Example: Sorting

- $P =$ "sort n numbers comparison-based"[a]
- **Inputs** $= S_n$ //permutations of $[n]$
- **Algos** $=$ e.g. suitable set of decision trees
- $C(A, I) =$ # of comparisons of A for input I

[a] n finite, though possibly $n \to \infty$ later.

Sorting (x, y, z)

		Adversary			
		$(1,2,3)$	$(3,1,2)$	$(2,3,1)$	$\dots$
Algorithm Designer	$x < y$ then $y < z$ then* $z < x$	2	3	3	
	$y < z$ then $z < x$ then* $x < y$	3	2	3	
	$\dots$				

* Only if needed.

Randomised Complexity and Yao's Principle

Definition: Randomised Complexity of a Problem

$$\mathcal{C} := \min_{\mathcal{A} \text{ dist. on } \textbf{Algos}} \max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}}[C(A, I)] \qquad \text{designer moves first}$$

$$\overset{\text{Loomis}}{=} \max_{\mathcal{I} \text{ dist. on } \textbf{Inputs}} \min_{A \in \textbf{Algos}} \mathbb{E}_{I \sim \mathcal{I}}[C(A, I)] \qquad \text{adversary moves first}$$

Yao's Principle: (Upper and) Lower Bounds on $\mathcal{C}$

Let $\mathcal{A}_0$ be a distribution on **Algos** and $\mathcal{I}_0$ a distribution on **Inputs**. Then

$$\max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}_0}[C(A, I)] \overset{\text{(old news)}}{\geq} \mathcal{C} \overset{\text{"Yao's Principle"}}{\geq} \min_{A \in \textbf{Algos}} \mathbb{E}_{I \sim \mathcal{I}_0}[C(A, I)].$$

Tightness: Loomis implies that "$=$" is possible.

$\hookrightarrow$ Can attain (tight) lower bounds on $\mathcal{C}$ by thinking about deterministic algorithm only!

Content

Computational Problem: $\overline{\wedge}$-Tree-Evaluation

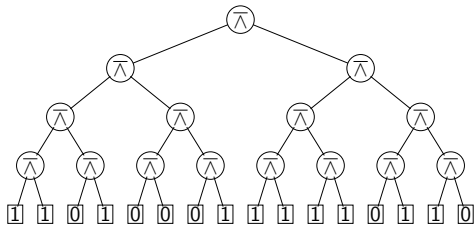
Problem: Evaluate $\overline{\wedge}$-Tree of depth d

- **Inputs** $= \{0,1\}^n$ for $n = 2^d$. Specify bits at leafs.
- **Algos** $=$ Algorithms computing value at root.
- $C(A, I) = \#$ bits of I that A examines
  $\hookrightarrow$ query complexity of A on I

Goal

Bound randomised query complexity

$$\mathcal{C} = \min_{\mathcal{A} \text{ dist. on } \textbf{Algos}} \max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}}[C(A, I)].$$



Computational Problem: $\overline{\wedge}$-Tree-Evaluation

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Goal

Bound randomised query complexity

$$\mathcal{C} = \min_{\mathcal{A} \text{ dist. on } \textbf{Algos}} \max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}}[C(A, I)].$$

Example and possible formalisation of **Algos** (that we won't use)

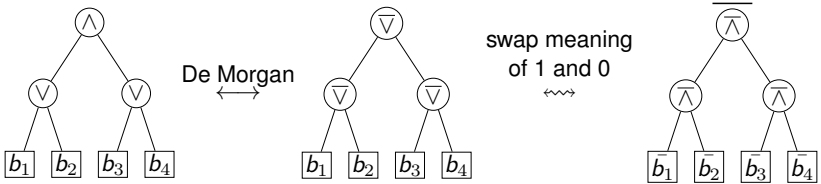
Each $A \in$ **Algos** corresponds to a *decision tree*. In the example:

- $C(A, (1,0,1,0)) = 4$
- $C(A, (0,1,0,1)) = 2$

Each leaf queried at most once per path
$\Rightarrow$ depth $\leq n \Rightarrow |\textbf{Algos}| < \infty$

What we already know

$\wedge$-$\vee$-trees are $\overline{\vee}$-trees are $\overline{\wedge}$-trees



What we already know

$\wedge$-$\vee$-trees are $\overline{\vee}$-trees are $\overline{\wedge}$-trees

Deterministic Query Complexity is n (Sheet 2, Exercise 3)

For all $A \in$ **Algos** there exists $I \in$ **Inputs** such that $C(A, I) = n$.

Randomised Query Complexity is $\mathcal{O}(n^{\log_4(3)}) \approx \mathcal{O}(n^{0.792})$ (Lecture "The Power of Randomness")

Let $\mathcal{A}$ be the randomised algorithm that evaluates one of the two depth $d-1$ subtrees at random (recursively) and, if that yields 1, also evaluates the other subtree (recursively).

$$\max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}}[C(A, I)] = \mathcal{O}(3^{d/2}) = \mathcal{O}(n^{\log_4(3)}).$$

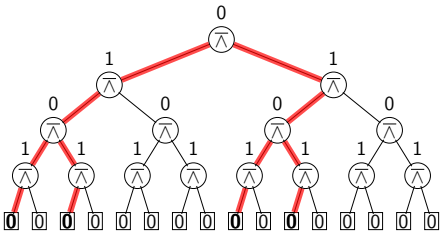
Goal: Show lower bound of $\Omega(\varphi^d) \approx \Omega(n^{0.694})$ using Yao's Principle (φ is the golden ratio).

Remark: actual complexity is $\Theta(n^{\log_4(3)})$, but that's more difficult.

Warm Up: A simple lower bound

Observation

For any even $d \in \mathbb{N}$ and $A \in$ **Algos** we have $C(A, (0, \ldots, 0)) \geq 2^{d/2}$.



Proof

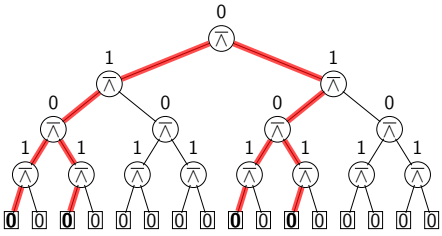
- in the end A knows that the root is 0.
- knowing a 0 requires knowing that both children are 1.
- Knowing a 1 requires knowing of one child that it is 0.

$\hookrightarrow$ A knows of $\geq 2^{d/2}$ leafs that they are 0 and must have checked them.

Warm Up: A simple lower bound

Observation

For any even $d \in \mathbb{N}$ and $A \in$ **Algos** we have $C(A, (0, \ldots, 0)) \geq 2^{d/2}$.

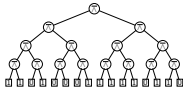


Corollary: Randomised Complexity is $\Omega(\sqrt{n})$

$$
\begin{aligned}
\mathcal{C} &= \min_{\mathcal{A} \text{ dist. on } \textbf{Algos}} \max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}}[C(A, I)] \\
&\geq \min_{\mathcal{A} \text{ dist. on } \textbf{Algos}} \mathbb{E}_{A \sim \mathcal{A}}[C(A, (0, \ldots, 0))] \\
&= \min_{A \in \textbf{Algos}} [C(A, (0, \ldots, 0))] \\
&\geq 2^{d/2} = 2^{\log_2(n)/2} = n^{1/2}.
\end{aligned}
$$

Note Yao's spirit: Lower bound on *randomised complexity* from result on *deterministic algorithms*.

A stronger lower bound



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Theorem (Tarsi 1984)

For any $p \in [0,1]$ simpleEval is optimal for input distribution $\mathcal{I}_p$, i.e.

$$\min_{A \in \textbf{Algos}} \mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(A, I)] = \mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(\text{simpleEval}, I)].$$

Lemma

Let $\varphi = \frac{\sqrt{5}+1}{2}$ be the golden ratio and $p_0 = \varphi - 1$. Then

$$\mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(\text{simpleEval}, I)] = (1 + p_0)^d = \varphi^d.$$

Corollary: $\mathcal{C} = \Omega(\varphi^d) \approx \Omega(n^{0.694})$

$$\mathcal{C} \overset{\text{Yao}}{\geq} \min_{A \in \textbf{Algos}} \mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(A, I)] \overset{\text{Tarsi}}{=} \mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(\text{simpleEval}, I)]$$

$$\overset{\text{Lemma}}{=} \varphi^d = \varphi^{\log_2 n} = n^{\log_2 \varphi} \approx n^{0.694}.$$

Independent Bernoulli Inputs

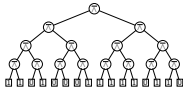
Let $\mathcal{I}_p = \text{Ber}(p)^n$ be the distribution where leafs are assigned independently values with distribution $\text{Ber}(p)$.

Deterministic Algorithm

Algorithm simpleEval(T)**:**

- **if** $T = \text{leaf}(b)$ **then**
  - **return** b
- **else**
  - $(T_\ell, T_r) \leftarrow T$
  - **if** simpleEval(T_ℓ) $= 0$ **then**
    - **return** 1
  - **else**
    - **return** $\neg$simpleEval(T_r)

Proof of Lemma: Cost of simpleEval on $\mathcal{I}_{p_0}$



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Lemma

Let $\varphi = \frac{\sqrt{5}+1}{2}$ be the golden ratio and $p_0 = \varphi - 1$. Then

$$\mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(\text{simpleEval}, I)] = (1+p_0)^d = \varphi^d.$$

Proof.

- $p_0 = \frac{\sqrt{5}-1}{2}$ is the solution to $p = 1 - p^2$.
- If $a, b \sim \text{Ber}(p_0)$ then $a \overline{\wedge} b \sim \text{Ber}(1-p_0^2) = \text{Ber}(p_0)$.
- For $I \sim \mathcal{I}_{p_0}$ the probability that an *internal* tree node evaluates to 1 is p_0.
- Let $c_d := \mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(\text{simpleEval}, I)]$ for trees of depth d. Then
  - $c_0 = 1$ // tree of depth 0 is just the leaf
  - $c_d = c_{d-1} + p_0 \cdot c_{d-1} = (1+p_0)c_{d-1} \overset{\text{Ind.}}{=} (1+p_0)(1+p_0)^{d-1} = (1+p_0)^d$
    // Always one recursive call, with probability p a second one.

□

Deterministic Algorithm

Algorithm simpleEval(T):

```
if T = leaf(b) then
    return b
else
    (T_ℓ, T_r) ← T
    if simpleEval(T_ℓ) = 0 then
        return 1
    else
        return ¬simpleEval(T_r)
```

Content

Tarsi's Theorem

Theorem (Tarsi 1984)

For any $p \in [0, 1]$ simpleEval is optimal for input distribution $\mathcal{I}_p$, i.e.

$$\min_{A \in \textbf{Algos}} \mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(A, I)] = \mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(\text{simpleEval}, I)].$$

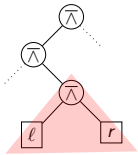
Proof idea:

- Take optimal Algorithm A.
- Transform A into simpleEval step by step.
- Show: Expected query complexity never increases.

Lemma: Evaluating Superleafs like simpleEval

Definition: Superleafs

A *superleaf* consists of two sibling leafs and their parent.



Lemma

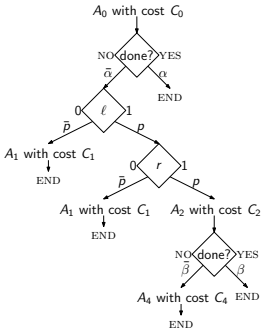
For any $p \in [0,1]$ and any $A \in$ **Algos** there exists $A' \in$ **Algos** such that

- $\mathbb{E}_{I \sim \mathcal{I}_p}[C(A', I)] \leq \mathbb{E}_{I \sim \mathcal{I}_p}[C(A, I)]$
- A' behaves on any superleaf $T = (\ell, r)$ like simpleEval:
  - i never visits r before ℓ
  - ii never visits r if $\ell = 0$
  - iii immediately visits r after visiting ℓ if $\ell = 1$

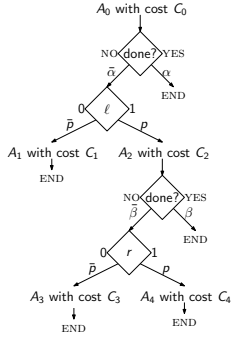
Proof Idea

- We fix every superleaf one by one. Let T be superleaf that needs fixing.
- Property i: Switch roles of ℓ and r if needed. Does not change the expected cost.
- Property ii: r does not contribute to result. Not visiting r *reduces* expected cost.
- Property iii: More difficult. See next slide.

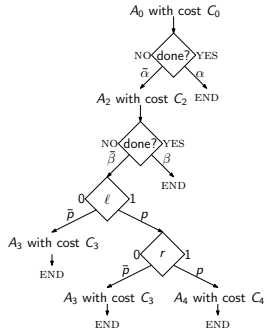
Modified Algorithm B



Original Algorithm A



Modified Algorithm D



$$C_A := \mathbb{E}[C(A,I)] = \mathbb{E}[C_0 + \bar{\alpha} \cdot (1 + \bar{p}C_1 + p \cdot (C_2 + \bar{\beta}(1 + \bar{p}C_3 + pC_4)))]$$
$$C_B := \mathbb{E}[C(B,I)] = \mathbb{E}[C_0 + \bar{\alpha} \cdot (1 + \bar{p}C_1 + p \cdot (1 + \bar{p}C_1 + p(C_2 + \bar{\beta}C_4)))]$$
$$C_D := \mathbb{E}[C(D,I)] = \mathbb{E}[C_0 + \bar{\alpha} \cdot (C_2 + \bar{\beta}(1 + \bar{p}C_3 + p(1 + \bar{p}C_3 + pC_4)))]$$

$$(C_B - C_A) + p \cdot (C_D - C_A) = \ldots = 0$$
$$\Rightarrow C_B - C_A \leq 0 \vee C_D - C_A \leq 0$$
$\Rightarrow B$ or D (or both) are at least as good as A and both visit superleaf (ℓ, r) as desired.

Theorem (Tarsi 1984)

For any $p \in [0,1]$ simpleEval is optimal for input distribution $\mathcal{I}_p$, i.e.

$$\min_{A \in \textbf{Algos}} \mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(A, I)] = \mathbb{E}_{I \sim \mathcal{I}_{p_0}}[C(\text{simpleEval}, I)].$$

We use induction on d. For $d = 0$ simpleEval is clearly optimal. Let now $d \geq 1$.
Let $A \in$ **Algos** be an algorithm minimising $\mathbb{E}_{I \sim \mathcal{I}_p}[C(A, I)]$.
By Lemma: There exists $A' \in$ **Algos** that behaves like simpleEval on superleafs such that

$$\mathbb{E}_{I \sim \mathcal{I}_p}[C(A', I)] \leq \mathbb{E}_{I \sim \mathcal{I}_p}[C(A, I)].$$

Let L' be the number of superleafs visited by A' and L the number of superleafs visited by simpleEval.
Superleafs evaluate to 1 with probability $1 - p^2$ independently and are in a complete binary tree of depth $d - 1$.

Apply induction for $d' = d - 1$ and $p' = 1 - p^2$.

$$\mathbb{E}_{I \sim \mathcal{I}_p}[L] \overset{\text{Ind.}}{\leq} \mathbb{E}_{I \sim \mathcal{I}_p}[L'].$$

The expected cost for evaluating a superleaf is $1 + p$.
Hence

$$\mathbb{E}_{I \sim \mathcal{I}_p}[C(A', I)] = (1 + p)\mathbb{E}[L']$$
$$\mathbb{E}_{I \sim \mathcal{I}_p}[C(A, I)] = (1 + p)\mathbb{E}[L]$$

Finally we obtain:

$$\mathbb{E}_{I \sim \mathcal{I}_p}[C(\text{simpleEval}, I)] = (1 + p)\mathbb{E}[L] \leq (1 + p)\mathbb{E}[L']$$
$$= \mathbb{E}_{I \sim \mathcal{I}_p}[C(A', I)] \leq \mathbb{E}_{I \sim \mathcal{I}_p}[C(A, I)].$$

Hence, simpleEval is optimal for $\mathcal{I}_p$. □

Content

Ski Rental – A Prototypical Online Problem

Setting: You are on a ski trip

Trip lasts for unknown number of days $I \in \mathbb{N}$ ("as long as there is snow").

Every day, if no skis bought yet:

- RENT skis for one day for cost 1 *or*
- BUY skis for cost $B \in \mathbb{N}$.

Goal: Minimise Competitive Ratio

The *competitive ratio* of distribution $\mathcal{A}$ on **Algos** is

$$C_{\mathcal{A}} = \sup_{I \in \textbf{Inputs}} \frac{\mathbb{E}_{A \sim \mathcal{A}}[C(A, I)]}{\mathrm{OPT}(I)}.$$

Framing using Online Algorithms

- **Inputs** $= \mathbb{N}$: number of days (not known in advance)
- **Algos** $= \mathbb{N}$: specify day for choosing BUY
- cost for $A \in$ **Algos** on $I \in$ **Inputs**:

$$C(A, I) = \begin{cases} I & \text{if } I < A \\ A - 1 + B & \text{otherwise.} \end{cases}$$

- cost of optimum *offline* solution

$$\mathrm{OPT}(I) = \begin{cases} I & \text{if } I < B \\ B & \text{otherwise.} \end{cases}$$

Break-Even is the best deterministic algorithm

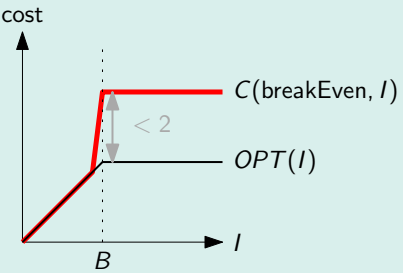
Observation

The algorithm breakEven $:= B$ has competitive ratio $\frac{2B-1}{B} \approx 2$.
All other $A \in$ **Algos** have competitive ratio ≥ 2.

Recall

B is the cost to BUY.

Proof



The worst ratio for breakEven is attained for input $I = B$.

$$C_{\text{breakEven}} = \sup_{I \in \mathbb{N}} \frac{C(\text{breakEven}, I)}{\text{OPT}(I)} = \frac{C(\text{breakEven}, B)}{\text{OPT}(B)}$$
$$= \frac{B - 1 + B}{B} = \frac{2B - 1}{B}.$$

Break-Even is the best deterministic algorithm

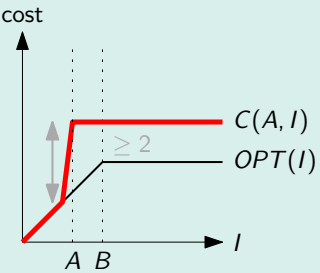
Observation

The algorithm breakEven := B has competitive ratio $\frac{2B-1}{B} \approx 2$.
All other $A \in$ **Algos** have competitive ratio ≥ 2.

Recall

B is the cost to BUY.

Proof



The worst ratio for $A \in$ **Algos** with $A < B$ is attained for input $I = A$.

$$C_A = \sup_{I \in \mathbb{N}} \frac{C(A,I)}{\mathrm{OPT}(I)} = \frac{C(A,A)}{\mathrm{OPT}(A)} = \frac{A-1+B}{A} = 1 + \frac{B-1}{A} \geq 1+1 = 2.$$

Break-Even is the best deterministic algorithm

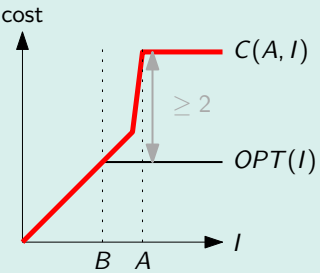
Observation

The algorithm breakEven $:= B$ has competitive ratio $\frac{2B-1}{B} \approx 2$.
All other $A \in$ **Algos** have competitive ratio ≥ 2.

Recall

B is the cost to BUY.

Proof



The worst ratio for $A \in$ **Algos** with $A > B$ is attained for input $I = A$.

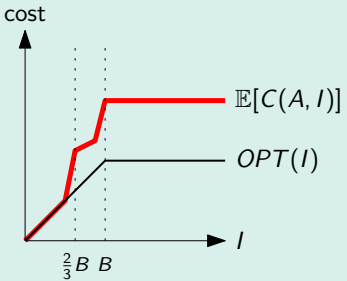
$$C_A = \sup_{I \in \mathbb{N}} \frac{C(A,I)}{\mathrm{OPT}(I)} = \frac{C(A,A)}{\mathrm{OPT}(A)} = \frac{A-1+B}{B} = 1 + \frac{A-1}{B} \geq 1+1 = 2.$$

A randomised algorithm can beat break-even

Observation (assuming wlog that B is a multiple of 3)

The randomised algorithm $\mathcal{A} = \mathcal{U}(\{\frac{2}{3}B, B\})$ has competitive ratio $\approx 1 + \frac{5}{6}$.

