Karlsruhe Institute of Technology

Probability & Computing

Probability Amplification

Probability Amplification

Definition: A **Monte Carlo Algorithm** is a randomized algorithm with bounded running time that, for each input, answers correctly with probability at least $p \in (0,1)$.

- In decision problems p is the probability of giving the correct answer
  - **One-sided error**: either *false-biased* or *true-biased*
  - **Two-sided error**: *no bias*
- In optimization problems p is the probability of finding the optimum

Algo Output	Correct Answer ✗	Correct Answer ✓
✗	true neg	false neg
✓	false pos	true pos

Definition: **Probability amplification** is the process of increasing the success probability of a Monte Carlo algorithm by using multiple runs.

Probability Amplification for true-biased algorithms

Exercise: For two-sided error.

- Execute independently t times.
  - If ✓ at least once: Return ✓. (surely correct)
  - Otherwise: Return ✗. $\Pr[\text{"correct"}] \geq 1-(1-p)^t \geq 1-e^{-pt}$

$1+x \leq e^x$ for $x \in \mathbb{R}$

Probability Amplification

Definition: A **Monte Carlo Algorithm** is a randomized algorithm with bounded running time that, for each input, answers correctly with probability at least $p \in (0, 1)$.

- In decision problems p is the probability of giving the correct answer
  - **One-sided error**: either *false-biased* or *true-biased*
  - **Two-sided error**: *no bias*
- In optimization problems p is the probability of finding the optimum

Algo Output \ Correct Answer	✗	✓
✗	true neg	false neg
✓	false pos	true pos

Definition: **Probability amplification** is the process of increasing the success probability of a Monte Carlo algorithm by using multiple runs.

Probability Amplification for optimization algorithms

- Execute independently t times.
  - output best result

$$\Pr[\text{“optimal”}] \geq 1 - (1-p)^t \geq 1 - e^{-pt}$$

The Clustering Problem

Input

- Set P of points in a feature space (e.g., $\mathbb{R}^d$)
- Similarity measure $\sigma\colon P \times P \mapsto \mathbb{R}_+$

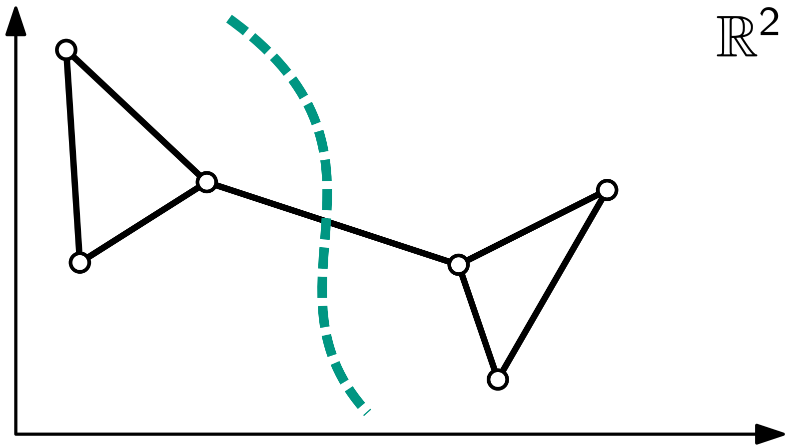
Output: $P_1, \dots, P_k$ such that

- Points within a P_i have high similarity
- Points in distinct P_i, P_j have low similarity

Applications: Compression, medical diagnosis, etc.

Approach: Model as graph

- Each point is a node
- Edges between all node pairs, with the weight given by the similarity of the two nodes
- Find *cut-set* (edges to remove) of minimal weight such that the graph decomposes into k components.



Example

- six points in $\mathbb{R}^2$
- σ is the inversed Euclidean distance
- partition into two sets

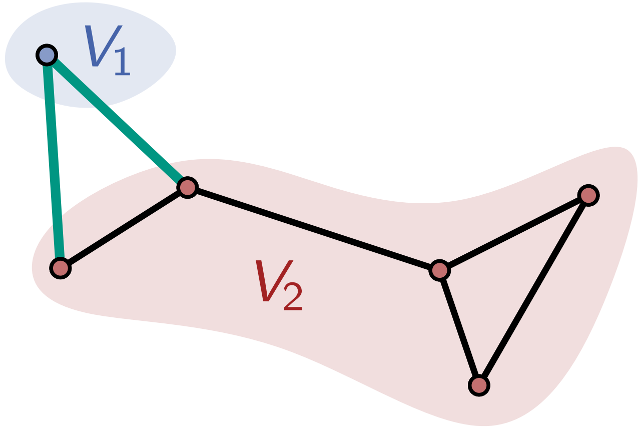
Today

$k = 2$ and $\sigma\colon P \times P \mapsto \{0, 1\}$

Computing Min Cuts

Cuts

- $G = (V, E)$ an unweighted, undirected, connected graph
- *Cut*: partition of V into non-empty parts V_1, V_2.
- *Cut-set*: set of edges with endpoints in V_1 and V_2
- *Weight of a cut*: size of the cut-set (or sum of weights in a weighted graph)