Proof



The competitive ratio of $\mathcal{A}$ "spikes" for inputs $\frac{2}{3}B$ and B. It is decreasing in between and constant after B.

$$\mathbb{E}_{A\sim\mathcal{A}}[C(A, \tfrac{2}{3}B)] = \underbrace{\tfrac{2}{3}B - 1}_{\text{rent}} + \underbrace{\tfrac{1}{2}(1+B)}_{\text{rent or buy}} < \tfrac{7}{6}B, \qquad \mathrm{OPT}(\tfrac{2}{3}B) = \tfrac{2}{3}B,$$

$$\mathbb{E}_{A\sim\mathcal{A}}[C(A, B)] = \underbrace{B}_{\text{buy}} + \underbrace{\tfrac{2}{3}B - 1}_{\text{rent}} + \underbrace{\tfrac{1}{2}(\tfrac{1}{3}B)}_{\text{maybe rent}} < \tfrac{11}{6}B, \qquad \mathrm{OPT}(B) = B.$$

Hence $C_{\mathcal{A}} = \sup_{I\in\mathbb{N}} \dfrac{\mathbb{E}_{A\sim\mathcal{A}}[C(A, I)]}{\mathrm{OPT}(I)} \leq \max\left\{\dfrac{7/6}{2/3}, \dfrac{11/6}{1}\right\} = \max\left\{\dfrac{7}{4}, \dfrac{11}{6}\right\} = \dfrac{11}{6}.$

What's next?

Goal: Lower bound

No randomised algorithm has competitive ratio better than $\frac{e}{e-1} \approx 1.582$.

Yao's Principle for Online Algorithms

Theorem (see Online Optimization Lecture, Corollary 3.8, Prof. Yann Disser, Darmstadt, 2023)

For any distribution $\mathcal{A}_0$ on **Algos** and any distribution $\mathcal{I}_0$ on **Inputs** we have

$$C_{\mathcal{A}_0} \stackrel{\text{def}}{=} \sup_{I \in \textbf{Inputs}} \frac{\mathbb{E}_{A\sim\mathcal{A}_0}[C(A,I)]}{\text{OPT}(I)} \stackrel{(*)}{\geq} \frac{\mathbb{E}_{I\sim\mathcal{I}_0, A\sim\mathcal{A}_0}[C(A,I)]}{\mathbb{E}_{I\sim\mathcal{I}_0}[\text{OPT}(I)]} \geq \frac{\inf_{A\in\textbf{Algos}} \mathbb{E}_{I\sim\mathcal{I}_0}[C(A,I)]}{\mathbb{E}_{I\sim\mathcal{I}_0}[\text{OPT}(I)]}.$$

Steps to see (*)

- Prove that $\frac{a+c}{b+d} \leq \max(\frac{a}{b}, \frac{c}{d})$ for $a, b, c, d > 0$.
- Conclude that $\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \leq \max_{i\in[n]} \frac{a_i}{b_i}$ for $a_i, b_i > 0$
- Conclude that for *random* $I \in [n]$ that $\frac{\mathbb{E}[a_I]}{\mathbb{E}[b_I]} \leq \max_{i\in[n]} \frac{a_i}{b_i}$.
- Conclude (*).

Yao's Principle for Online Algorithms

Theorem (see Online Optimization Lecture, Corollary 3.8, Prof. Yann Disser, Darmstadt, 2023)

For any distribution $\mathcal{A}_0$ on **Algos** and any distribution $\mathcal{I}_0$ on **Inputs** we have

$$C_{\mathcal{A}_0} \overset{\text{def}}{=} \sup_{I \in \textbf{Inputs}} \frac{\mathbb{E}_{A\sim\mathcal{A}_0}[C(A,I)]}{\mathrm{OPT}(I)} \overset{(*)}{\geq} \frac{\mathbb{E}_{I\sim\mathcal{I}_0, A\sim\mathcal{A}_0}[C(A,I)]}{\mathbb{E}_{I\sim\mathcal{I}_0}[\mathrm{OPT}(I)]} \geq \frac{\inf_{A\in\textbf{Algos}} \mathbb{E}_{I\sim\mathcal{I}_0}[C(A,I)]}{\mathbb{E}_{I\sim\mathcal{I}_0}[\mathrm{OPT}(I)]}.$$

Remark

- Yao's principle exists for other settings as well.
- Tightness typically follows from duality of optimisation problems or fixed point theorems.
  (though I'm not sure how it works here)

A hard distribution for Ski-Rental: Intuition

$$\mathcal{I}_0 := \mathrm{Geom}_1(\tfrac{1}{B}).$$

Why $\mathcal{I}_0$?

- distribution is **memoryless**, i.e. $\Pr_{I\sim\mathcal{I}_0}[I = i + t \mid I > i] = \Pr_{I\sim\mathcal{I}_0}[I = t]$.
  Assume no skis bought on day i: Minimising expected *future* cost is the same problem as on day 1.
  $\hookrightarrow$ wlog: either buy right away or not at all.
- **expectation** tuned such that

$$\mathbb{E}_{I\sim\mathcal{I}_0}[C(\text{never buy}, I)] = \mathbb{E}_{I\sim\mathcal{I}_0}[C(\text{immediately buy}, I)] = B.$$

  $\hookrightarrow$ all strategies equally good

A hard distribution for Ski-Rental: Analysis

Lemma

Let $\mathcal{I}_0 := \text{Geom}(\frac{1}{B})$ *and* $q := 1 - \frac{1}{B} =$ "$\Pr[❄]$". *Then*

i $\mathbb{E}_{I\sim\mathcal{I}_0}[C(A, I)] = B$ *for all* $A \in \mathbb{N}$.

ii $\mathbb{E}_{I\sim\mathcal{I}_0}[\text{OPT}(I)] = B(1 - (1 - \frac{1}{B})^B)$.

Tail sum formula:

For random variable X with values in $\mathbb{N}$:

$$\mathbb{E}[X] = \sum_{j\geq 1} \Pr[X \geq j].$$

Proof of (i)

$$\begin{aligned}\mathbb{E}_{I\sim\mathcal{I}_0}[C(A, I)] &= \underbrace{\Pr[I \geq A] \cdot B}_{\text{buy}} + \sum_{i=1}^{A-1} \underbrace{\Pr[I \geq i] \cdot 1}_{\text{rent}} \\ &= q^{A-1} \cdot B + \sum_{i=1}^{A-1} q^{i-1} = q^{A-1} \cdot B + \sum_{i=0}^{A-2} q^i = q^{A-1} \cdot B + \frac{1 - q^{A-1}}{1 - q} \\ &= q^{A-1} \cdot B + (1 - q^{A-1})B = B.\end{aligned}$$

A hard distribution for Ski-Rental: Analysis

Lemma

Let $\mathcal{I}_0 := \text{Geom}(\frac{1}{B})$ *and* $q := 1 - \frac{1}{B} =$ "$\Pr[❄]$". *Then*

i. $\mathbb{E}_{I\sim\mathcal{I}_0}[C(A, I)] = B$ *for all* $A \in \mathbb{N}$.

ii. $\mathbb{E}_{I\sim\mathcal{I}_0}[\text{OPT}(I)] = B(1 - (1 - \frac{1}{B})^B)$.

Tail sum formula:

For random variable X with values in $\mathbb{N}$:

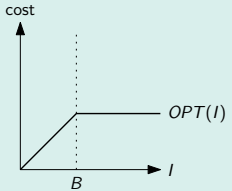
$$\mathbb{E}[X] = \sum_{j\geq 1} \Pr[X \geq j].$$

Proof of (ii)

$$\mathbb{E}_{I\sim\mathcal{I}_0}[\text{OPT}(I)] \overset{\text{TSF}}{=} \sum_{j\geq 1} \Pr[\text{OPT}(I) \geq j] = \sum_{j=1}^{B} \Pr[\text{OPT}(I) \geq j]$$

$$= \sum_{j=1}^{B} \Pr[I \geq j] = \sum_{j=1}^{B} q^{j-1} = \sum_{j=0}^{B-1} q^j$$

$$= \frac{1 - q^B}{1 - q} = B(1 - (1 - \tfrac{1}{B})^B).$$



Note: $\text{OPT}(I) = I$ for $I \in [B]$.

A hard distribution for Ski-Rental: Analysis

Lemma

Let $\mathcal{I}_0 := \text{Geom}(\frac{1}{B})$ *and* $q := 1 - \frac{1}{B} =$ "$\Pr[❄]$"*. Then*

i. $\mathbb{E}_{I\sim\mathcal{I}_0}[C(A,I)] = B$ *for all* $A \in \mathbb{N}$.

ii. $\mathbb{E}_{I\sim\mathcal{I}_0}[\text{OPT}(I)] = B(1-(1-\frac{1}{B})^B)$.

Tail sum formula:

For random variable X with values in $\mathbb{N}$:

$$\mathbb{E}[X] = \sum_{j\geq 1} \Pr[X \geq j].$$

Lower bound for Ski-Rental

By Yao's theorem any randomised algorithm $\mathcal{A}$ for ski-rental has competitive ratio at least

$$C_{\mathcal{A}} \stackrel{\text{def}}{=} \sup_{I\in\textbf{Inputs}} \frac{\mathbb{E}_{A\sim\mathcal{A}}[C(A,I)]}{\text{OPT}(I)} \stackrel{\text{Yao}}{\geq} \frac{\inf_{A\in\textbf{Algos}} \mathbb{E}_{I\sim\mathcal{I}_0}[C(A,I)]}{\mathbb{E}_{I\sim\mathcal{I}_0}[\text{OPT}(I)]} = \frac{B}{B(1-(1-\frac{1}{B})^B)} = \frac{1}{1-(1-\frac{1}{B})^B}.$$

For large B the lower bound converges to $\lim_{B\to\infty} \frac{1}{1-(1-\frac{1}{B})^B} = \frac{1}{1-1/e} = \frac{e}{e-1} \approx 1.582$.

Upper bound for Ski-Rental

Remark: The lower bound is tight (Karlin et al. 1994)

There exists a distribution $\mathcal{A}$ on $[B]$ such that $c_{\mathcal{A}} \leq \frac{e}{e-1}$.

Applications

Very basic online question:

Should I pay a small possibly recurring cost or a large one time cost?

Occurs in:

- Cache management.
- Networking.
- Scheduling.
- ...

Conclusion

Algorithm Design as a Two-Player Game

- “we” choose algorithm to minimise cost
- “adversary” chooses input to maximise cost
- Nash/Loomis: It does not matter who moves first *if* mixed strategy is allowed for first player.

Yao’s Principle

Lower bound on worst-case expected cost of *any randomised algorithm* $\mathcal{A}_0$ by analying *any deterministic algorithm* on specific input distribution $\mathcal{I}_0$.

$$\max_{I \in \textbf{Inputs}} \mathbb{E}_{A \sim \mathcal{A}_0}[C(A, I)] \geq \mathcal{C} \geq \min_{A \in \textbf{Algos}} \mathbb{E}_{I \sim \mathcal{I}_0}[C(A, I)].$$

Can narrow down randomised complexity $\mathcal{C}$ of underlying problem from both sides.

Appendix: Possible Exam Questions I

Game theory:

- What is a two-player game in the game-theoretic sense?
- What is a Nash equilibrium?
- Does a Nash equilibrium always exist?
- What is a zero-sum game?
- What does Nash's Theorem state (for two-player zero-sum games)?
- What does Loomis' Theorem state?
- Prove Loomis' Theorem! (challenging task)

Yao's principle:

- What is the connection between game theory and the design of algorithms?
- How is randomised complexity (with respect to a cost function C) usually defined? What alternative viewpoint does Loomis' Theorem provide?
- State Yao's Principle! What is it useful for?

Appendix: Possible Exam Questions II

Application to $\overline{\wedge}$-trees:

- What goal did we set ourselves when evaluating $\overline{\wedge}$-trees? (minimising query complexity)
- What worst-case cost can be achieved with a deterministic algorithm?
- Can randomised algorithms do better? How?
- It is rather easy to see that the randomised complexity is $\Omega(\sqrt{n})$. How?
- We also saw a tighter analysis. What components did it have? In particular: how does Yao's principle come into play?
- What does Tarsi's theorem state?

Ski rental problem:

- State the Ski Rental Problem.
- What do we call this type of problem? (*online* problem)
- Name some applications of the Ski Rental Problem.
- How is the competitive ratio defined?

Appendix: Possible Exam Questions III

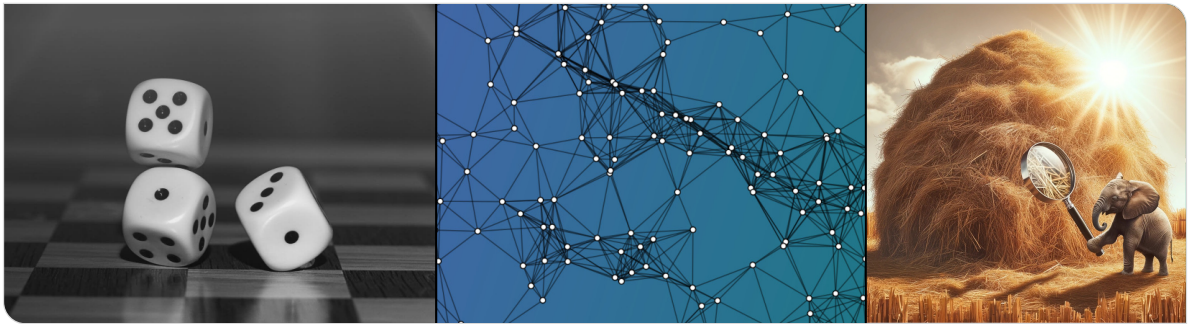
- What is the best deterministic algorithm? How can one see this?
- Is there a randomised algorithm that can beat the break-even point? (idea only)
- State Yao's principle for online algorithms.
- Which input distribution did we assume for the lower bound for ski rental? What is the intuition?
- What costs arise for online and offline algorithms for this input distribution? What can we conclude about the competitive ratio?

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Probabilistic Method

Stefan Walzer | WS 2025/2026



The Name of the Game

The Probabilistic Method (pioneered by Paul Erdős)

Show that something exists by proving that it has a positive probability of arising from a random process.

- Used to proved statements that don't involve randomness at all.
- Probabilistic arguments replace combinatorial arguments.

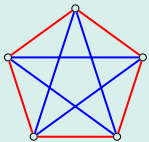
First Example: Ramsey Numbers

Definition: Ramsey Number

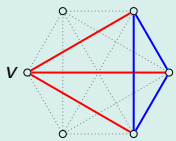
$R(k,k) := \min\{n \in \mathbb{N} \mid$ any red-blue colouring of the edges of K_n contains a monochromatic k-clique$\}$. [a]

[a] The general definition of $R(r,b)$ asks for red r-clique or blue b-clique.

$R(3,3) > 5$



$R(3,3) \leq 6$



- v has 3 incident edges of the same colour
- wlog that colour is red
- if there is no red triangle then w_1, w_2, w_3 form a blue triangle.

Hence: $R(3,3) = 6$.

First Example: Ramsey Numbers

Definition: Ramsey Number

$R(k,k) := \min\{n \in \mathbb{N} \mid$ any red-blue colouring of the edges of K_n contains a monochromatic k-clique$\}$.

Theorem: $R(k,k) > 2^{k/2}$ for $k \geq 6$. // actually $k \geq 3$ suffices

Proof.

- *To show*: Edges of K_n with $n \leq 2^{k/2}$ can be coloured while avoiding a monochromatic k-clique.
- *Plan*: Show that *uniformly random colouring* avoids monochromatic k-clique with positive probability.
- There are $\binom{n}{k}$ k-cliques. Each is monochromatic with probability $2^{-\binom{k}{2}+1}$.
- The number M of monochromatic k-cliques satisfies:

$$\mathbb{E}[M] = \binom{n}{k} \cdot 2^{-\binom{k}{2}+1} \leq \frac{n^k}{k!} \cdot 2^{-k^2/2+k/2+1} \leq \frac{(2^{k/2})^k}{(k/2)^{k/2}} \cdot 2^{-k^2/2} 2^{k/2} 2 = 2\left(\frac{4}{k}\right)^{k/2} < 1.$$

Since $\mathbb{E}[M] < 1$ it is possible that $M = 0$. In particular a colouring with no monochromatic k-cliques exists. □

Expectation Argument

We have implicitly used:

$\Pr[X \leq \mathbb{E}[X]] > 0$ and $\Pr[X \geq \mathbb{E}[X]] > 0$.

Probabilistic Method with Expectation Argument

Show that an object x with $f(x) \overset{\leq}{\geq} b$ exists by proving that a random object X satisfies $\mathbb{E}[f(X)] \overset{\leq}{\geq} b$.

Simple Use Case

Any graph $G = (V, E)$ admits a cut of weight at least $|E|/2$.

Proof.

- Assign each $v \in V$ to V_1 or V_2 uniformly at random.
- Each edge crosses the cut (V_1, V_2) with probability $1/2$.
- $\mathbb{E}[\text{weight of } (V_1, V_2)] = \mathbb{E}\left[\sum_{e \in E}[e \text{ crosses } (V_1, V_2)]\right] = \sum_{e \in E} \Pr[e \text{ crosses } (V_1, V_2)] = |E| \cdot \frac{1}{2}$. □

Example: Independent Sets

Theorem

Let $G = (V, E)$ with $n = |V|$, $m = |E|$ and $m \geq \frac{n}{2}$.

Then there exists an independent set of size $\frac{n^2}{4m}$. // $= \frac{n}{2 \cdot \text{average degree}}$

Proof.

- sampleAndReject computes an independent set $V^+ \setminus V^-$.
- $\mathbb{E}[|V^+|] = n \cdot \frac{n}{2m} = \frac{n^2}{2m}$.
- $\mathbb{E}[|V^-|] \leq \sum_{\{u,v\} \in E} \Pr[u \in V^+, v \in V^+] = \sum_{\{u,v\} \in E} \left(\frac{n}{2m}\right)^2 = \frac{n^2}{4m}$.
- $\mathbb{E}[|V^+ \setminus V^-|] = \mathbb{E}[|V^+|] - \mathbb{E}[|V^-|] \geq \frac{n^2}{2m} - \frac{n^2}{4m} = \frac{n^2}{4m}$. □

Algorithm sampleAndReject**:**

```
// pick random vertex set:
V⁺ ← ∅
for v ∈ V do
    with probability n/(2m) do
        V⁺ ← V⁺ ∪ {v}
// destroy induced edges:
V⁻ ← ∅
for {u, v} ∈ E do
    if u ∈ V⁺ and v ∈ V⁺ then
        V⁻ ← V⁻ ∪ {u} // or v
return V⁺ \ V⁻
```

Remark: sampleAndReject seems suitable for a parallel / distributed setting.

Lovász Local Lemma

Context

Given: Family $\mathcal{E} = \{E_1, \ldots, E_n\}$ of "bad" events with $\Pr[E_i] \leq p < 1$.

Want: Show $\Pr[\bar{E}_1 \cap \ldots \cap \bar{E}_n] = \Pr[\text{none of } \mathcal{E}] > 0$.

Observation: Easy if $\mathcal{E}$ is independent

If $\mathcal{E}$ is an independent family then $\Pr[\text{none of } \mathcal{E}] = \prod_{i=1}^{n} \Pr[\bar{E}_i] \geq (1-p)^{|\mathcal{E}|} > 0$.

Observation: Expectation arguments only gets us so far

If $np < 1$ then $\mathbb{E}[\#\text{events from } \mathcal{E} \text{ occuring}] \leq np < 1$, hence $\Pr[\text{none of } \mathcal{E}] > 0$.
If $np = 1$ then $\Pr[\text{none of } \mathcal{E}] = 0$ is possible, e.g. $X \sim \mathcal{U}([n])$ and $E_i := \{X = i\}$.

Lovász Local Lemma (László Lovász and Paul Erdős, 1973)

If each $E \in \mathcal{E}$ has $\Pr[E] \leq p$ and depends on at most d events[a] from $\mathcal{E}$ and $4pd \leq 1$ then $\Pr[\text{none of } \mathcal{E}] > 0$.