Today Goal: Compute a Min-Cut

i.e. a cut of minimum weight or cut-set of minimum size
the weight of the min-cut is known as the edge-connectivity of G

- Known deterministic strategies have worst case running time $\Omega(n^3)$.
- We'll see randomised algorithm with running time $O(n^2 \cdot \log^3(n))$.

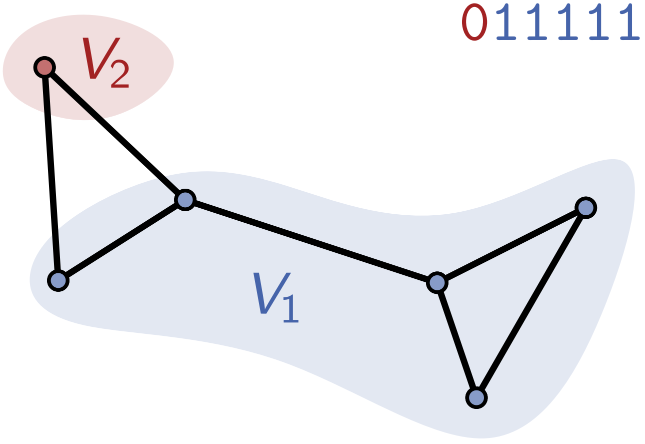
A Trivial Algorithm: Random Cut

Observation: There are $2^{n-1} - 1$ cuts in a graph with n nodes.

$(2^n - 2)/2$

- Number of possible assignments of n nodes to 2 parts
- Partitions with empty parts that do not represent cuts
- Swapping parts does not yield a new partition

011111



Algorithm: Random Cut

- Return a uniformly random cut.
- Minor challenge: How to uniformly sample cuts?
  - Represent cut using bit-string
  - Have to uniformly sample bit-string *while avoiding* `11...1` and `00...0`?
    - intution: sample from $\mathcal{U}(\{0,1\}^n)$ and use rejection sampling
    - actually for bounded running time: declare failure rather than sampling again
    - samples each cut with probability $1/2^{n-1}$

Random Cut: Analysis

Running time: $O(n)$ much better than the $\Omega(n^3)$ in the deterministic setting , but...

Success probability: $\geq 1/2^{n-1}$ "$=$" if there is only one min-cut.

$\rightarrow$ exponentially small!

Amplification

- Repeat the algorithm to obtain t independent random cuts, return the smallest

$$\Pr[\text{"min cut found"}] \geq 1 - \left(1 - 1/2^{n-1}\right)^t \geq 1 - e^{-t/2^{n-1}}$$

$\boxed{1 + x \leq e^x \text{ for } x \in \mathbb{R}}$

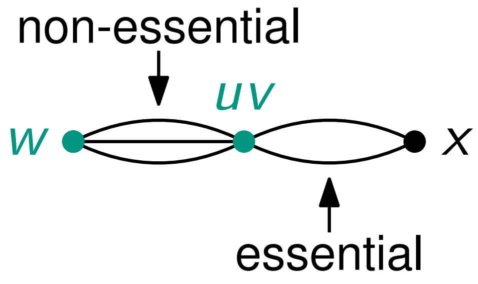
- For $t = 2^{n-1}$ min cut found with constant probability $1 - 1/e \approx 0.63$
- For $t = 2^{n-1} \cdot \ln(n)$ min cut found with high probability $1 - 1/n$

this is terrible so far...

Karger’s Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops
- A (multi) graph with two nodes has a unique cut-set



Contraction Algorithm

not part of a min-cut

- Motivation: distinguish *non-essential* as well as *essential* edges (part of a min-cut) & hope there are few essential ones

Karger$(G_0 = (V_0, E_0))$

for $i = 1$ to $n-2$ **do** $\quad // \ O(n)$
  sample $e \sim \mathcal{U}(E_{i-1})$ $\quad // \ O(1)$
  $G_i \leftarrow G_{i-1}$.**contract**(e) $\quad // \ O(n)$
return unique cut-set in G_{n-2}

- Running time in $O(n^2)$
- Can be implemented to run in $O(m)$

Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i =$ “C in G_i”

Observation: min-degree $\geq k$

(holds for all G_i due to 1st observation)

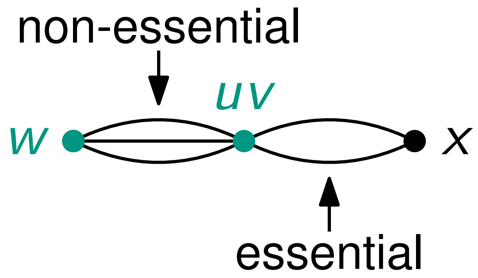
$$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m} \geq 1 - \frac{k}{nk/2} = 1 - \frac{2}{n}$$

$$m = \frac{1}{2} \sum_{v \in V} \deg(v) \geq \frac{1}{2} \sum_{v \in V} k \geq \frac{1}{2} nk$$

Karger's Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops
- A (multi) graph with two nodes has a unique cut-set



Contraction Algorithm

- Motivation: distinguish $\overbrace{\textit{non-essential}}^{\text{not part of a min-cut}}$ as well as *essential* edges $\}$ part of a min-cut
  & hope there are few essential ones

Karger$(G_0 = (V_0, E_0))$
- **for** $i = 1$ to $n - 2$ **do** // $O(n)$
  - sample $e \sim \mathcal{U}(E_{i-1})$ // $O(1)$
  - $G_i \leftarrow G_{i-1}$.**contract**(e) // $O(n)$
- **return** unique cut-set in G_{n-2}

- Running time in $O(n^2)$
- Can be implemented to run in $O(m)$

Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i$ = "C in G_i"

Observation: min-degree $\geq k$

$\Pr[\mathcal{E}_1] \geq 1 - \frac{2}{n}$ (holds for all G_i due to 1st observation)

$\Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \geq 1 - \frac{2}{n-1} \longrightarrow \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}] \geq 1 - \frac{2}{n-i+1}$

$$\Pr[\mathcal{E}_{n-2}] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_{n-2} \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{n-3}]$$

$$\geq \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\cdots\left(\frac{2}{\cancel{4}}\right)\left(\frac{1}{\cancel{3}}\right)$$

$$= \frac{2}{n\,(n-1)} \geq \frac{2}{n^2}$$

Karger's Algorithm Amplified

Theorem: On a graph with n nodes, Karger's algorithm runs in $O(n^2)$ time and returns a minimum cut with probability at least $\frac{2}{n^2}$.

$$\Pr[\text{"min-cut found"}] \geq 1 - \exp(-\tfrac{2}{n^2} \cdot t) = 1 - \frac{1}{n}$$

for $t = \frac{n^2}{2} \ln(n)$

Success probability $\geq p$
Number of repetitions t
Amplified prob. $\geq 1 - e^{-pt}$

Corollary: On a graph with n nodes, $O(n^2 \log(n))$ Karger repetitions run in $O(n^4 \log(n))$ total time and return a min-cut with high probability. Much better than exp. time of Randomized Cut!

Sidenote: Number of minimum cuts

- Let $C_1, \ldots, C_\ell$ be all the min-cuts in G and $\underbrace{\mathcal{E}^i_{n-2}}_{\text{disjoint, since the algorithm returns only one cut}}$ for $i \in [\ell]$ be the event that C_i is returned by Karger's algorithm
- Just seen: $\Pr[\mathcal{E}^i_{n-2}] \geq \frac{2}{n^2}$

$$1 \geq \Pr\left[\bigcup_{i\in[\ell]} \mathcal{E}^i_{n-2}\right] = \sum_{i\in[\ell]} \Pr[\mathcal{E}^i_{n-2}] \geq \frac{2\cdot\ell}{n^2}$$

Observation: $\ell \leq \frac{n^2}{2}$.

More Amplification: Karger-Stein

Motivation

- Probability that a min-cut survives i contractions

$$\Pr[\mathcal{E}_i] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}]$$
$$\geq \left(1-\frac{2}{n}\right)\left(1-\frac{2}{n-1}\right)\left(1-\frac{2}{n-2}\right)\cdots\left(1-\frac{2}{n-i+2}\right)\left(1-\frac{2}{n-i+1}\right)$$
$$= \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\cdots\left(\frac{n-i}{\cancel{n-i+2}}\right)\left(\frac{n-i-1}{\cancel{n-i+1}}\right)$$
$$= \frac{(n-i)(n-i-1)}{n(n-1)} \geq \frac{(n-i-1)(n-i-1)}{n \cdot n} = \left(1-\frac{i+1}{n}\right)^2.$$

- Probability becomes very small only towards the very end.