[a] Little challenge: State what this means formally.

Example for Lovász Local Lemma (by Wikipedia User *Kevinatilusa*)

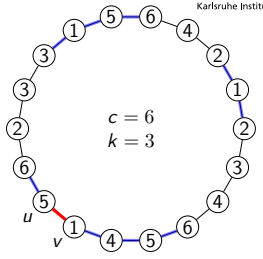
Lovász Local Lemma (László Lovász and Paul Erdős, 1973)

If each $E \in \mathcal{E}$ has $\Pr[E] \leq p$ and depends on at most d events from $\mathcal{E}$ and $4pd \leq 1$ then $\Pr[\text{none of } \mathcal{E}] > 0$.

Setting

Consider a necklace of ck beads with k beads of each of c colours.
An *independent rainbow* is a set of beads

- containing one bead of each colour // rainbow
- and not containing a pair of adjacent beads. // independent



Claim: If $k \geq 16$ then an independent rainbow always exists. // $k \geq 11$ also suffices

Consider any necklace. Let R contain a random bead of each color. // Goal: $\Pr[R \text{ independent}] > 0$.
One bad event per pair of adjacent beads:

$$E_{\{u,v\}} := \{u \in R \wedge v \in R\}, \quad \Pr[E] \leq \tfrac{1}{k^2} =: p.$$

$E_{\{u,v\}}$ depends on $E_{\{u',v'\}}$ only if u' or v' share the colour of u or v.
$2k$ relevant beads, hence $4k-2$ relevant pairs.

$$\Rightarrow d = 4k-2, \qquad 4pd \leq 4\tfrac{1}{k^2}(4k-2) < \tfrac{16}{k} \leq 1.$$

$$\Pr[R \text{ independent}] = \Pr[\text{none of } (E_{\{u,v\}})_{u,v}] \overset{\text{LLL}}{>} 0.$$

Proof of Lovász Local Lemma

Lovász Local Lemma (László Lovász and Paul Erdős, 1973)

If each $E \in \mathcal{E}$ has $\Pr[E] \leq p$ and depends on at most d events from $\mathcal{E}$ and $4pd \leq 1$ then $\Pr[\text{none of } \mathcal{E}] > 0$.

Claim: $\forall S \subseteq \mathcal{E} : \forall E^* \in \mathcal{E} \setminus S : \Pr[E^* \mid \text{none of } S] \leq 2p$.

Proof of LLL using the Claim.

$\Pr[\text{none of } \mathcal{E}] = \prod_{i=1}^{n} \Pr[\bar{E}_i \mid \text{none of } \{E_1, \ldots, E_{i-1}\}] \geq (1-2p)^n \overset{4pd \leq 1}{\geq} 2^{-n} > 0.$ □

Proof of Lovász Local Lemma

Lovász Local Lemma (László Lovász and Paul Erdős, 1973)

If each $E \in \mathcal{E}$ has $\Pr[E] \leq p$ and depends on at most d events from $\mathcal{E}$ and $4pd \leq 1$ then $\Pr[\text{none of } \mathcal{E}] > 0$.

Claim: $\forall S \subseteq \mathcal{E} : \forall E^* \in \mathcal{E} \setminus S : \Pr[E^* \mid \text{none of } S] \leq 2p$.

Proof of the Claim by Induction on $|S|$.

- Base case: If $|S| = 0$ then $\Pr[E^* \mid \text{none of } \varnothing] = \Pr[E^*] \leq p \leq 2p$. ✓ Let now $|S| > 0$.
- Partition $S = I \dot\cup D$ such that E^* is independent of I and $1 \leq |D| \leq d$. // $\leq d$ possible by assumption, > 0 is our choice.
- $\Pr[\text{none of } D \mid \text{none of } I] = 1 - \Pr[\bigcup_{E \in D} E \mid \text{none of } I] \overset{\text{UB}}{\geq} 1 - \sum_{E \in D} \underbrace{\Pr[E \mid \text{none of } I]}_{\leq 2p \text{ (Induction, using } |I| < |S|)} \geq 1 - 2dp \overset{4pd \leq 1}{\geq} \frac{1}{2}$. ($\bigstar$).

$$\Pr[E^* \mid \text{none of } S] = \frac{\Pr[E^* \wedge \text{none of } S]}{\Pr[\text{none of } S]} \leq \frac{\Pr[E^* \wedge \text{none of } I]}{\Pr[\text{none of } D \mid \text{none of } I]\Pr[\text{none of } I]}$$

$$= \frac{\Pr[E^*]\Pr[\text{none of } I]}{\Pr[\text{none of } D \mid \text{none of } I]\Pr[\text{none of } I]} \leq \frac{p}{\Pr[\text{none of } D \mid \text{none of } I]} \overset{(\bigstar)}{\leq} \frac{p}{1/2} = 2p. \qquad \square$$

Summary

What the Probabilistic Method is all About

- Goal: Prove the existence of objects with certain properties.
- Use probabilistic language as a tool.

Vanilla Variant:

Goal: Show that $P \subseteq \Omega$ is not empty.

1. Define a random object $X \in \Omega$.
2. Show: $\Pr[X \in P] > 0$.
3. Conclude: $\exists x \in \Omega : x \in P$.

Variant with Expectation Argument

Goal: Show that $f : \Omega \to \mathbb{R}$ has maximum at least q.

1. Define a random object $X \in \Omega$.
2. Show: $\mathbb{E}[f(X)] \geq q$.
3. Conclude: $\exists x \in \Omega : f(x) \geq q$.

Variant with Lovász Local Lemma

Goal: Show that $P \subseteq \Omega$ is not empty.

1. Define random object X.
2. Define family $\mathcal{E}$ of bad events such that $\bigcap_{E \in \mathcal{E}} \bar{E} \Rightarrow X \in P$.
4. Show that $E \in \mathcal{E}$ satisfies $\Pr[E] \leq p$.
5. Show $E \in \mathcal{E}$ depends on at most d other events from $\mathcal{E}$.
6. Show $4dp \leq 1$.
7. Conclude with LLL: $\exists x : x \in P$.

Appendix: Possible Exam Questions I

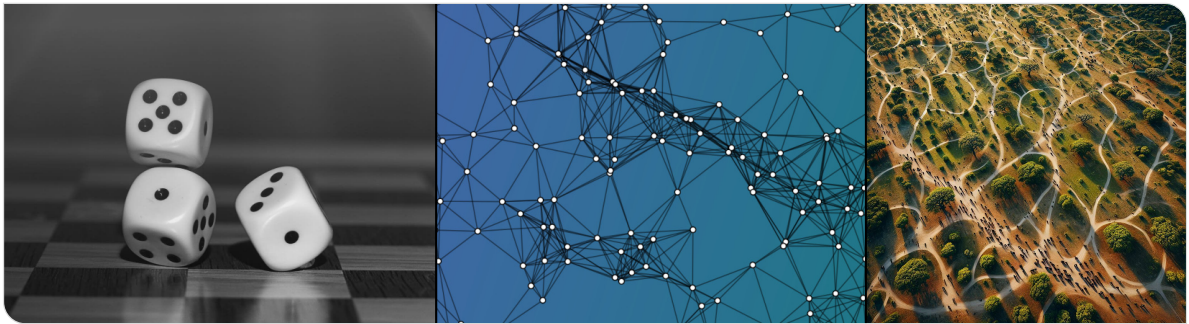
- What is the goal of the probabilistic method?
- Concerning the basic method:
  - What kind of “creative step” is required, and what must then be computed?
  - Illustrate the method with an example.
- Concerning expectation arguments:
  - What kind of “creative step” is required, and what must then be computed?
  - Illustrate the method with an example.
  - We showed that every graph has a cut of weight $|E|/2$. How?
  - We showed that every graph has an independent set of size $\frac{n^2}{4m}$. How?
- Concerning the Lovász Local Lemma:
  - State the lemma.
  - What is the connection to the probabilistic method?
  - We showed that colored graphs have independent rainbow sets of a certain size. How did we proceed?

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Random Graphs

Stefan Walzer | WS 2025/2026



Organisation: Oral Exam

Dates and Time Slots

	19.2.	*date of the last lecture*
We	11.03.	Oral Exams
Th	12.03.	Oral Exams
Fr	13.03.	Oral Exams
We	25.3.	Oral Exams
Th	26.3.	Oral Exams
Fr	27.3.	Oral Exams

Other dates may be possible on request.

Available time slots:

- 10:00, 10:25, 10:50, 11:15
- 14:00, 14:25, 14:50, 15:15

- Registration via our secretary:
  - Anja Blancani (blancani@kit.edu)
  - cc to me (stefan.walzer@kit.edu)
  - Please specify:
    - your full name
    - matriculation number
    - subject of study (Studienfach)
    - version of the exam regulation (Version der Prüfungsordnung)
- Cancellation also via our secretary
- Location: Stefan's Office (50.34, Room 209).
- duration: 20 minutes
- scope: content of lectures *and exercises*

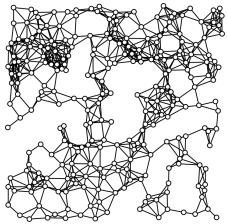
Content

Motivation 1: Average Case Analysis

Theory-Practice Gap

Minimum Vertex Cover is APX-hard $\overset{???}{\longleftrightarrow}$ small vertex covers can often be computed efficiently in practice

$\rightsquigarrow$ relevant graph classes (e.g. social networks) are not worst-case.



Bridging the Gap

1. Define a distribution $\mathcal{G}$ on graphs.
   - $\mathcal{G}$ should be realistic, i.e. model real world instances
   - $\mathcal{G}$ should have simple mathematical structure
2. Consider randomised complexity of handling $G \sim \mathcal{G}$.

Goals

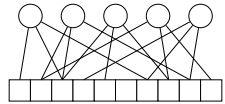
- model real world instances
- identify useful properties of these instances
- build algorithms exploiting these properties

Motivation 2: Data Structure Design

Stay tuned

Random graphs occur naturally in

- cuckoo hash tables
- retrieval data structures
- perfect hash functions



Motivation 3: Probabilistic Method

Probabilistic Method for Graph Theory

Show that graphs with a property P exist by showing that a random graph G satisfies $\Pr[G \text{ has } P] > 0$.

(we're not doing this)

Content

The Erdős-Renyi Model and Related Distributions

Original Erdős-Renyi Model $G(n, m)$: "Uniformly random graph with n nodes and m edges"

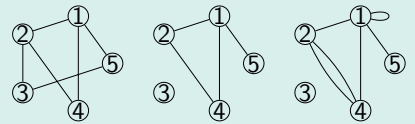
Gilbert Model $G(n, p)$: "Every edge with probability p"

Uniform Endpoint Model $G^{\text{UE}}(n, m)$: "randomly attach the $2m$ endpoints of edges"

Definition

Let $n \in \mathbb{N}$, $0 \leq m \leq \binom{n}{2}$. We use $G(n, m)$ to refer to a graph sampled uniformly from the set of all graphs with vertex set $[n]$ and m edges.

Example: $n = 5, m = 6$



probability $1/\binom{\binom{n}{2}}{m}$ 0 0

The Erdős-Renyi Model and Related Distributions

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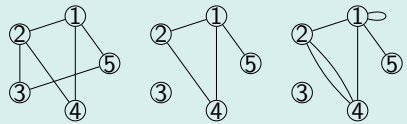
Gilbert Model $G(n, p)$: "Every edge with probability p"

Uniform Endpoint Model $G^{\text{UE}}(n, m)$: "randomly attach the $2m$ endpoints of edges"

Definition

Let $n \in \mathbb{N}$ and $p \in (0, 1)$. We use $G(n, p)$ to refer to a graph with vertex set $[n]$ that contains each of the $\binom{n}{2}$ possible edges with probability p, independently from other edges.

Example: $n = 5$



probability $p^6(1-p)^4$ $p^4(1-p)^6$ 0

The Erdős-Renyi Model and Related Distributions

Original Erdős-Renyi Model $G(n, m)$: "Uniformly random graph with n nodes and m edges"

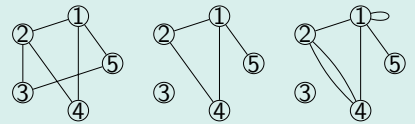
Gilbert Model $G(n, p)$: "Every edge with probability p"

Uniform Endpoint Model $G^{\text{UE}}(n, m)$: "randomly attach the $2m$ endpoints of edges"

Definition

Let $n, m \in \mathbb{N}$ and $v_1, \ldots, v_{2m} \sim \mathcal{U}([n])$. We use $G^{\text{UE}}(n, m)$ to refer to a multi-graph with vertex set $[n]$ and a multiset of edges that contains a copy of $\{v_{2i-1}, v_{2i}\}$ for each $i \in [m]$.

Example: $n = 5, m = 6$



probability $6! \cdot 2^6 \cdot 5^{-12}$ 0 $6! \cdot 2^4 \cdot 5^{-12}$

The Erdős-Renyi Model and Related Distributions

Original Erdős-Renyi Model $G(n, m)$: "Uniformly random graph with n nodes and m edges"

Gilbert Model $G(n, p)$: "Every edge with probability p"

Uniform Endpoint Model $G^{\text{UE}}(n, m)$: "randomly attach the $2m$ endpoints of edges"

Remarks

- for $p = m/\binom{n}{2}$ the three distributions are similar in many ways
- the original Erdős-Renyi model is often inconvenient to work with
- the uniform endpoint model is non-standard (we'll need it in later chapters)

Plan for the Next Few Slides: Sparse Graphs

Focus on Expected Degree $\lambda \in \mathcal{O}(1)$

- for $G(n, m)$ choose $m = \frac{\lambda n}{2}$ $\Rightarrow$ average vertex degree $\frac{2m}{n} = \lambda$
- for $G(n, p)$ choose $p = \frac{\lambda}{n-1}$ $\Rightarrow$ expected vertex degree $(n-1) \cdot p = \lambda$
- for $G^{\text{UE}}(n, m)$ choose $m = \frac{\lambda n}{2}$ $\Rightarrow$ average vertex degree $\frac{2m}{n} = \lambda$ // loops contribute 2 to a vertex degree

Goals

- Build intuition for properties of Erdős-Renyi graphs.
- Get a feeling for how to work with them.
- For simplicity: Focus on the Gilbert model only.

Selected Properties of Erdős-Renyi Graphs

On the next few slides we consider:

Vertex Degrees

For large n, the degree of a given vertex is approximately Poisson distributed.

Local Structure

The neighbourhood around a vertex resembles a Galton-Watson tree.

Degree Statistics

The number of vertices of each degree is highly concentrated around its expectation.

Largest Connected Component

Size of the largest component is highly predictable.

Content

Degree Distribution of Erdős-Reyni Graphs

Exercise: Degrees are approximately Poisson distributed

For each $n \in \mathbb{N}$ consider $G(n, \lambda/n)$ and the degree $X_n \sim \text{Bin}(n-1, \lambda/n)$ of vertex 1. Moreover, let $X \sim \text{Pois}(\lambda)$. Then

$$X_n \xrightarrow{d} X \text{ for } n \to \infty.$$

The same holds for $G(n, \lfloor \lambda n/2 \rfloor)$ and $G^{\text{UE}}(n, \lambda n/2)$.

Content

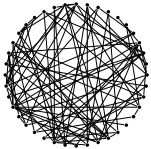
The Number N_d of Vertices of Degree N_d

Notation

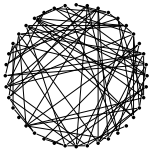
- Let $d \in \mathbb{N}$, $\lambda > 0$. We consider $G(n, \lambda/n)$. // Gilbert model
- Let $N_d := |\{v \in [n] \mid \deg(v) = d\}|$

Is N_d highly concentrated?

- Note: $(\deg(v))_{v \in [n]}$ are correlated.
- Otherwise N_d would have a binomial distribution and we could use Chernoff bounds.



d	0	1	2	3	4	5	6	7	8	9
N_d	0	2	8	6	7	7	3	2	3	1



d	0	1	2	3	4	5	6	7	8	9
N_d	1	2	5	8	9	11	3	1	0	0

Concentration of N_d

Lemma (Near Independence of Degrees)

Let $u \neq v$ be two vertices of $G(n, \lambda/n)$. Then $\Pr[\deg(u) = d, \deg(v) = d] = \Pr[\deg(u) = d] \Pr[\deg(v) = d] \pm \Theta(1/n)$.

Proof

Let $\deg'(u) = \deg(u) - [\{u, v\} \in E]$ be the degree of u when ignoring $\{u, v\}$ if present. Then

$$\Pr[\deg(u) \neq \deg'(u)] = \Pr[\{u, v\} \in E] = \lambda/n = \Theta(1/n).$$

The same holds for $\deg'(v) = \deg(v) - [\{u, v\} \in E]$. We conclude:

$$\begin{aligned}\Pr[\deg(v_1) = d, \deg(v_2) = d] &= \Pr[\deg'(v_1) = d, \deg'(v_2) = d] \pm \Theta(1/n) \\ &= \Pr[\deg'(v_1) = d] \Pr[\deg'(v_2) = d] \pm \Theta(1/n) = \Pr[\deg(v_1) = d] \Pr[\deg(v_2) = d] \pm \Theta(1/n).\end{aligned}$$

Concentration of N_d

Lemma (Near Independence of Degrees)

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Theorem

$\Pr[|N_d - np_d| \geq n^{2/3}] = \mathcal{O}(n^{-1/3})$ where $p_d = \Pr[\deg(1) = d] \approx e^{-\lambda} \frac{\lambda^d}{d!}$.

Proof

$\mathbb{E}[N_d] = np_d$ $\qquad$ $\mathbb{E}[N_d^2] = n^2 p_d^2 \pm \mathcal{O}(n)$ $\qquad$ $\mathrm{Var}(N_d) = \mathcal{O}(n)$.

$$\mathbb{E}[N_d] = \mathbb{E}[\sum_{v \in [n]} [\deg(v) = d]] = n \cdot \Pr[\deg(1) = d] = n \cdot p_d.$$

Concentration of N_d

Lemma (Near Independence of Degrees)

Let $u \neq v$ be two vertices of $G(n, \lambda/n)$. Then $\Pr[\deg(u) = d, \deg(v) = d] = \Pr[\deg(u) = d] \Pr[\deg(v) = d] \pm \Theta(1/n)$.

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Proof

$\mathbb{E}[N_d] = np_d \qquad \mathbb{E}[N_d^2] = n^2 p_d^2 \pm \mathcal{O}(n) \qquad \mathrm{Var}(N_d) = \mathcal{O}(n).$

$$\mathbb{E}[N_d^2] = \mathbb{E}\big[(\sum_{v \in [n]} [\deg(v) = d])^2\big] = \mathbb{E}\big[\sum_{u \in [n]} \sum_{v \in [n]} [\deg(u) = d, \deg(v) = d]\big]$$

$$= \sum_{u \in [n]} \sum_{v \in [n]} \Pr[\deg(u) = d, \deg(v) = d] = \sum_{u \in [n]} \Pr[\deg(u) = d] + \sum_{u \in [n]} \sum_{v \neq u} \Pr[\deg(u) = d, \deg(v) = d]$$

$$= n \cdot p_d + n \cdot (n-1) \cdot (p_d^2 \pm \mathcal{O}(1/n)) = n^2 p_d^2 \pm \mathcal{O}(n).$$

Concentration of N_d

Lemma (Near Independence of Degrees)

Let $u \neq v$ be two vertices of $G(n, \lambda/n)$. Then $\Pr[\deg(u) = d, \deg(v) = d] = \Pr[\deg(u) = d] \Pr[\deg(v) = d] \pm \Theta(1/n)$.

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Proof

$\mathbb{E}[N_d] = np_d$ $\qquad$ $\mathbb{E}[N_d^2] = n^2 p_d^2 \pm \mathcal{O}(n)$ $\qquad$ $\mathrm{Var}(N_d) = \mathcal{O}(n)$.

$$\mathrm{Var}(N_d) = \mathbb{E}[N_d^2] - \mathbb{E}[N_d]^2 \leq n^2 p_d^2 + \mathcal{O}(n) - (np_d)^2 = \mathcal{O}(n).$$

Concentration of N_d

Lemma (Near Independence of Degrees)

Let $u \neq v$ be two vertices of $G(n, \lambda/n)$. Then $\Pr[\deg(u) = d, \deg(v) = d] = \Pr[\deg(u) = d] \Pr[\deg(v) = d] \pm \Theta(1/n)$.

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Proof

$\mathbb{E}[N_d] = np_d$ $\qquad$ $\mathbb{E}[N_d^2] = n^2 p_d^2 \pm \mathcal{O}(n)$ $\qquad$ $\mathrm{Var}(N_d) = \mathcal{O}(n)$.

$$\text{Hence: } \Pr[|N_d - np_d| \geq n^{2/3}] = \Pr[|N_d - \mathbb{E}[N_d]| \geq n^{2/3}] \overset{\text{Cheb.}}{\leq} \frac{\mathrm{Var}(N_d)}{n^{4/3}} = \mathcal{O}(n^{-1/3}).$$

Content

Erdős-Renyi Graphs have Few Cycles

Theorem: There are few short cycles in Erdős-Renyi graphs

Let C_k be the number of cycles of length k in $G(n, \lambda/n)$ where $k, \lambda = \Theta(1)$. Then $\mathbb{E}[C_k] \leq \frac{\lambda^k}{2k} = \Theta(1)$.

Proof.

The number of potential cycles is $\underbrace{n(n-1)\cdot\ldots\cdot(n-k+1)}_{\text{sequences } (v_1,\ldots,v_k)} \cdot \underbrace{\frac{1}{k}\cdot\frac{1}{2}}_{\substack{\text{startpoint and direction} \\ \text{irrelevant}}}$

The probability that $(v_1, \ldots, v_k, v_1)$ is a cycle is $(\lambda/n)^k$. Hence:

$$\mathbb{E}[C_k] \leq \frac{n^k}{2k}\left(\frac{\lambda}{n}\right)^k = \frac{\lambda^k}{2k}.$$

□

The Galton-Watson Branching Process

Definition

Let $\mathcal{D}$ be a distribution on $\mathbb{N}_0$ and $X_{i,j} \sim \mathcal{D}$ for $i, j \in \mathbb{N}$. Define $Z_0 = 1$ and $Z_i = \sum_{j=1}^{Z_{i-1}} X_{i,j}$ for $i \geq 1$.

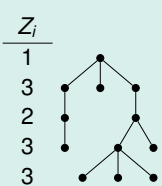
Intuition

- Start with a population of size $Z_1 = 1$.
- Each individual has a random number of decendents.
- Key question: What is the probability of extinction, i.e. for $\lim_{i \to \infty} Z_i = 0$?

Exercise: Galton-Watson Process with $\mathcal{D} = \text{Pois}(\lambda)$

If $\lambda \leq 1$ then the process goes extinct with probability 1.
If $\lambda > 1$ then the process survives with probability $s_\lambda > 0$.

Galton-Watson *Tree*



Z_i
1
3
2
3
3
0

$X_{i,j}$	1	2	3	4	...
1	3	1	0	2	...
2	1	0	1	3	...
3	1	2	2	0	...
4	0	3	0	0	...
5	0	0	0	2	...
⋮	⋮	⋮	⋮	⋮	

Local Structure of Erdős-Renyi Graphs

Theorem: The Neighbourhood of v looks like a Galton Watson Tree

Let $R = \mathcal{O}(1)$. Let H be an (ordered) tree of depth R given by a sequence $c_1, \ldots, c_k$ specifiying the number of children of nodes in all layers except the last, in level order.
Let $\mathrm{GWT}(\lambda)|_R$ be the first R layers of a $\mathrm{Pois}(\lambda)$-Galton-Watson tree.
Let $G(n, \lambda/n)|_{v,R}$ be the (ordered) subgraph of $G(n, \lambda/n)$ induced by vertices with distance $\leq R$ from v.

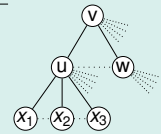
$$\Pr[\mathrm{GWT}(\lambda)|_R = H] \overset{\text{(i)}}{=} \prod_{i=1}^{k} \Pr_{X \sim \mathrm{Pois}(\lambda)}[X = c_i] = \prod_{i=1}^{k} e^{-\lambda} \frac{\lambda^{c_i}}{c_i!} \overset{\text{(ii)}}{\approx} \Pr[G(n, \lambda/n)|_{v,R} = H].$$

Example for H

$(c_1, c_2, c_3) = (2, 3, 0)$

Proof of (ii) by Example: The following has to "go right" for $G(n, \lambda/n)|_{v,R} = H$

random variable	desired outcome	probability
$\deg(v) \sim \mathrm{Bin}(n-1, \lambda/n) \approx \mathrm{Pois}(\lambda)$	2	$\approx e^{-\lambda} \frac{\lambda^2}{2!}$
$[\{u, w\} \in E]$	0	$1 - \frac{\lambda}{n} \approx 1$
$\deg(u) - 1 \sim \mathrm{Bin}(n-3, \lambda/n) \approx \mathrm{Pois}(\lambda)$	3	$\approx e^{-\lambda} \frac{\lambda^3}{3!}$
$\deg(w) - 1 \sim \mathrm{Bin}(n-3, \lambda/n) \approx \mathrm{Pois}(\lambda)$	0	$\approx e^{-\lambda} \frac{\lambda^0}{0!}$
$[\{x_1, x_2\} \in E \vee \{x_2, x_3\} \in E \vee \{x_1, x_3\} \in E]$	0	$(1 - \frac{\lambda}{n})^3 \approx 1$



Local Structure of Erdős-Renyi Graphs

Theorem: The Neighbourhood of v looks like a Galton Watson Tree

Let $R = \mathcal{O}(1)$. Let H be an (ordered) tree of depth R given by a sequence $c_1, \ldots, c_k$ specifiying the number of children of nodes in all layers except the last, in level order.
Let $\mathrm{GWT}(\lambda)|_R$ be the first R layers of a $\mathrm{Pois}(\lambda)$-Galton-Watson tree.
Let $G(n, \lambda/n)|_{v,R}$ be the (ordered) subgraph of $G(n, \lambda/n)$ induced by vertices with distance $\leq R$ from v.

$$\Pr[\mathrm{GWT}(\lambda)|_R = H] \overset{\text{(i)}}{=} \prod_{i=1}^{k} \Pr_{X \sim \mathrm{Pois}(\lambda)}[X = c_i] = \prod_{i=1}^{k} e^{-\lambda} \frac{\lambda^{c_i}}{c_i!} \overset{\text{(ii)}}{\approx} \Pr[G(n, \lambda/n)|_{v,R} = H].$$

Example for H

$(c_1, c_2, c_3) = (2, 3, 0)$

Corollaries

- $G(n, \lambda/n)|_{v,R} \xrightarrow{d} \mathrm{GWT}(\lambda)|_R$ // convergence in distribution for $n \to \infty$
- The number N_H of "copies" of H in $G(n, \lambda/n)$ satisfies $\mathbb{E}[N_H] \approx n \cdot \prod_{i=1}^{k} e^{-\lambda} \frac{\lambda^{c_i}}{c_i!}$.
  Concentration of N_H can be proved much like we proved concentration of N_d earlier.

Content

How does $G(n, \lambda/n)$ look like for different λ?

Karlsruhe Institute of Technology

Theorem: Sudden Emergence of the Giant Component (Erdős, Renyi 1960)

Consider $G(n, \lambda/n)$. The following holds with probability approaching 1 for $n \to \infty$.

i. If $\lambda < 1$ then $G(n, \lambda/n)$ only has components of size $\mathcal{O}(\log n)$.
   Each component is a tree or pseudotree. // pseudotree: connected and # edges = # vertices
   $\hookrightarrow$ Intuition: GWT(λ) dies out with probability 1.

ii. If $\lambda > 1$ then $G(n, \lambda/n)$ has one "giant" component of size $\approx s(\lambda) \cdot n$.
   $\hookrightarrow$ Intuition: $s(\lambda) > 0$ is the probability that GWT(λ) is infinite
   $\approx$ probability that fixed vertex is in giant component

iii. If $\lambda = 1$ then the largest component of $G(n, \lambda/n)$ has size $\Theta(n^{2/3})$.
   $\hookrightarrow$ Intuition: ?

wild pseudotree
https://crowspath.org/love-trees/

Content

Locality: A Property of Networks in Practice

Observation: Locality in Practice

Take social networks. A friend of my friend is more likely to be my friend than a random person.

Definition: Locality[1]

$L = \Pr[\{u, w\} \in E \mid \{v, u\} \in E \wedge \{v, w\} \in E]$ where v, u, w are distinct (random) vertices.

Similar numbers are sometimes called *clustering coefficient*.

Observation: No Locality in Erdős-Renyi Random Graphs

In $G(n, \lambda/n)$ we have $L = \frac{\lambda}{n} = \mathcal{O}(\frac{1}{n})$.

Next: Random *Geometric* Graphs with $L = \Omega(1)$.

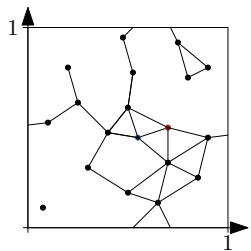
Random Geometric Graphs

Definition: Random Geometric Graph (RGG)

An RGG is obtained by distributing vertices in a metric space and connecting any two vertices with a probability depending on their distance.

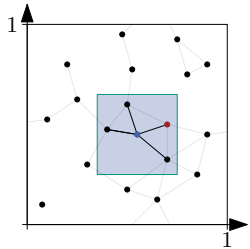
Simple Example: $G^{\mathbb{T}^2}(n, r)$

- number of vertices: n
- space: 2-dimensional torus $\mathbb{T}^2 = [0, 1)^2$
  // standard unit square is more common but less simple
- metric: L_∞ // L_2 is more common but less simple
  $\hookrightarrow \text{dist}((x_1, y_1), (x_2, y_2)) = \max(\text{dist}(x_1, x_2), \text{dist}(y_1, y_2))$.
- vertex distribution: for $v \in [n]$: $P_v \sim \mathcal{U}(\mathbb{T}^2)$
- edge "probability" is 0 or 1: $\{u, v\} \in E \Leftrightarrow \text{dist}(P_u, P_v) \leq r$
  // not random when P_u and P_v are given



Degree Distribution of $G^{\mathbb{T}^2}(n, r)$

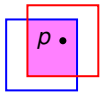
- Consider arbitrary $v \in [n]$.
- By symmetry of $\mathbb{T}^2$ each outcome of P_v behaves the same.
- $\Pr[\{u, v\} \in E] = \Pr[P_u \text{ is in the } 2r \times 2r \text{ square centered at } P_v] = 4r^2$.
- Hence $\deg(v) \sim \text{Bin}(n-1, 4r^2)$ and $\mathbb{E}[\deg(v)] = 4r^2(n-1)$.



Locality in $G^{\mathbb{T}^2}(n, r)$

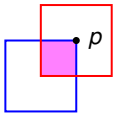
Observation

Let $p, q \sim \mathcal{U}([-0.5, 0.5]^2)$ and S_p the unit square around p.
Then $\Pr[q \in S_p] = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} (1 - |x|)(1 - |y|) dx\, dy = \frac{9}{16}$.

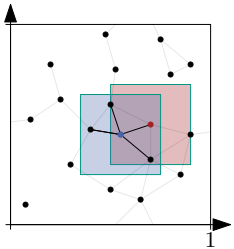


general case

best case



worst case



Corollary

By "rescaling" the observation we get $L = \Pr[\underbrace{\{u, w\} \in E}_{P_w \text{ in square around } P_u} \mid \underbrace{\{v, u\} \in E \wedge \{v, w\} \in E}_{P_u, P_w \text{ in square around } P_v}] = \frac{9}{16} = \Omega(1)$.

Poissonisation

Poissonised Variant $G^{\mathbb{T}^2}_{\text{Pois}}(n,r)$ of $G^{\mathbb{T}^2}(n,r)$

Replace the point set with a Poisson point process on $\mathbb{T}^2$ with intensity n.
$\hookrightarrow$ i.e. region of size λ contains $\text{Pois}(\lambda n)$-many points, independent for disjoint regions

Note: $G^{\mathbb{T}^2}_{\text{Pois}}(n,r) \overset{\text{d}}{=} G^{\mathbb{T}^2}(N,r)$ where $N \sim \text{Pois}(n)$.

Advantages

- No long-distance correlations. ✓
- $\text{Pois}(4r^2)$-distributed degrees. ✓

Disadvantages

- Less natural in practice. ✗
- Number of vertices $N \sim \text{Pois}(n)$ not fixed. ✗

De-Poissonisation (an analogous result holds for de-Poissonising balls-into-bins)

Let P be a graph property. If P is very unlikely for $G^{\mathbb{T}^2}_{\text{Pois}}(n,r)$ then P is unlikely for $G^{\mathbb{T}^2}(n,r)$:

$$\Pr[G^{\mathbb{T}^2}(n,r) \in P] = \Pr[G^{\mathbb{T}^2}_{\text{Pois}}(n,r) \in P \mid N = n] \leq \frac{\Pr[G^{\mathbb{T}^2}_{\text{Pois}}(n,r) \in P]}{\Pr[N=n]} = \Theta(n^{1/2}) \Pr[G^{\mathbb{T}^2}_{\text{Pois}}(n,r) \in P].$$

Content

Scale-Free Networks

Semi-Formal Definition

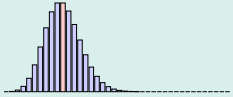
A scale-free network is a graph with a degree distribution that follows a power law (in an asymptotic sense)

Practical Consequence

There are vertices of very high degree (*hubs*).

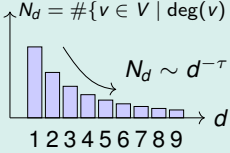
Contrast: Erdős-Renyi

exponentially decreasing tail.



for $\lambda = 10$.

Power Laws



$\tau \leq 1$: not a distribution

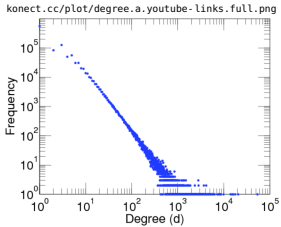
$1 < \tau \leq 2$: distribution, but $\mathbb{E}[\deg(v)] = \infty$

$2 < \tau \leq 3$: $\mathbb{E}[\deg(v)] < \infty$, but $\mathrm{Var}(\deg(v)) = \infty$

$3 < \tau \leq 4$: variance $< \infty$, but higher moments are ∞

$\tau \in (2, 3]$ is especially popular

"Youtube"



Scale-Free Networks

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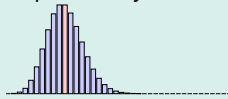


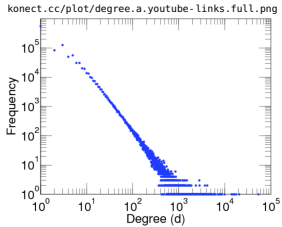
$\text{Pois}(\lambda)$ for $\lambda = 10$.

The Name “Scale-Free”

From *Barabási: “Linked: The New Science of Networks”, 2002.*

In a random network [...] the vast majority of nodes have the same number of links [...]. Therefore, a random network has a characteristic scale in its node connectivity [...]. In contrast, the absence of a peak in a power-law degree distribution implies that [...] we see a continuous hierarchy of nodes, spanning from rare hubs to the numerous tiny nodes. There is no intrinsic scale in these networks. This is the reason my research group started to describe networks with power-law degree distribution as scale-free.

“Youtube”



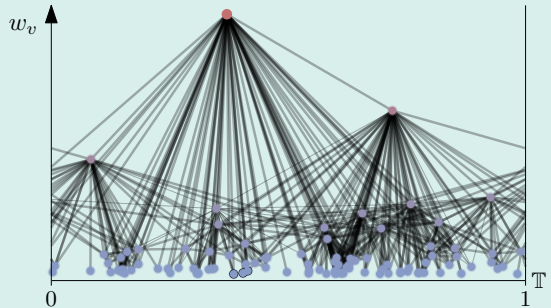
A Scale-Free Random Geometric Graph

Reminder: Random Geometric Graph (RGG)

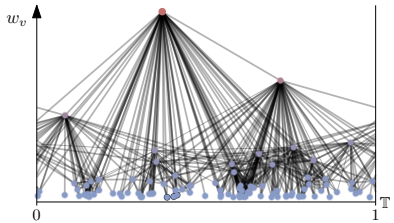
Distribute vertices in a metric space and connect any two vertices with a probability depending on their distance.

Definition: Geometric Inhomogeneous Random Graph (GIRG), Special Case

- number of vertices: n
- desired average degree $\lambda > 0$
- metric space $\mathbb{T}$ // more generally: $\mathbb{T}^d$ for $d \in \mathbb{N}$
- for each v: position $x_v \sim \mathcal{U}(\mathbb{T})$
- for each v: weight $w_v \sim \text{Par}(\tau - 1, 1)$
  the Pareto distribution is a power law distribution with exponent τ
- $\{u, v\} \in E \Leftrightarrow \text{dist}(x_u, x_v) \leq \frac{\lambda}{n} w_u w_v$
  $\Leftrightarrow \frac{n}{\lambda w_v} \leq \frac{w_u}{\text{dist}(x_u, x_v)}$.



How GIRGs are Useful



GIRGs are Scale-Free

$\mathbb{E}[\deg(v) \mid w_v] = \Theta(w_v)$ and $\deg(v)$ follows a power law if w_v does.

GIRGs are a Good Model for Real World Networks (Bläsius, Fischbeck, 2022)

- consider two graph parameters: locality and heterogeneity ($\approx \log \mathrm{Var}(\deg(v))$).
- in many contexts, a real network behaves like a GIRG with the same parameters

On the External Validity of Average-Case Analyses of Graph Algorithms, ESA 2022.

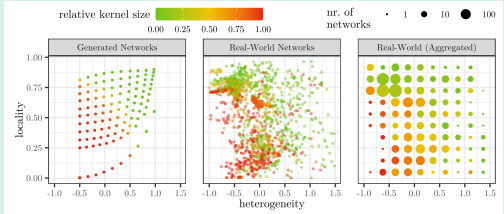


Figure 7 The relative kernel size of the vertex cover domination rule.

Hyperbolic Geometric Graphs

Poincaré Model of Hyperbolic Geometry

Illustration by M.C. Escher, Circle Limit III, 1959.

(All creatures are congruent in hyperbolic space.)

Result (Bläsius, Friedrich, Katzmann, 2021)

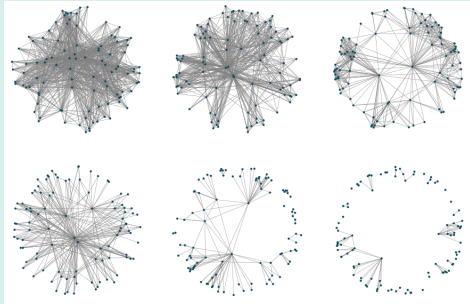
Vertex Cover can be Approximated on HGGs.

Efficiently Approximating Vertex Cover on Scale-Free Networks with Underlying Hyperbolic Geometry, ESA 2021.

Hyperbolic Random Graph (HGGs)

Sample points with bias towards the centre.
Connect points if distance is beneath a threshold.

https://commons.wikimedia.org/wiki/File:Hyperbolic_graphs.png



Can yield power law distribution for node degrees.

Preferential Attachment and the Barabasi-Albert Model

How a Graph is Grown Over Time

- There is a parameter $m \in \mathbb{N}$.
- start with any graph on $\geq m$ nodes.
- add new nodes one by one
  - new node is connected to m existing nodes
  - existing nodes are selected with probability proportional to their degree

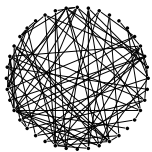
Why the Model is Interesting

- node degrees approach a power law distribution with exponent 3
- model may explain *why* scale-free networks emerge in practice

Summary

Erdős-Renyi Random Graphs

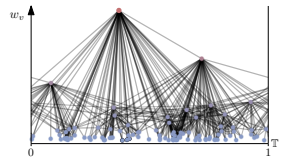
- simplest type of random graphs
- “Erdős-Renyi” refers to various related models
- arise in certain data structures (stay tuned)
- look locally like Poisson Galton-Watson Trees
- no locality or high-degree vertices



Random Graphs for Average Case Analsis

Mimic properties of real world networks:

- locality // a friend of my friend is often my friend
  - arises naturally in random geometric graphs
- “scale-freeness” $\approx$ existence of hubs
  - assign weights to vertices (in GIRGs)
  - use hyperbolic geometry
  - use preferential attachment



Appendix: Possible Exam Questions I

- What is meant by the theory–practice gap in the context of graph algorithms?
- What might a theoretician try to overcome the gap?
- What is the classical model of Erdős and Rényi?
  - Which variants of the Erdős–Rényi model did we consider?
  - What can be said about the distribution of $\deg(v)$ when we set $\mathbb{E}[\deg(v)] = \lambda$?
  - What can be said about $N_d = |\{v \in [n] \mid \deg(v) = d\}|$?
  - We studied the R-neighborhood $G(n, \lambda/n)|_{v,R}$ of a vertex v.
    - What holds for the distribution of $G(n, \lambda/n)|_{v,R}$, and why?
    - What is a Galton–Watson tree?
    - What can be said about the extinction probability of a Poisson Galton–Watson tree?
  - What is meant by the "sudden emergence of the giant component"? State the result formally.
  - We considered a quantity L, called locality. How is it defined?
  - What locality do Erdős–Rényi graphs have?
- Name properties that distinguish real-world networks from Erdős–Rényi graphs.
  - Give an example of a geometric random graph. What is the locality in this model?

Appendix: Possible Exam Questions II

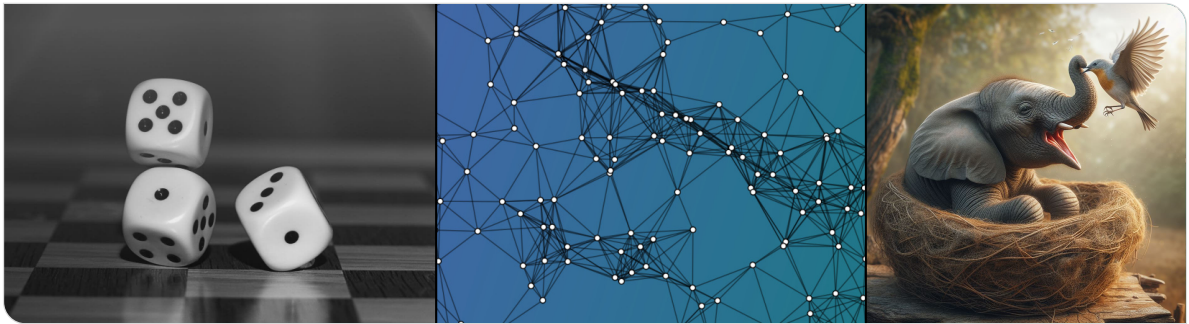
- In what way might a poissonized model be more convenient?
- When is a network “scale-free”?
  - Give an example of a real-world network to which this property is attributed.
  - Describe at least two ways in which networks of this type can be generated.

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Cuckoo Hashing

Stefan Walzer | WS 2025/2026



Evaluation

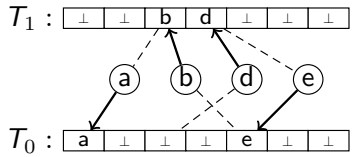
https://onlineumfrage.kit.edu/evasys/online.php?p=RF73W

Content

Classic Cuckoo Hashing

Setup

$S \subseteq D$	key set of size n
T_0, T_1	two tables of size m
$h_0, h_1 \sim \mathcal{U}([m]^D)$	two hash functions (SUHA)
$\frac{n}{m} = 1 - \beta$	for some $\beta > 0$
	(⚠ load factor $\alpha = \frac{n}{2m}$)



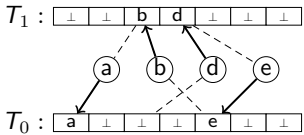
Algorithm lookup(x):
  return $x \in \{T_0[h_0(x)], T_1[h_1(x)]\}$

Algorithm delete(x):
  if $T_0[h_0(x)] = x$ **then**
    $T_0[h_0(x)] \leftarrow \perp$
  else if $T_1[h_1(x)] = x$ **then**
    $T_1[h_1(x)] \leftarrow \perp$

Algorithm insert(x):
  for $i = 0$ **to** LIMIT **do**
    $b \leftarrow i \bmod 2$
    swap(x, $T_b[h_b(x)]$)
    if $x = \perp$ **then**
      return SUCCESS
  return FAILURE

Cuckoo Hashing Theorem

Algorithm insert(x)**:**
- **for** $i = 0$ **to** LIMIT **do**
  - $b \leftarrow i \bmod 2$
  - swap(x, $T_b[h_b(x)]$)
  - **if** $x = \perp$ **then**
    - **return** SUCCESS
- **return** FAILURE



Theorem (Analysis with LIMIT $= \infty$)

Assume we insert all $x \in S$ and then another key y. Let E be the event that this succeeds and

$$T = \begin{cases} \text{insertion time of } y & \text{if } E \text{ occurs} \\ 0 & \textit{otherwise} \end{cases}$$

Then (i) $\Pr[E] = 1 - \mathcal{O}(1/m)$ and (ii) $\mathbb{E}[T] = \mathcal{O}(1)$.

Theorem (full analysis, not here)

If we
- set LIMIT $= \Omega(\log n)$ appropriately
- rebuild the table with fresh hash functions when LIMIT is reached

we obtain a hash table where lookup and delete take $\mathcal{O}(1)$ time and insert takes *expected* $\mathcal{O}(1)$ time.

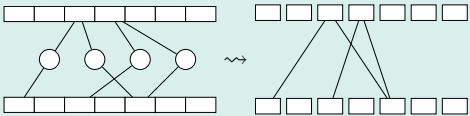
Proof of i: Success probability is $1 - \mathcal{O}(1/m)$

The Cuckoo Graph

Consider the bipartite *cuckoo graph*

$$G = ([m], [m], \{(h_0(x), h_1(x)) \mid x \in S\})$$

the key x corresponds to the edge $(h_0(x), h_1(x))$ and each table position to a vertex.



// Duplicate edges possible, don't worry about it.

Connection to Erdős-Renyi Graphs

G is a bipartite Erdős-Renyi variant

- much like $G^{\text{UE}}(2m, n)$ // uniform endpoint model
- a bit like $G(2m, n)$ // original Erdős-Renyi
- a bit like $G(2m, n/(2m^2))$ // Gilbert

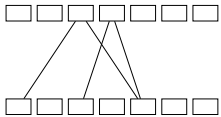
Confusing: n is a number of edges and $2m$ a number of vertices.

Exercise

Design a variant of cuckoo hashing such that the "Sudden Emergence" result for the $G^{\text{UE}}(m, n)$ model implies success for load factor $\alpha < \frac{1}{2}$.

Next: Completely self-contained analysis without reference to Erdős-Renyi.

Proof of i: Success probability is $1 - \mathcal{O}(1/m)$



Keys and buckets in the infinite loop

Assume $\bar{E}$ occurs, i.e. an insertion fails due to an infinite loop. Let $G^* = (V^*, E^*)$ be the subgraph of G with

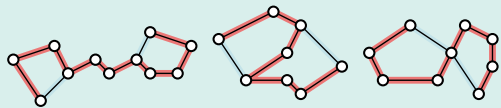
- V^*: table positions touched in the infinite loop
- E^*: keys touched in the infinite loop.

Properties of G^*:

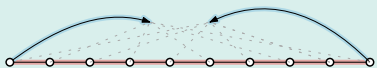
- connected
- $|E^*| = |V^*| + 1$ // can you see why?
- $\deg_{E^*}(v) \geq 2$ for $v \in V^*$.

Possibilities for G^*

There are three options:

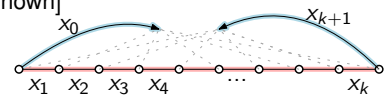


In all three cases: Simple path through $|V^*|$ and two extra edges connecting inwards:



Proof of i: Success probability is $1 - \mathcal{O}(1/m)$

$$\begin{aligned}
\Pr[\bar{E}] &= \Pr[\exists \text{path as shown}] \\
&= \Pr[\exists k \in \mathbb{N} : \exists x_0, \ldots, x_{k+1} \in S : x_0, \ldots, x_{k+1} \text{ form a path as shown}] \\
&\overset{\text{union bound}}{\leq} \sum_{k=1}^{n} \sum_{x_0,\ldots,x_{k+1} \in S} \Pr[x_0, \ldots, x_{k+1} \text{ form a path as shown}] \\
&\leq \sum_{k=1}^{n} \underbrace{n^{k+2}}_{\text{a}} \cdot \underbrace{2}_{\text{b}} \cdot \underbrace{\tfrac{1}{m^{k+1}}}_{\text{c}} \cdot \underbrace{\left(\tfrac{k+1}{2m}\right)^2}_{\text{d}} \\
&\leq \tfrac{1}{2} \sum_{k=1}^{n} m^{k+2-k-1-2} (1-\beta)^{k+2} (k+1)^2 \\
&\leq \tfrac{1}{2m} \sum_{k=1}^{\infty} (1-\beta)^{k+2} (k+1)^2 = \tfrac{1}{m} \cdot \mathcal{O}\left(\tfrac{1}{\beta^3}\right) = \mathcal{O}\left(\tfrac{1}{m}\right) \quad \square
\end{aligned}$$



- a Choose sequence of $k+2$ keys.
- b Choose to start in top or bottom table.
- c Neighbouring keys share a hash.
- d Two bordering keys connect back inward.

Proof of i: Success probability is $1 - \mathcal{O}(1/m)$

$$\begin{aligned}
\Pr[\bar{E}] &= \Pr[\exists \text{path as shown}] \\
&= \Pr[\exists k \in \mathbb{N} : \exists x_0, \ldots, x_{k+1} \in S : x_0, \ldots, x_{k+1} \text{ form a path as shown}] \\
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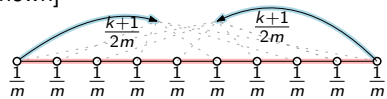
&\leq \tfrac{1}{2m} \sum_{k=1}^{\infty} (1-\beta)^{k+2} (k+1)^2 = \tfrac{1}{m} \cdot \mathcal{O}\left(\tfrac{1}{\beta^3}\right) = \mathcal{O}\left(\tfrac{1}{m}\right) \quad \square
\end{aligned}$$



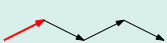
- a Choose sequence of $k + 2$ keys.
- b Choose to start in top or bottom table.
- c Neighbouring keys share a hash.
- d Two bordering keys connect back inward.

Proof of ii: Expected insertion time is $\mathcal{O}(1)$

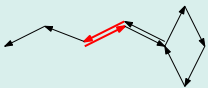
Lemma

If the insertion of y takes $t \in \mathbb{N}$ steps then the cuckoo graph G contained (previously) a path of length $\lceil (t-2)/3 \rceil$ starting from $h_0(y)$ or from $h_1(y)$.

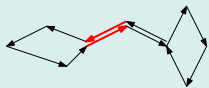
Proof.



no turning back
$\rightsquigarrow$ path of length $t-1$
starting from $h_0(y)$



turn back once
$\rightsquigarrow$ path of length $\lceil (t-2)/3 \rceil$
starting from $h_0(y)$ or $h_1(y)$



turn back twice
impossible: insertion would fail

□

Proof of ii: Expected insertion time is $\mathcal{O}(1)$ (continued)

$$\mathbb{E}[T] = \sum_{t \geq 1} \Pr[T \geq t]$$ tail sum formula

$$\leq \sum_{t \geq 1} \Pr[\exists \text{path of length } \lceil (t-2)/3 \rceil \text{ starting from } h_0(y) \text{ or } h_1(y)]$$ by Lemma

$$\leq 2 \cdot \sum_{t \geq 1} \Pr[\exists \text{path of length } \lceil (t-2)/3 \rceil \text{ starting from } h_0(y)]$$ union bound + symmetry

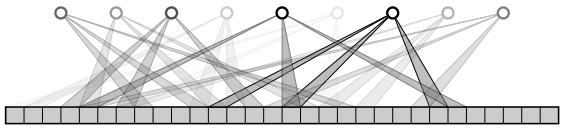
$$\leq 2\left(2 + 3 \cdot \sum_{t \geq 1} \Pr[\exists \text{path of length } t \text{ starting from } h_0(y)]\right)$$ $$\sum_{i \geq 1} f(\lceil t/3 \rceil) = 3 \cdot f(1) + 3 \cdot f(2) + \ldots$$

$$\leq 4 + 6 \cdot \sum_{t \geq 1} \sum_{x_1, \ldots, x_t \in S} \Pr[x_1, \ldots, x_t \text{ form path starting from } h_0(y)]$$ union bound

$$\leq 4 + 6 \cdot \sum_{t \geq 1} n^t m^{-t} \leq 6 \sum_{t \geq 0} (1-\beta)^t = 6/\beta = \mathcal{O}(1). \quad \square$$

Content

Cuckoo Hashing with one table and k hash functions



$n \in \mathbb{N}$ keys
$m \in \mathbb{N}$ table size
$\alpha = \frac{n}{m}$ load factor
$h_1, \ldots, h_k \sim \mathcal{U}([m]^D)$ hash functions
↪ Could also use a separate table per hash function.

randomWalkInsert(x)

while $x \neq \perp$ **do** // TODO: limit
  sample $i \sim \mathcal{U}([k])$
  swap($x, T[h_i(x)]$)

(some improvements possible)

Theorem (without proof)

For each $k \in \mathbb{N}$ there is a **threshold** c_k^* such that:

- if $\alpha < c_k^*$ all keys can be placed with probability $1 - \mathcal{O}(\frac{1}{m})$.
- if $\alpha > c_k^*$ **not** all keys can be placed with probability $1 - \mathcal{O}(\frac{1}{m})$.

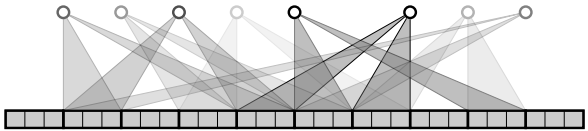
$c_2^* = \frac{1}{2}$, $c_3^* \approx 0.92$, $c_4^* \approx 0.98, \ldots$

Theorem (Bell, Frieze, 2024; retracted in 2025; re-announced for 2026)

If $k \geq 3$ and $\alpha < c_k^*$ then, conditioned on a high probability event[a], the expected insertion time is $\mathcal{O}(1)$.

[a] Without this conditioning, randomWalkInsert might be trapped in an infinite loop.

Cuckoo Hashing with k buckets of size ℓ

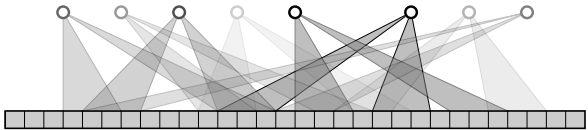


- picture illustrates $k = 2, \ell = 3$
- $k = 2$ has best cache efficiency
- larger ℓ improves space efficiency

Thresholds for the load factor

$\ell \backslash k$	2	3	4	5	6	7
1	0.5	0.9179352767	0.9767701649	0.9924383913	0.9973795528	0.9990637588
2	0.8970118682	0.9882014140	0.9982414840	0.9997243601	0.9999568737	0.9999933439
3	0.9591542686	0.9972857393	0.9997951434	0.9999851453	0.9999989795	0.9999999329
4	0.9803697743	0.9992531564	0.9999720661	0.9999990737	0.9999999721	0.9999999992
5	0.9895513619	0.9997746588	0.9999958681	0.9999999374	0.9999999992	≈ 1
6	0.9940727066	0.9999281468	0.9999993570	0.9999999956	≈ 1	≈ 1

Cuckoo Hashing with k unaligned blocks of size ℓ



- picture illustrates $k = 2, \ell = 3$
- note: unaligned block may cross cache line boundary

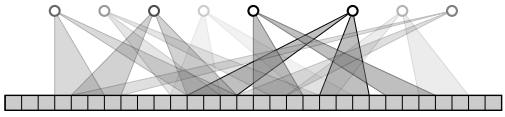
Thresholds for the load factor (slightly better than for buckets)

$\ell \backslash k$	2	3	4	5	6	7
1	0.5	0.9179352767	0.9767701649	0.9924383913	0.9973795528	0.9990637588
2	0.9649949234	0.9968991072	0.9996335076	0.9999529036	0.9999937602	0.9999991631
3	0.9944227538	0.9998255112	0.9999928198	0.9999996722	0.9999999843	0.9999999992
4	0.9989515932	0.9999896830	0.9999998577	0.9999999977	≈ 1	≈ 1
5	0.9997922174	0.9999993863	0.9999999972	≈ 1	≈ 1	≈ 1
6	0.9999581007	0.9999999635	0.9999999999	≈ 1	≈ 1	≈ 1

Back to Approximate Membership: Cuckoo Filters

General Idea

- construct cuckoo table for key set $S \subseteq D$
- store $\text{fp}(x)$ instead of $x \in S$ for random *fingerprint function* $\text{fp} : D \to \{0,1\}^r$
- query(x) checks for $\text{fp}(x)$ in positions associated with x



State of the Art Variant[1]

- uses $k = 2$ unaligned blocks of size $\ell = 2$
  - threshold ≈ 0.965
  - queries check 4 positions
- false positive probability $\varepsilon \leq 4 \cdot 2^{-r} = 2^{-r+2}$
- space $\approx \frac{r}{0.965} n = 1.04(\log_2(1/\varepsilon) + 2)n$ bits
  // compared to Bloom $\approx 1.44 \log_2(1/\varepsilon)n$ bits

Supporting Insertions and Deletions

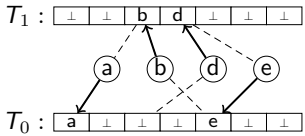
Complication: Need to evict fingerprints $\text{fp}(x)$ without knowing x.
Solution: Positions and fingerprints are related. Tricky details.

[1] Schmitz, Zentgraf, Rahman: Smaller and More Flexible Cuckoo Filters, ALENEX 2026.

Conclusion

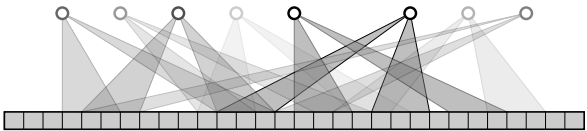
Classic Cuckoo Hashing

- hash table with *worst case* constant access times
- analysis considers paths in graphs similar to the Erdős-Renyi model



Practical Cuckoo Hashing

- uses buckets (or unaligned blocks) e.g. $k = 2$ buckets of size $\ell = 8$
- better than conventional hash tables, if high load factors are needed
- cuckoo filters are state of the art dynamic AMQ data structures
- *very* difficult to analyse



Anhang: Mögliche Prüfungsfragen I

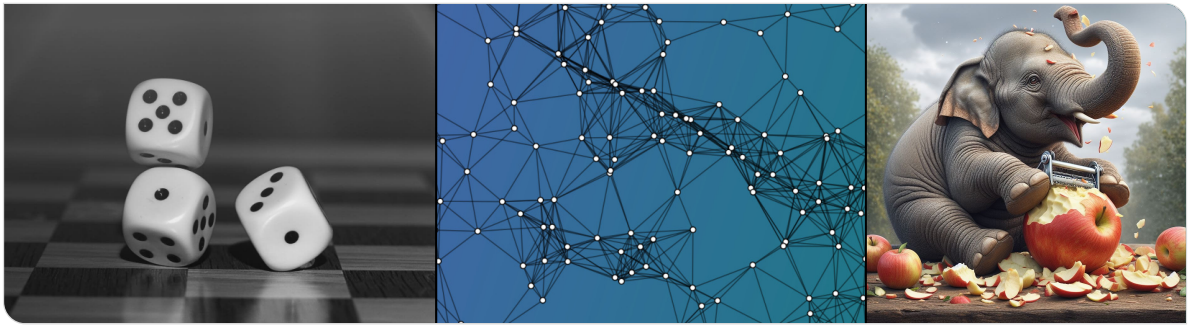
- Was ist und was kann Cuckoo Hashing?
  - Was ist die Grundidee? Wie funktionieren die Operationen?
  - Worauf ist bei der Wahl der Tabellengröße / beim Load Factor zu achten?
  - Was kann man über die Laufzeit der Operationen sagen?
  - Welche Vorteile und Nachteile ergeben sich im Vergleich zu anderen Techniken wie linearem Sondieren?
- Analyse:
  - Eine Einfügung, die fehlschlägt, entspricht gewissen Strukturen im Cuckoo-Graphen. Welchen?
  - Wie haben wir gezeigt, dass solche Strukturen unwahrscheinlich sind?
  - Wie haben wir die erwartete Einfügezeit abgeschätzt?

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – The Peeling Algorithm

Stefan Walzer | WS 2025/2026



Static Hash Tables

Static Hash Table

`construct`(S): builds table T with key set S no insertions or deletions after construction

`lookup`(x): checks if x is in T or not

Constructing cuckoo hash tables:

- solved by Khosla 2013: “Balls into Bins Made Faster”
- matching algorithm resembling preflow push
- expected running time $\mathcal{O}(n)$, finds placement whenever one exists
- not in this lecture

Greedily constructing cuckoo hash tables

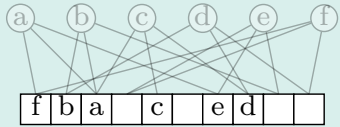
- Peeling: simple algorithm but sophisticated analysis
- interesting applications beyond hash tables (see “retrieval” in next lecture)

Content

The Peeling Algorithm

constructByPeeling($S \subseteq D$, $h_1, h_2, h_3 \in [m]^D$)

$T \leftarrow [\bot, \ldots, \bot]$ // empty table of size m
while $\exists i \in [m] : \exists$ *exactly one* $x \in S : i \in \{h_1(x), h_2(x), h_3(x)\}$ **do**
  // x is only unplaced key that may be placed in i
  $T[i] \leftarrow x$
  $S \leftarrow S \setminus \{x\}$
if $S = \varnothing$ **then**
  return T
else
  return NOT-PEELABLE



Exercise

- Success of constructByPeeling does not depend on choices for i made by while.
- constructByPeeling can be implemented in linear time.

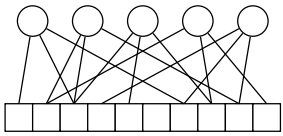
Peelability and the Cuckoo Graph

Cuckoo Graph and Peelability

- The **Cuckoo Graph** is the bipartite graph

$$G_{S,h_1,h_2,h_3} = (S, [m], \{(x, h_i(x)) \mid x \in S, i \in [3]\})$$

- Call G_{S,h_1,h_2,h_3} **peelable** if constructByPeeling(S, h_1, h_2, h_3) succeeds.
- If $h_1, h_2, h_3 \sim \mathcal{U}([m]^D)$ then the distribution of G_{S,h_1,h_2,h_3} does not depend on S. We then simply write $G_{m,\alpha m}$.
  - m □-nodes and $\lfloor \alpha m \rfloor$ -○-nodes
  - think: α is constant and $m \to \infty$.



$G_{m,\alpha m}$

Peeling simplified (not computing placement)

while ∃ □*-node of degree* 1 **do**
  remove it and its incident ○

G is peelable if and only if
this algorithm removes all ○-nodes.

Content

Peeling Theorem

Peeling Threshold

Let $c_3^\Delta = \min_{y \in [0,1]} \frac{y}{3(1-e^{-y})^2} \approx 0.81$.

Theorem (today's goal)

Let $\alpha < c_3^\Delta$. Then $\Pr[G_{m,\alpha m} \text{ is peelable}] = 1 - o(1)$.

Remark: More is known.

- For "$\alpha < c_3^\Delta$" we get "peelable" with probability $1 - \mathcal{O}(1/m)$.
- For "$\alpha > c_3^\Delta$" we get "not peelable" with probability $1 - \mathcal{O}(1/m)$.
- Corresponding thresholds c_k^Δ for $k \geq 3$ hash functions are also known.

Exercise: What about $k = 2$?

Peeling does not reliably work for $k = 2$ for any $\alpha > 0$.

Peeling Theorem: Proof outline

Theorem (today's goal)

Let $\alpha < c_3^\Delta$. Then $\Pr[G_{m,\alpha m} \text{ is peelable}] = 1 - o(1)$.

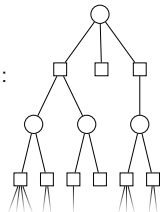
Proof Idea

The random (possibly) infinite tree T_α can be peeled for $\alpha < c_3^\Delta$ and T_α is locally like $G_{m,\alpha m}$.

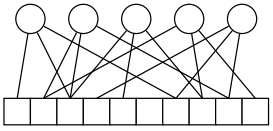
Steps

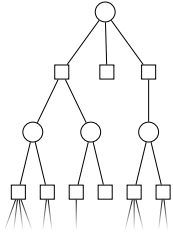
I. What is T_α?
II. What does peeling mean in this setting?
III. What role does c_3^Δ play?
IV. What does it mean for T_α to be locally like $G_{m,\alpha m}$?
V. What is the probability that a fixed key of $G_{m,\alpha m}$ is peeled?
VI. What is the probability that *all* keys of $G_{m,\alpha m}$ are peeled?

T_α :



What is T_α?





Observations for the finite Graph $G_{m,\alpha m}$

- each ◯ has 3 □ as neighbours (rare exception: $h_1(x), h_2(x), h_3(x)$ not distinct)
- each □ has random number X of ◯ as neighbours with $X \sim \text{Bin}(3n, \frac{1}{m}) = \text{Bin}(3\lfloor \alpha m \rfloor, \frac{1}{m})$. In an exercise we have seen that

$$X \xrightarrow{d} Y \text{ for } Y \sim \text{Pois}(3\alpha) \text{ and } m \to \infty$$

// i.e. $\forall i : \Pr[X = i] \xrightarrow{m\to\infty} \Pr[Y = i]$.

Definition of the (possibly) infinite random tree T_α

- root is ◯ and has three □ as children
- each □ has random number of ◯ children, sampled $\text{Pois}(3\alpha)$ (independently for each □).
- each non-root ◯ has two □ as children.

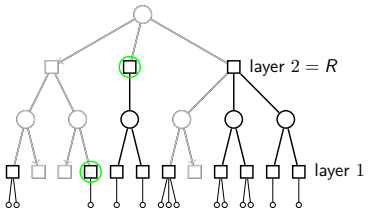
Remark: T_α is finite with positive probability > 0, e.g. when the first three $\text{Pois}(3\alpha)$ random variables come out as 0. But T_α is also infinite with positive probability.

ii What does peeling mean in this setting?

Peeling Algorithm

while $\exists$ *childless* □*-node* **do**
  remove it and its incident ○

↪ not well defined outcome on T_α!
↪ but well defined on T_α^R!



Peel only the first $R \in \mathbb{N}$ layers

- Let T_α^R be the first $2R + 1$ levels of T_α.
- *R layers* of □-nodes, labeled bottom to top.
- Run peeling on T_α^R (later $R \to \infty$).

↪ Why not consider the first $2R$ levels? (without $+1$)

Only care whether root is removed (root represents arbitrary node in $G_{m,\alpha m}$)

We may then simplify the peeling algorithm.

- replace "□-node of degree 1" condition with stronger "childless □-node".
  - prevents peeling of □-nodes with one child and no parent
  - no matter: such nodes are disconnected from the root anyway
- whether node is peeled only depends on subtree
  ↪ one bottom up pass suffices for peeling

ii What does peeling mean in this setting? (2)

Observation

Let $q_R = \Pr[\text{root survives when peeling } T_\alpha^R]$.
The values q_R are decreasing in R.

Peeling Algorithm

while $\exists$ *childless* □*-node* **do**
  remove it and its incident ○

Proof.

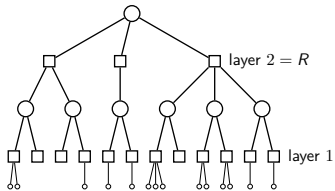
Assume when peeling T_α^R the sequence $\vec{x} = (x_1, \ldots, x_k)$ is a valid sequence of □-node choices. Then $\vec{x}$ is also valid when peeling T_α^{R+1}.

$$\text{peeling } T_\alpha^R \text{ removes the root} \Rightarrow \text{peeling } T_\alpha^{R+1} \text{ removes the root}$$

$$\text{root survives when peeling } T_\alpha^{R+1} \Rightarrow \text{peeling } T_\alpha^R \text{ removes the root}$$

$$q_{R+1} \leq q_R$$

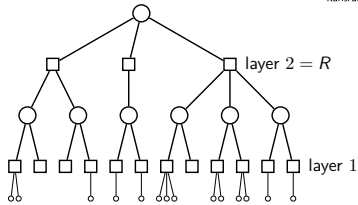
□



ii What does peeling mean in this setting? (3)

Peeling T_α^R bottom up

for $i = 1$ **to** R **do** // □-layers bottom to top
  for *each* □*-node v in layer i* **do**
    if *v has no children* **then**
      remove v and its parent ○



Survival probabilities $p_i :=$ Pr[□-node in layer i is *not* peeled]

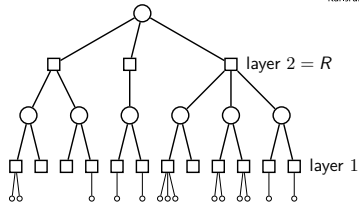
$$p_1 = \Pr[\square\text{-node has at least 1 child}]$$
$$= \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y > 0] = 1 - e^{-3\alpha}.$$
$$p_i = \Pr[\text{layer } i \ \square\text{-node } v \text{ has at least 1 } \textit{surviving} \text{ child}]$$
$$= \Pr_{X\sim\mathrm{Pois}(3\alpha p_{i-1}^2)}[X > 0] = 1 - e^{-3\alpha p_{i-1}^2}.$$

$Y :=$ number of (initial) children of v
$X :=$ number of surviving children of v
each child ○-node survives if both its □-children from layer $i-1$ survive $\rightsquigarrow$ probability p_{i-1}^2.
$\Rightarrow Y \sim \mathrm{Pois}(3\alpha)$ and $X \sim \mathrm{Bin}(Y, p_{i-1}^2)$.
$\Rightarrow X \sim \mathrm{Pois}(3\alpha p_{i-1}^2)$. $\rightsquigarrow$ exercise!

ii What does peeling mean in this setting? (3)

Peeling T_α^R bottom up

for $i = 1$ **to** R **do** // □-layers bottom to top
  for *each* □*-node* v *in layer* i **do**
    if v *has no children* **then**
      remove v and its parent ○



Survival probabilities $p_i :=$ Pr[□-node in layer i is *not* peeled]

$$p_1 = \Pr[\text{□-node has at least 1 child}]$$
$$= \Pr_{Y\sim\text{Pois}(3\alpha)}[Y > 0] = 1 - e^{-3\alpha}.$$
$$p_i = \Pr[\text{layer } i \text{ □-node } v \text{ has at least 1 } \textit{surviving} \text{ child}]$$
$$= \Pr_{X\sim\text{Pois}(3\alpha p_{i-1}^2)}[X > 0] = 1 - e^{-3\alpha p_{i-1}^2}.$$

□-survival probabilities. With $p_0 := 1$ we have

$$p_i = \begin{cases} 1 & \text{if } i = 0 \\ 1 - e^{-3\alpha p_{i-1}^2} & \text{if } i = 1, 2, \ldots \end{cases}$$

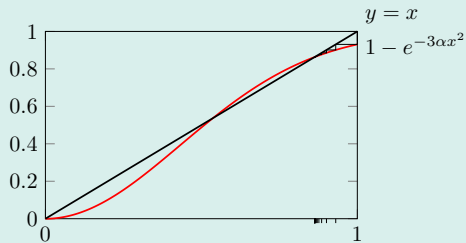
Moreover: $q_R := \Pr[\text{root survives}] = p_R^3$.

iii What role does $c_3^{\Delta} \approx 0.81$ play?

$$p_i = \begin{cases} 1 & \text{if } i = 0 \\ 1 - e^{-3\alpha p_{i-1}^2} & \text{if } i = 1, 2, \ldots \end{cases}$$

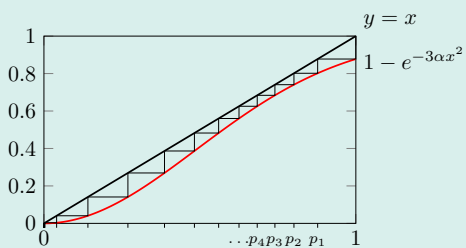
$\hookrightarrow$ consider $f(x) = 1 - e^{-3\alpha x^2}$

Case 1: $\exists x > 0 : f(x) = x$.



$$\Rightarrow \lim_{i \to \infty} p_i = x^* = \max\{x \in [0,1] \mid f(x) = x\}.$$

Case 2: $\forall x \in (0, 1] : f(x) < x$



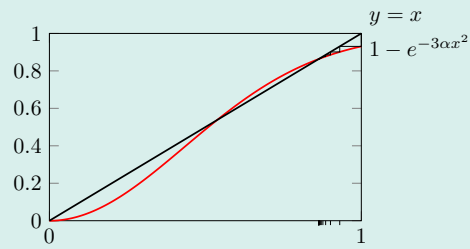
$$\Rightarrow \lim_{i \to \infty} p_i = 0.$$

iii What role does $c_3^\Delta \approx 0.81$ play?

$$p_i = \begin{cases} 1 & \text{if } i = 0 \\ 1 - e^{-3\alpha p_{i-1}^2} & \text{if } i = 1, 2, \dots \end{cases}$$

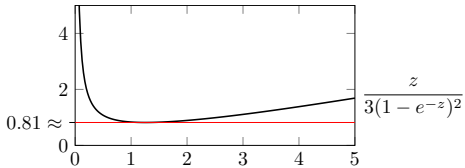
$\hookrightarrow$ consider $f(x) = 1 - e^{-3\alpha x^2}$

Case 1: $\exists x > 0 : f(x) = x$.



$$\Rightarrow \lim_{i\to\infty} p_i = x^* = \max\{x \in [0,1] \mid f(x) = x\}.$$

$$\begin{aligned}
\text{Case 1} &\Leftrightarrow \exists x > 0 : x = 1 - e^{-3\alpha x^2} \\
&\Leftrightarrow \exists x > 0 : x^2 = (1 - e^{-3\alpha x^2})^2 \\
&\Leftrightarrow \exists z > 0 : \frac{z}{3\alpha} = (1 - e^{-z})^2 \;//\; z = 3\alpha x^2 \\
&\Leftrightarrow \exists z > 0 : \alpha = \frac{z}{3(1 - e^{-z})^2} \\
&\Leftrightarrow \alpha \geq \min_{z>0} \frac{z}{3(1 - e^{-z})^2} =: c_3^\Delta \approx 0.81
\end{aligned}$$



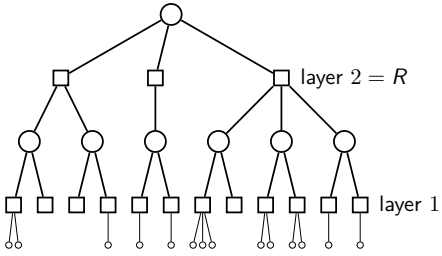
iii Interim Conclusion: What we learned about peeling T_α

Lemma

For $\alpha < c_3^\Delta \approx 0.81$ we have

- $\lim_{i \to \infty} p_i = 0$.
- $\lim_{R \to \infty} q_R = \lim_{R \to \infty} p_R^3 = 0$.

"Root rarely survives for large R."



iv What does it mean for T_α to be locally like $G_{m,\alpha m}$?

Neighbourhoods in T_α and G

Let $R \in \mathbb{N}$. We consider

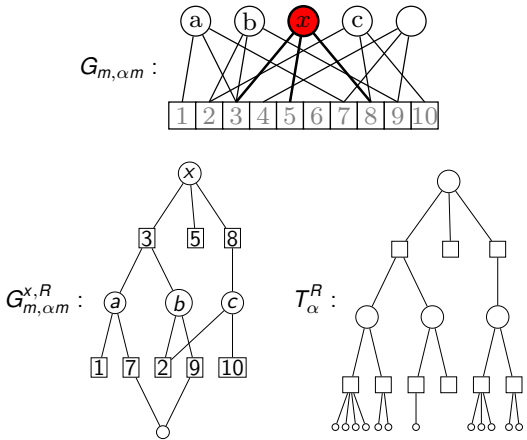
- T_α^R as before and
- for any fixed $x \in S$ the subgraph $G_{m,\alpha m}^{x,R}$ of $G_{m,\alpha m}$ induced by all nodes with distance at most $2R$ from x.

Lemma

For any $R \in \mathbb{N}$ and $m \to \infty$ we have

$$G_{m,\alpha m}^{x,R} \xrightarrow{d} T_\alpha^R$$

i.e. $\forall T : \lim_{m\to\infty} \Pr[G_{m,\alpha m}^{x,R} = T] = \Pr[T_\alpha^R = T]$.



iv Distribution of T_α^R

Lemma

Let T_y be a possible outcome of T_α^R given by a finite sequence $y = (y_1, \dots, y_k) \in \mathbb{N}_0^k$ specifying the number of children of $\square$-nodes in level order. Then

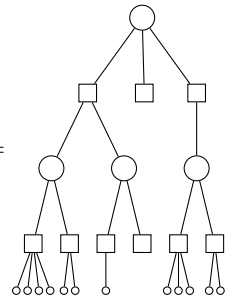
$$\Pr[T_\alpha^R = T_y] = \prod_{i=1}^{k} \Pr_{Y \sim \mathrm{Pois}(3\alpha)}[Y = y_i].$$

Remark on the meaning of "$=$"

We consider rooted unlabeled graphs where neighbours of vertices are ordered.

e.g. for $y = (2, 0, 1, 4, 2, 1, 0, 3, 2)$:

$T_y =$



iv No cycles in $G^{x,R}_{m,\alpha m}$

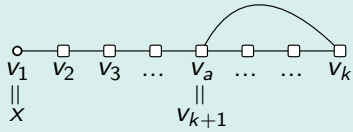
Lemma

Assume $R = \mathcal{O}(1)$. The probability that $G^{x,R}_{m,\alpha m}$ contains a cycle is $\mathcal{O}(1/m)$.

Proof.

If $G^{x,R}_{m,\alpha m}$ contains a cycle then we have

- a sequence $(v_1 = x, v_2, \ldots, v_k, v_{k+1} = v_a)$ of nodes with $a \in [k]$
- of length $k \leq 4R$ (consider BFS tree for x and additional edge in it)
- for each $i \in \{1, \ldots, k\}$ an index $j_i \in \{1, 2, 3\}$ of the hash function connecting v_i and v_{i+1}. (If $a = k - 1$ then $j_k \neq j_{k-1}$.)



$$\Pr[\exists \text{cycle in } G^{x,R}_{m,\alpha m}] \leq \Pr[\exists 2 \leq k \leq 4R : \exists v_2, \ldots, v_k : \exists a \in [k] : \exists j_1, \ldots, j_k \in [3] : \forall i \in [k] : h_{j_i} \text{ connects } v_i \text{ to } v_{i+1}]$$

$$\leq \sum_{k=2}^{4R} \sum_{v_2,\ldots,v_k} \sum_{a=1}^{k} \sum_{j_1,\ldots,j_k} \prod_{i=1}^{k} \Pr[h_{j_i} \text{ connects } v_i \text{ to } v_{i+1}] \leq \sum_{k=2}^{4R} (\max\{m, n\})^{k-1} \cdot k \cdot 3^k \left(\tfrac{1}{m}\right)^k = \tfrac{1}{m} \sum_{k=2}^{4R} k \cdot 3^k = \mathcal{O}(1/m). \quad \square$$

iv Distribution of $G^{x,R}_{m,\alpha m}$

Lemma

Let T_y be a possible outcome of T^R_α as before. Then

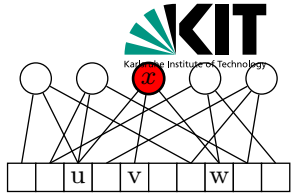
$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

"Proof by example", using T_y shown on the right.

The following things have to "go right" for $G^{x,R}_{m,\alpha m} = T_y$.

a $h_1(x), h_2(x), h_3(x)$ pairwise distinct: probability $\xrightarrow{m\to\infty}$ 1
↪ non-distinct would give cycle of length 2. Unlikely by lemma.

Note: $3\lfloor\alpha m\rfloor - 3$ remaining hash values $\sim \mathcal{U}([m])$.



iv Distribution of $G^{x,R}_{m,\alpha m}$

Lemma

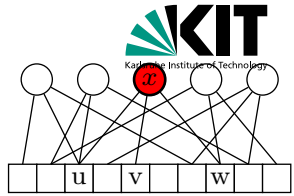
Let T_y be a possible outcome of T^R_α as before. Then

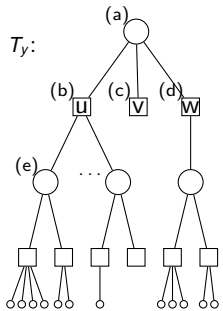
$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

"Proof by example", using T_y shown on the right.

b Exactly $y_1 = 2$ of the remaining hash values are u.
↪ $\Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor-3,\frac{1}{m})}[Y = 2] \xrightarrow{m\to\infty} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = 2]$. → exercise
Moreover: The two hash values must belong to 2 distinct keys. Probability $\xrightarrow{m\to\infty}$ 1.
↪ non-distinct would give cycle of length 2.

Note: The $3\lfloor\alpha m\rfloor - 5$ remaining hash values are $\sim \mathcal{U}([m] \setminus \{u\})$. → exercise





iv Distribution of $G^{x,R}_{m,\alpha m}$

Lemma

Let T_y be a possible outcome of T^R_α as before. Then

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \xrightarrow{m\to\infty} \prod_{i=1}^{k} \Pr_{Y\sim\text{Pois}(3\alpha)}[Y = y_i].$$

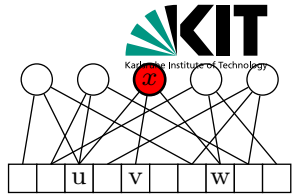
"Proof by example", using T_y shown on the right.

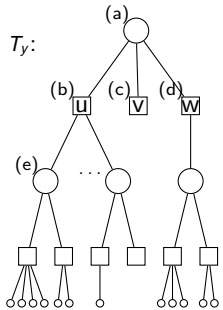
c None of the remaining hash values are v.
$\hookrightarrow \Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor-5,\frac{1}{m-1})}[Y = 0] \xrightarrow{m\to\infty} \Pr_{Y\sim\text{Pois}(3\alpha)}[Y = 0]$.

Note: The $3\lfloor\alpha m\rfloor - 5$ remaining hash values are $\sim \mathcal{U}([m] \setminus \{u, v\})$.

d One of the remaining hash values is w.
$\hookrightarrow \Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor-5,\frac{1}{m-2})}[Y = 1] \xrightarrow{m\to\infty} \Pr_{Y\sim\text{Pois}(3\alpha)}[Y = 1]$.

...





iv Distribution of $G^{x,R}_{m,\alpha m}$

Lemma

Let T_y be a possible outcome of T^R_α as before. Then

$$\Pr_{h_1,h_2,h_3\sim\mathcal{U}([m]^D)}[G^{x,R}_{m,\alpha m} = T_y] \overset{m\to\infty}{\longrightarrow} \prod_{i=1}^{k} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

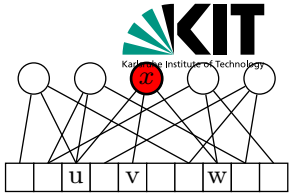
Proof sketch in general (some details ommitted)

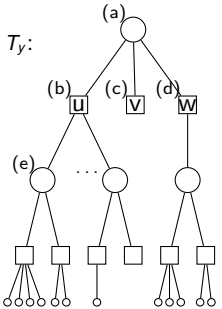
- General case at i-th □-node. Want: probability that $G^{x,R}_{m,\alpha m}$ continues to match T_y. Note: T_y is fixed, so i and the number c_i of previously revealed hash values is bounded.

$$\Pr_{Y\sim Bin(3\lfloor\alpha m\rfloor - c_i, \frac{1}{m-i+1})}[Y = y_i] \overset{m\to\infty}{\longrightarrow} \Pr_{Y\sim\mathrm{Pois}(3\alpha)}[Y = y_i].$$

  Moreover, those y_i hash values must belong to distinct fresh keys. Probability $\overset{m\to\infty}{\longrightarrow} 1$
  ↪ otherwise we'd have a cycle.

- General case for ◯-node. The two children must be fresh: probability $\overset{m\to\infty}{\longrightarrow} 1$
  ↪ otherwise there would be a cycle.





v Probability that a specific key survives peeling

Lemma

Let $\alpha < c_3^{\Delta}$. Let x be any ◯-node in $G_{m,\alpha m}$ as before (chosen before sampling the hash functions). Let

$$\mu_m := \Pr_{h_1,h_2,h_3 \sim \mathcal{U}([m]^D)}[x \text{ is removed when peeling } G_{m,\alpha m}].$$

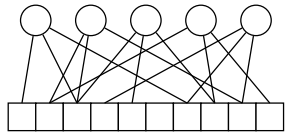
Then $\lim_{m\to\infty} \mu_m = 1$.

$\mu_m := \Pr[x \text{ is removed when peeling } G_{m,\alpha m}] \xrightarrow{m\to\infty} 1$

Let $\delta > 0$ be arbitrary. We will show $\lim_{m\to\infty} \mu_m \geq 1 - 2\delta$.
Let $R \in \mathbb{N}$ be such that $q_R < \delta$. — possible because $\lim_{R\to\infty} q_R = 0$
$\mathcal{Y}^R := \{\text{all possibilities for } T^R_\alpha\}$
$\mathcal{Y}^R_{\text{peel}} := \{T \in \mathcal{Y}^R \mid \text{peeling } T \text{ removes the root}\}$ — note: $\Pr[T^R_\alpha \notin \mathcal{Y}^R_{\text{peel}}] = q_R \leq \delta$.
Let $\mathcal{Y}^R_{\text{fin}} \subseteq \mathcal{Y}^R$ be a *finite* set such that $\Pr[T^R_\alpha \notin \mathcal{Y}^R_{\text{fin}}] \leq \delta$ — uses that $\mathcal{Y}^R$ is countable and $\sum_{T\in\mathcal{Y}^R} \Pr[T^R_\alpha = T] = 1$.

$$
\begin{aligned}
\lim_{m\to\infty} \mu_m &\geq \lim_{m\to\infty} \Pr[G^{x,R}_{m,\alpha m} \in \mathcal{Y}^R_{\text{peel}}] && \text{peeling only in } R\text{-neighbourhood of } x \text{ is “weaker”}\\
&\geq \lim_{m\to\infty} \Pr[G^{x,R}_{m,\alpha m} \in \mathcal{Y}^R_{\text{peel}} \cap \mathcal{Y}^R_{\text{fin}}]\\
&= \lim_{m\to\infty} \textstyle\sum_{T\in\mathcal{Y}^R_{\text{peel}}\cap\mathcal{Y}^R_{\text{fin}}} \Pr[G^{x,R}_{m,\alpha m} = T]\\
&= \textstyle\sum_{T\in\mathcal{Y}^R_{\text{peel}}\cap\mathcal{Y}^R_{\text{fin}}} \lim_{m\to\infty} \Pr[G^{x,R}_{m,\alpha m} = T] && \textit{finite} \text{ sums commute with limit}\\
&= \textstyle\sum_{T\in\mathcal{Y}^R_{\text{peel}}\cap\mathcal{Y}^R_{\text{fin}}} \Pr[T^R_\alpha = T] && \text{previous lemmas}\\
&= \Pr[T^R_\alpha \in \mathcal{Y}^R_{\text{peel}} \cap \mathcal{Y}^R_{\text{fin}}] = 1 - \Pr[T^R_\alpha \notin \mathcal{Y}^R_{\text{peel}} \cap \mathcal{Y}^R_{\text{fin}}]\\
&= 1 - \Pr[T^R_\alpha \notin \mathcal{Y}^R_{\text{peel}} \vee T^R_\alpha \notin \mathcal{Y}^R_{\text{fin}}] && \text{De Morgan's laws: } \overline{A\cap B} = \overline{A}\cup\overline{B}\\
&\geq 1 - \Pr[T^R_\alpha \notin \mathcal{Y}^R_{\text{peel}}] - \Pr[T^R_\alpha \notin \mathcal{Y}^R_{\text{fin}}] \geq 1 - 2\delta. && \text{union bound: } \Pr[E_1 \vee E_2] \leq \Pr[E_1] + \Pr[E_2] \quad \square
\end{aligned}
$$

vi Proof of the Peeling Theorem



Theorem

Let $\alpha < c_3^{\Delta}$. Then $\Pr[G_{m,\alpha m} \text{ is peelable}] = 1 - o(1)$.

Proof

Let $n = \lfloor \alpha m \rfloor$ and $0 \leq s \leq n$ the number of ◯ nodes surviving peeling.

last lemma:	each ◯ survives with probability $o(1)$.
linearity of expectation	$\mathbb{E}[s] = n \cdot o(1) = o(n)$.
Exercise:	$\Pr[s \in \{1, \ldots, \delta n\}] = \mathcal{O}(1/m)$ if $\delta > 0$ is a small enough constant.
Markov:	$\Pr[s > \delta n] \leq \frac{\mathbb{E}[s]}{\delta n} = \frac{o(n)}{\delta n} = o(1)$.
finally:	$\Pr[s > 0] = \Pr[s \in \{1, \ldots, \delta n\}] + \Pr[s > \delta n] = \mathcal{O}(1/m) + o(1) = o(1)$. □

Conclusion

Peeling Process

- greedy algorithm for placing keys in cuckoo table
- works up to a load factor of $c_3^\Delta \approx 0.81$

We saw glimpses of important techniques

- *Local interactions in large graphs.* Also used in statistical physics.
- *Galton-Watson Processes / Trees.* Random processes related to T_α.
- *Local weak convergence.* How the finite graph $G_{m,\alpha m}$ is locally like T_α.

But wait, there's more!

- Further applications of peeling
  - retrieval data structures (next lecture)
  - perfect hash functions (next lecture)
  - set sketches
  - linear error correcting codes

Anhang: Mögliche Prüfungsfragen I

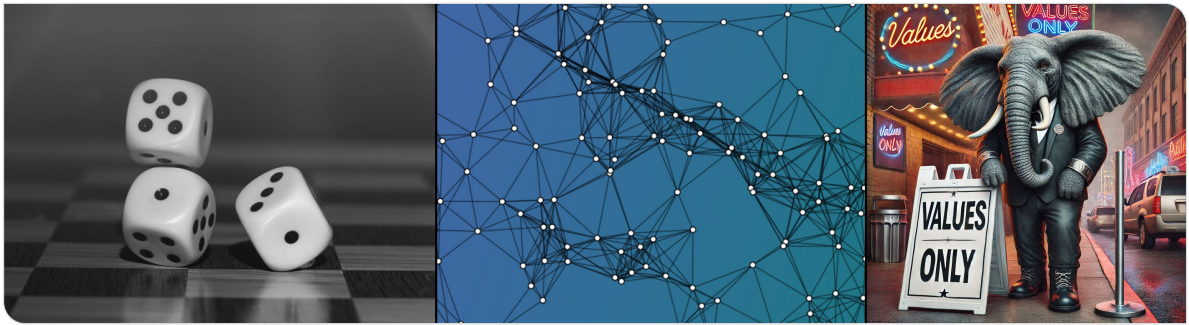
- Cuckoo Hashing und der Schälalgorithmus
  - (Wie) kann man Cuckoo Hashing mit mehr als 2 Hashfunktionen aufziehen?
  - Welcher Vorteil ergibt sich im Vergleich zu 2 Hashfunktionen?
  - Wie funktioniert der Schälalgorithmus zur Platzierung von Schlüsseln in einer Cuckoo Hashtabelle?
  - Schälen lässt sich als einfacher Prozess auf Graphen auffassen. Wie?
  - Was besagt das Hauptresultat, das wir zum Schälprozess bewiesen haben?
- Beweis des Schälsatzes. *Mir ist klar, dass der Beweis äußerst kompliziert ist.*
  - Im Beweis haben zwei Graphen eine Rolle gespielt ein endlicher und ein (potentiell) unendlicher. Wie waren diese Graphen definiert?
  - Welcher Zusammenhang besteht zwischen der Verteilung von T_α und der Verteilung von $G_{m,\alpha m}$?

Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Retrieval Data Structures

Stefan Walzer | WS 2025/2026



Results of the Course Evaluation

Summary of Results ($n = 17$)

- content somewhat difficult… (3.4 / 5)
- … but mostly clear (1.8 / 5)
- Amount of work is medium… (3.1 / 5)
- … but you learn a lot (1.6 / 5)
- learn materials are good (1.9 / 5)
- overall grade is 1.4

Full evaluation results are available on the website.

Content

The Static Retrieval Problem

The retrieval data type (for universe D, range $[k]$)

construct(f):

input:	function $f : S \to [k]$ // $f \subseteq D \times [k]$ where $S \subseteq D$ has size $n = \lvert S \rvert$
output:	data structure R.

eval$_R(x)$:

input:	$x \in D$
output:	some value in $[k]$
requirement:	**eval**$_R(x) = f(x)$ for all $x \in S$

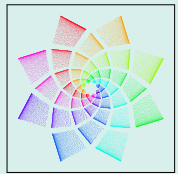
The price to pay

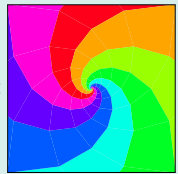
- R cannot be used to decide "is $x \in S$?"
- **eval**$_R(x)$ is *unspecified* if $x \notin S$.

Goals

- space requirement of R is $\mathcal{O}(n \log k)$ bits
  - possibly even $n \log_2(k) + o(n)$
  - ⚠ naively storing f needs $\Omega(n(\log(k) + \log(\lvert D \rvert)))$
- ideally running time of **eval**$_R$ is $\mathcal{O}(1)$
- ideally running time of **construct** is $\mathcal{O}(n)$

Intuition

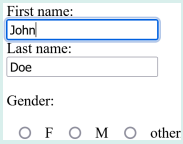
f : 

R : 

- R is a *continuation* of f
- information about the domain S is lost.

Motivation for Static Retrieval (somewhat contrived)

Task: Predict gender based on first name



- want $\geq 99\%$ accuracy
- client side only
- lightweight

Have large data base:

Annotated list of 10000 most common first names.

$$f : \{\text{Dave} \mapsto M, \text{Joanna} \mapsto F, \text{Christina} \mapsto F, \ldots\}$$

$\approx$ 10 bytes per name, too large to send to client.

Solution using retrieval

- send $R = \textbf{construct}(f)$ to client
  $\hookrightarrow \approx 1$ bit per name
- prefill gender with $\textbf{eval}_R(\text{firstName})$

Weaknesses:

May guess incorrectly if

- name is ambiguous (“Kim”, “Chris”)
- user is *other* / prefers not to say
- name not listed in f (e.g. “Crhistina”, “Inghean”)
  $\hookrightarrow$ would be better to *refrain from guessing*

Static Filters / AMQs

Exercise: Filters from Retrieval

Good retrieval data structures yield good static filters.

Content

Cuckoo-Style Retrieval using Peeling for $f : S \to \{0, \ldots, k-1\}$

Retrieval Data Structure $R = (h_1, h_2, h_3, A)$

- $A \in \{0, \ldots, k-1\}^m$ is array of cleverly chosen values
- $m = \frac{n}{0.81} = 1.23n$ //0.81 is peeling threshold $c_3^\triangle$
- $h_1, h_2, h_3 \sim \mathcal{U}([m]^D)$ //SUHA
- $\mathbf{eval}_R(x) := (A[h_1(x)] + A[h_2(x)] + A[h_3(x)]) \bmod k$

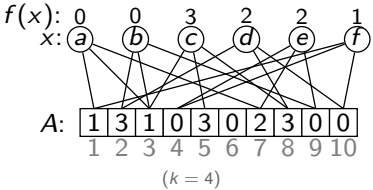
Performance

- space $1.23n\lceil \log_2(k) \rceil$ bits
- eval in $\mathcal{O}(1)$
- construct in $\mathcal{O}(n)$

How does **construct**(f) choose A?

If $A[j]$ is only used by x_i then setting $A[j]$ *in the end* takes care of x_i without affecting other keys.

- ↪ can forget about x_i "for now" and focus on the rest
- ↪ if configuration is peelable, this takes care of all keys



Equations (mod k for $k = 4$)

c: $A[5] := 3 - A[3] - A[8]$
d: $A[8] := 2 - A[2] - A[10]$
b: $A[2] := 0 - A[3] - A[9]$
a: $A[3] := 0 - A[1] - A[7]$
e: $A[7] := 2 - A[4] - A[9]$
f: $A[1] := 1 - A[4] - A[10]$

Cuckoo-Style Retrieval using Linear Algebra over the field $\mathbb{F}_2 = \{0, 1\}$

Cuckoo-style retrieval for $f : S \to \mathbb{F}_2^r$ with $|S| = n$

Pick $m \geq n$. Data structure is pair $R = (h : D \to \mathbb{F}_2^m, \vec{z} \in \mathbb{F}_2^{m \times r})$ such that $h(x)^T \cdot \vec{z} = f(x)$ for all $x \in S$.

$r = 3$
$m = 7$
$n = 5$

$$\begin{pmatrix} \text{———} h(x_1) \text{———} \\ \text{———} h(x_2) \text{———} \\ \text{———} h(x_3) \text{———} \\ \text{———} h(x_4) \text{———} \\ \text{———} h(x_5) \text{———} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \\ f(x_5) \end{pmatrix}$$

Goals when Choosing h

- i **success probability**: rows of matrix $(h(x))_{x \in S}$ must be linearly independent
- ii **construction time**: linear system should be easy to solve
- iii **query time**: products $h(x) \cdot z$ should be fast to compute
- iv **space**: $\alpha = \frac{n}{m}$ should be close to 1

What the peeling-based approach achieves

- i $1 - \mathcal{O}(1/m)$ (success iff peelable)
- ii $\mathcal{O}(n)$ (time to run peeling)
- iii $\mathcal{O}(1)$ (three memory accesses two $\oplus$-additions)
- iv $\alpha = 0.81$ (peeling threshold)

What more could we hope for?

- i -
- ii better cache efficiency
- iii better cache efficiency
- iv $\alpha = 1 - o(1)$

Teaser: “Ribbon Retrieval”

Standard Ribbon Retrieval

Idea: $\boldsymbol{h}(\boldsymbol{x})$ = “randomly placed block of $\mathcal{O}(\frac{\log n}{1-\alpha})$ random bits”

$$\begin{pmatrix} & & 0 & 1 & 1 & 0 & 0 & 1 & & & & & & \\ 1 & 0 & 1 & 0 & 0 & 1 & & & & & & & & \\ & & & & & 0 & 1 & 0 & 1 & 1 & 0 & & & \\ & & & & 1 & 0 & 0 & 0 & 1 & 0 & & & & \\ & 0 & 1 & 1 & 0 & 1 & 1 & & & & & & & \\ & & & & & & & & & 0 & 1 & 0 & 1 & 0 & 1 \\ & & 0 & 1 & 0 & 1 & 1 & 1 & & & & & & \\ & & & & & & 1 & 1 & 1 & 0 & 0 & 0 & & \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \\ z_{10} \\ z_{11} \\ z_{12} \\ z_{13} \\ z_{14} \\ z_{15} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 010 \\ 011 \\ 101 \\ 111 \\ 101 \\ 000 \\ 010 \\ 100 \end{pmatrix}$$

$\underbrace{\qquad\qquad}_{\frac{\log n}{1-\alpha}}$

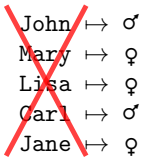
Bumped Ribbon Retrieval

Further tricks allow reducing the width of the blocks to $\log n$. End result after a lot of effort[a]:

- construction $\mathcal{O}(n \log n)$
  // cache efficient
- query $\mathcal{O}(r)$ // cache efficient
- bits per key: $r + \mathcal{O}(\frac{\log \log n}{\log^2 n})$.

[a] *Ribbon: Fast Succinct Static Retrieval and Approximate Membership*, Martin Dietzfelbinger, Peter C. Dillinger, Lorenz Hübschle-Schneider, Peter Sanders, Stefan Walzer, 2026.

Summary



Static Retrieval

- constructed for $f : S \to [k]$
- goal: $\approx \log_2(k)$ bits per key
- does not store S

$$\textbf{eval}_R(x) = \begin{cases} f(x) & \text{if } x \in S \\ \textit{unspecified} & \text{if } x \notin S \end{cases}$$

- insert & delete not supported

Dictionary / Hash Table

- constructed for $f : S \to [k]$ // $S \subseteq D$
- goal: $\approx \log_2(D) + \log_2(k)$ bits per key
- stores S

$$\textbf{lookup}_R(x) = \begin{cases} f(x) & \text{if } x \in S \\ \perp & \text{if } x \notin S \end{cases}$$

- insert & delete usually supported

Constructions discussed here

- cuckoo-style retrieval using peeling
- cuckoo-style retrieval using linear algebra with $\mathbb{F}_2$
- approaches using perfect hashing (next lecture)

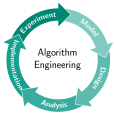
Remark: There is more...

- compressed retrieval data structures
- learned retrieval data structures
- active research @ITI Sanders!

Appendix: Possible Exam Questions

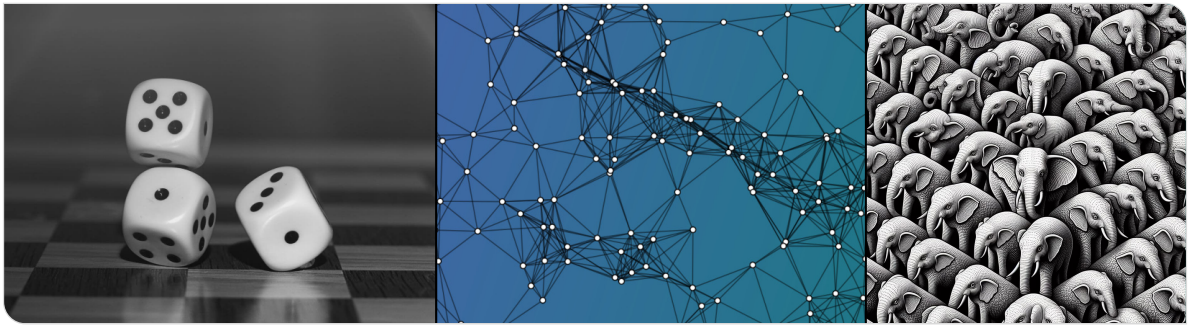
- What functionality does a retrieval data structure provide?
- What are the advantages and disadvantages compared to a standard hash table?
- What are applications of retrieval data structures?
- Regarding retrieval data structures using the fingerprint approach:
  - How is the data structure constructed? How does the query algorithm work?
  - What is “bumping” and why do we need it?
  - What are construction and access times, and what is the memory usage?
- Regarding retrieval data structures based on the peeling algorithm:
  - How is the data structure constructed? How does the query algorithm work?
  - What are construction and access times, and what is the memory usage?
- Regarding retrieval data structures based on linear algebra over the field $\mathbb{F}_2$:
  - What is the general framework? In what sense does the peeling-algorithm approach also fit into this framework?
  - What goals should one keep in mind when choosing the function h?
- Ribbon retrieval is not exam-relevant.

Karlsruhe Institute of Technology



Probability and Computing – Perfect Hashing

Stefan Walzer | WS 2025/2026



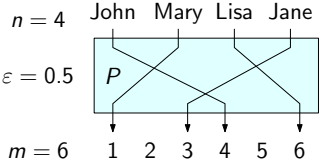
Content

Perfect Hash Function (PHF) and Minimal Perfect Hash Function (MPHF)

The PHF data type, for universe D, $\varepsilon \geq 0$

construct(S):
- input: $S \subseteq D$ of size $n = |S|$
- output: data structure P.

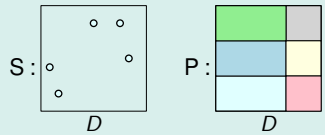
eval$_P(x)$:
- input: $x \in D$
- output: a number in $[m]$ where $m = (1+\varepsilon)n$
- requirement: $x \mapsto$ **eval**$_P(x)$ is injective on S



Goals

- ε small // $\varepsilon = 0$: *Minimal* perfect hash function (MPHF)
- space requirement of P is $\mathcal{O}(n)$ bits
  - $\approx 1.44n$ bits is necessary and sufficient for $\varepsilon = 0$
  - note: storing S might need $\Omega(n \log(|D|))$ bits.
- ideally: running time of **eval** is $\mathcal{O}(1)$
- ideally: running time of **construct** is $\mathcal{O}(n)$

Intuition



- P is partition of D that separates S
- details about S are lost.
- note: P is "perfect hash function" but need not be random

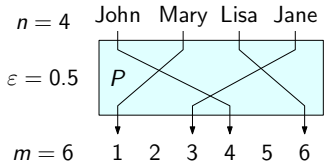
Motivation for (Minimal-) Perfect Hashing

Short IDs

Replace keys with short unique identifies

$\mathbf{eval}_P(\texttt{“creativeUserName@example.org”}) = 10241.$

$\mathbf{eval}_P(\texttt{“ACGGGTCAGTA”}) = 563.$ // k-mers in bioinformatics



Updatable retrieval: A hash table without keys

- assume we have MPHF P for S
- can store additional data $f(x) \in [k]$ on $x \in S$ in array of length m in position $\mathbf{eval}_P(x)$.
  $\hookrightarrow$ array takes $m\lceil \log_2(k) \rceil$ bits

⚠ Weaker than a normal hash table:

- S is static (values updateable)
- trying to access $f(x)$ for $x \notin S$ gives undefined result
- trying to update $f(x)$ for $x \notin S$ destroys information

Content

Space Lower Bound & Brute Force Construction

Exercise: $\approx n \log_2 e$ bits are sufficient

For any $S \subseteq D$ of size n we have $\Pr_{h \sim \mathcal{U}([n]^D)}[h \text{ is injective on } S] = \frac{n!}{n^n}$.

$\hookrightarrow$ success after trying $\approx \frac{n^n}{n!}$ random hash functions

$\hookrightarrow$ requires storing seed of $\log_2 \left(\frac{n^n}{n!}\right) \approx \log_2(e^n) \approx 1.44n$ bits.

Exercise: $\approx n \log_2 e$ bits are necessary

- Show that $\approx \frac{n^n}{n!}$ distinct MPHF data structures are needed to handle all possible inputs.
- Conclude that $\approx \log_2 \frac{n^n}{n!} \approx n \log_2 e$ bits are needed for worst-case inputs.

Plan for Today

- lots of ideas and intuition
- few details

Content

Content

Elias-Fano from the “Advanced Data Structure” Lecture

Theorem: Elias-Fano Encoding

An Elias-Fano data structure encodes a sequence $0 \leq a_1 \leq \ldots \leq a_n \leq u$ with

- access time $\mathcal{O}(1)$ // input $i \in [n]$, output $a_i \in [u]$
- space $n(2 + \lceil \log_2 \frac{u}{n} \rceil)$ // Intuition: mean *distance* between a_i and a_{i+1} is $\frac{u}{n}$; hence $\approx \log \frac{u}{n}$ bits per element

Idea (ignoring divisibility issues)

- Partition $[u]$ into n buckets of size $\frac{u}{n}$.
- Unary encode bucket sizes $\rightsquigarrow 2n$ bits
- keep lower $\log_2 \frac{u}{n}$ bits of entries in array
- fast access using select data structure

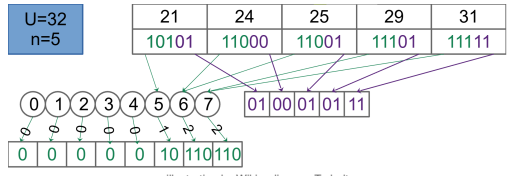


illustration by Wikipedia user Trobolt
https://commons.wikimedia.org/wiki/File:Elias-fano_1.svg

Content

“Repairing” a Non-Minimal PHF to make it Minimal

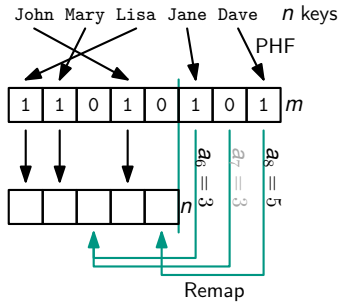
Idea

Remap keys in the $m - n$ extra slots into the gaps left within the first n slots.

Additional data structure

- Let $1 \leq a_{n+1} \leq \ldots \leq a_m \leq n$ be such that a used slot $i \in \{n+1, \ldots, m\}$ is assigned an empty slot $a_i \in [n]$, without collisions.
- Store $a_{n+1} \leq \ldots \leq a_m$ using Elias-Fano

$$\text{space} = (m-n)(2 + \lceil \log_2 \tfrac{n}{m-n} \rceil) = \varepsilon n \cdot (2 + \lceil \log_2 \tfrac{1}{\varepsilon} \rceil)$$



Content

From Exponential to Linear Time using Partioning

Lemma

Assume an MPHF has

- construction time $\mathcal{O}(e^{c_1 n})$
- space $c_2 n$ bits // optimal if $c_2 = \log_2 e$

Then for $\lambda > 0$ there exists MPHF with

- expected construction time $\mathcal{O}(n \frac{e^{\lambda(e^{c_1}-1)}}{\lambda})$ // "$\mathcal{O}(n)$"
- space $c_2 n + \mathcal{O}(n \frac{\log \lambda}{\lambda})$ // "$(c_2 + \varepsilon)n$" for any small ε

Idea: Partitioned MPHF

- Partition input set $S \subseteq D$ into $S_1, \ldots, S_k$ using hash function $h \sim \mathcal{U}([k]^D)$
- overall MPHF given by
  - MPHFs P_i on S_i for $i \in [k]$
  - prefix sums $\sigma_i = |S_1| + \cdots + |S_{i-1}|$ for $i \in [k]$
  $\hookrightarrow \mathbf{eval}_P(x) = \mathbf{eval}_{P_{h(x)}}(x) + \sigma_{h(x)}$

Analysis sketch for $k = \frac{n}{\lambda}$

- space for storing $\sigma_1, \ldots, \sigma_k$:
  - $\frac{n}{\lambda}(2 + \lceil \log \lambda \rceil)$ with Elias-Fano
- expected construction time for part 1:
  - $|S_1| \sim \mathrm{Bin}(n, \frac{\lambda}{n}) \approx \mathrm{Pois}(\lambda)$
  - $\mathbb{E}[e^{c_1 |S_1|}] \approx \sum_{i \geq 0} e^{-\lambda} \frac{\lambda^i}{i!} \cdot e^{c_1 i} =$
  $e^{-\lambda} \sum_{i \geq 0} \frac{(\lambda e^{c_1})^i}{i!} = e^{-\lambda} e^{\lambda e^{c_1}} = e^{\lambda(e^{c_1}-1)}.$

Remark: Perfect Partitioning

Variant: k-perfect hash function (k-PHF)

- similar to perfect hash function
- up to k keys mapped to the same value
- *minimal*: $m = \lceil \frac{n}{k} \rceil$

Useful in practice but not yet well understood. Active research @ITI Sanders.

Content

MPHF via Recursive Splitting

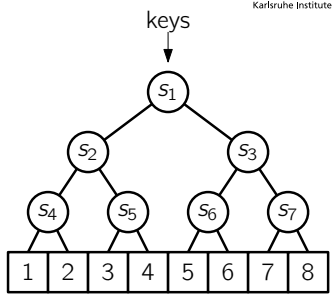
Idea (for $n = 2^d$)

Nodes in binary tree store seeds $s_1, \ldots, s_{n-1}$ of a hash function that split incoming keys perfectly in half.

Expected construction time $\mathcal{O}(n^{3/2})$

- splitting n keys in half:
  - work $\mathcal{O}(n)$ for trying a seed
  - success probability $\binom{n}{n/2} 2^{-n} = \Theta(1/\sqrt{n})$

  $\hookrightarrow$ expected work $\mathcal{O}(n^{3/2})$
- work for entire tree:
  $\mathcal{O}(n^{3/2} + 2 \cdot (\frac{n}{2})^{3/2} + 4 \cdot (\frac{n}{4})^{3/2} + \ldots) = \mathcal{O}(n^{3/2})$.



Issues (not addressed here)

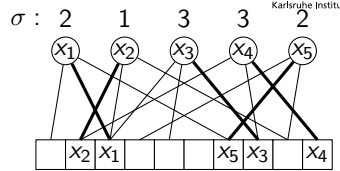
- query time $\Omega(\log n)$
- how to encode seeds compactly?
- what if n is not a power of 2?

Content

PHF via Cuckoo Hashing + Retrieval

Cuckoo Hashing (abstract reminder)

Let $S \subseteq D$ of size $n = |S|$ and $h_1, \ldots, h_k \sim \mathcal{U}([m]^D)$ where $\frac{n}{m} < c_k^*$ for some *threshold* c_k^*.
With high probability there exists $\sigma(x) \in [k]$ for each $x \in S$ such that $x \mapsto h_{\sigma(x)}(x)$ is injective on S.



(picture assumes $h_1(x) < h_2(x) < h_3(x)$)

Perfect Hash Function from Retrieval

- Store $\sigma : S \to [k]$ as retrieval data structure R
- (non-minimal) PHF $P = (R, h_1, \ldots, h_k)$ with

$$\mathbf{eval}_P(x) := h_{\mathbf{eval}_R(x)}(x).$$

Example with $k = 3$

- use $\frac{n}{m} < c_3^{\Delta} \approx 0.81 \rightsquigarrow \varepsilon \approx 0.23$
  $\hookrightarrow$ placement found using peeling
- space needed for P is the space for R:
  $\approx 1.23n\lceil \log_2(k) \rceil = 2.26n$ bits
  with peeling-based retrieval from last time
- can use the same hashes and peeling twice!

Small Heavily Overloaded Cuckoo Table (ShockHash)

Cuckoo Hashing + Retrieval with $k = 2$ and $n = m$

- only $\approx n$ bits in space optimal retrieval data structure
- success probability $\approx (2/e)^n$ // not obvious
  $\hookrightarrow$ need seed of expected size $\approx \log((e/2)^n)$
- total space $\approx n \log_2(e/2) + n = n \log_2(e) = \text{OPT}$.

Running Time

- $n(e/2)^n = \mathcal{O}(1.36^n)$
- can be improved to $\mathcal{O}(1.17^n)$

recall: naive brute force is $\mathcal{O}(e^n) = \mathcal{O}(2.72^n)$

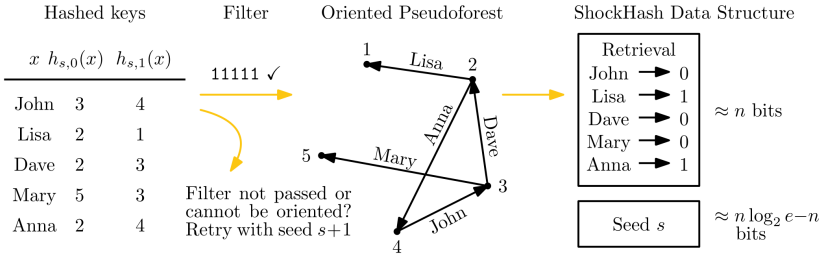
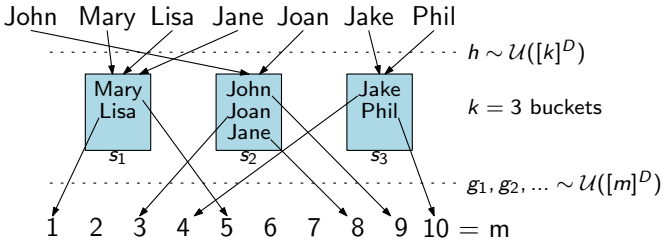


illustration by Hans-Peter Lehmann. "Filter" checks if all n positions are hit at least once to more quickly reject seeds

Content

(M)PHF via Bucket Placement



Perfect Hash Function $P = (k, h, (g_i)_{i \in \mathbb{N}}, (s_1, \ldots, s_k))$

- $\textbf{eval}_P(x) := g_{s_{h(x)}}(x)$
- $s_1, \ldots, s_k$ are found one by one using trial and error
- *huge design space*

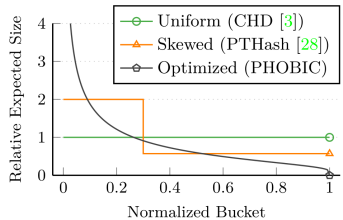
Observation 1

Placing buckets in order of decreasing size reduces construction time.

// in the example: choose s_2 first.

Observation 2

Deliberately non-uniform partitioning reduces construction time.

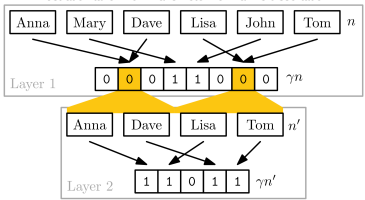


Expected bucket size functions from "PHOBIC: Perfect Hashing with Optimized Bucket Sizes and Interleaved Coding", Hermann et al.

Content

PHF via Fingerprinting and Bumping

Illustration taken from Hans-Peter Lehmann's dissertation.



$P(x)$	1	2	3	4	5	6
x	Mary	John	Anna	Lisa	Dave	Tom

- use fingerprint function $f : D \to [\gamma n]$
- bitvector $B[1..\gamma n]$ indicates unique fingerprints
- keys with colliding fingerprints are *bumped* to next layer
- rank data structure maps fingerprint to its rank in layer

Lemma

i The expected space requirement of this approach is $\gamma e^{1/\gamma} + o(1)$ bits per key.

ii Space is minimised for $\gamma = 1$ with $\approx e$ bits per key.

Proof sketch of (i).

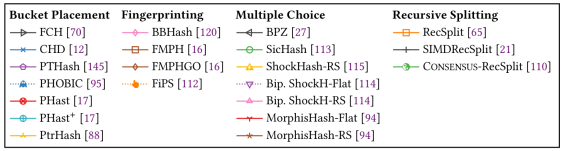
A key has a unique fingerprint in the first layer with probability

$$\Pr_{X \sim \mathrm{Bin}(n-1, \frac{1}{\gamma n})}[X = 0] \approx \Pr_{X \sim \mathrm{Pois}(1/\gamma)}[X = 0] = e^{-1/\gamma}.$$

In expectation $n \cdot e^{-1/\gamma}$ keys handled with γn bits $\rightsquigarrow \gamma e^{1/\gamma}$ bits per key.

Content

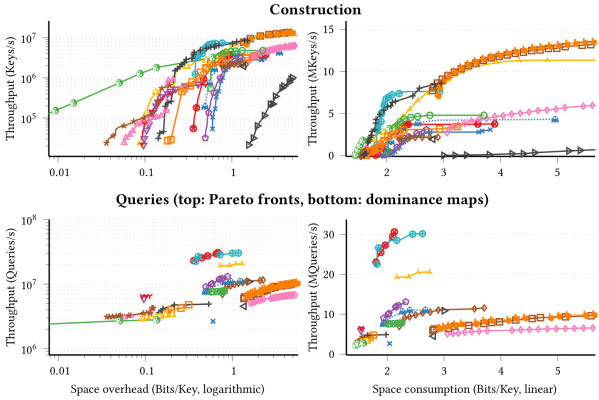
What's Best in Practice?



- See "Modern Minimal Perfect Hashing: A Survey" by Hans-Peter Lehmann et al.
- Active research @ITI Sanders.

Oversimplified takeaways

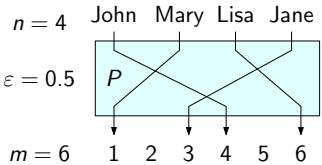
- least space: recursive splitting
- fastest queries: bucket placement
- fastest construction: fingerprinting



Summary

Perfect Hash Function (PHF) for $S \subseteq D$

- maps D to range $[m]$
- "perfect" on S, i.e. no collisions on S
- *minimal* perfect (MPHF) if $m = n := |S|$
  - $n \log_2 e$ bits necessary and sufficient for MPHF



Motivation

- map long keys to short unique IDs
- updatable retrieval // "hash table without keys"

Techniques

- Partitioning
- Recursive Splitting
- Fingerprinting & Bumping
- Bucket Placement
- Cuckoo + Retrieval

Appendix: Possible Exam Questions I

- What is a (minimum) perfect hash function?
  - What makes for a *good* (M)PHF?
- In what sense does a PHF realise a “hash table without keys”?
- How much space does an MPHF require at least?
  - How can $\approx n \log_2 e$ bits achieved?
  - What is the idea for the proof that $\approx n \log_2 e$ bits are necessary?
- Techniques for constructing (M)PHFs:
  - Why can MPHF constructions with exponential running time still be useful?
  - What is the idea of recursive splitting? What running time was achieved?
  - How did we use cuckoo hashing and retrieval to construct a PHF?
    - Offer a concrete construction. What ε and what space requirement do you get?
  - How does bucket placement work?
    - What refinement regarding partitioning into buckets did we hint at?
  - How does fingerprinting with bumping work?
    - What is the bumping probability?
    - What is the space requirement?

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