KargerStein$(G_0 = (V_0, E_0))$
- **if** $|V_0| = 2$ **then** return unique cut-set
- **for** $i = 1$ to $s = |V_0| - \frac{|V_0|}{\sqrt{2}} - 1$ **do**
  - sample $e \sim \mathcal{U}(E_{i-1})$
  - $G_i \leftarrow G_{i-1}$.**contract**(e)
- $C_1 \leftarrow$ **KargerStein**(G_s) // inde-
- $C_2 \leftarrow$ **KargerStein**(G_s) // pendent // runs
- **return** smaller of C_1, C_2

- Idea: stop when a min-cut is still likely to exist and recurse
- After $s = n - n/\sqrt{2} - 1$ steps we have

$$\Pr[\mathcal{E}_s] \geq \left(1 - \frac{n - n/\sqrt{2}}{n}\right) = \left(1-(1-1/\sqrt{2})\right)^2 = (1/\sqrt{2})^2 = \frac{1}{2}$$

Karger-Stein: Running Time

Recursion

- After $t = n - n/\sqrt{2} - 1$ steps the number of nodes is $n/\sqrt{2} + 1$

$$T(n) = 2T\left(\frac{n}{\sqrt{2}} + 1\right) + O(n^2)$$

Solution (essentially by Master Theorem)

$$T(n) = O(n^2 \log n)$$

KargerStein$(G_0 = (V_0, E_0))$

- // $O(1)$ — **if** $|V_0| = 2$ **then** return unique cut-set
- // $O(n)$ — **for** $i = 1$ to $s = |V_0| - \frac{|V_0|}{\sqrt{2}} - 1$ **do**
  - // $O(1)$ — sample $e \sim \mathcal{U}(E_{i-1})$
  - // $O(n)$ — $G_i \leftarrow G_{i-1}.$**contract**(e)
- $C_1 \leftarrow$ **KargerStein**(G_s) // inde-
- $C_2 \leftarrow$ **KargerStein**(G_s) // pendent // runs
- **return** smaller of C_1, C_2

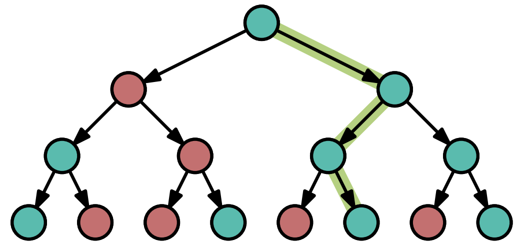
Karger-Stein: Success Probability

Know: Each call to Karger-Stein breaks the min-cut with probability at most $\frac{1}{2}$.

before calling itself recursively

Auxiliary Problem

Given complete binary tree of height d where each node is randomly coloured red or green (with probability $\frac{1}{2}$ each).
What is the probability p_d that a green root-to-leaf path exists?



$p_0 = 1/2$ // root green $\qquad p_d = \frac{1}{2}(1-(1-p_{d-1})^2)$ // root green, **not** no path in both left and right subtree

Claim: $p_d \geq \frac{1}{d+2}$. Proof by induction.

$$p_0 = \frac{1}{2} = \frac{1}{0+2} \quad \checkmark$$

$$p_d = \frac{1}{2}(1-(1-p_{d-1})^2) \geq \frac{1}{2}(1-(1-\frac{1}{d+1})^2) = \frac{1}{2}(\frac{2}{d+1} - \frac{1}{(d+1)^2})$$

$$= \frac{1}{2} \cdot \frac{2d+2-1}{(d+1)^2} = \frac{1}{2} \cdot \frac{2d+1}{d^2+2d+1} \geq \frac{1}{2} \cdot \frac{2d}{d^2+2d} = \frac{1}{d+2} \quad \text{// for } 1 \leq a \leq b \text{ we have } \frac{a}{b} \geq \frac{a-1}{b-1}$$

Corollary: Karger-Stein succeeds with probability at least $p_{\log_{\sqrt{2}}(n)} = \frac{1}{O(\log n)}$.

Karger-Stein Amplified

Karlsruhe Institute of Technology

Theorem: On a graph with n nodes, Karger-Stein runs in $O(n^2 \log(n))$ time and returns a minimum cut with probability at least $1/O(\log(n))$.

Amplification

$$\Pr[\text{“min-cut found”}] \geq 1 - \exp\left(-\frac{t}{O(\log(n))}\right) \underset{\uparrow}{=} 1 - O\left(\frac{1}{n}\right)$$

for $t = \log^2(n)$

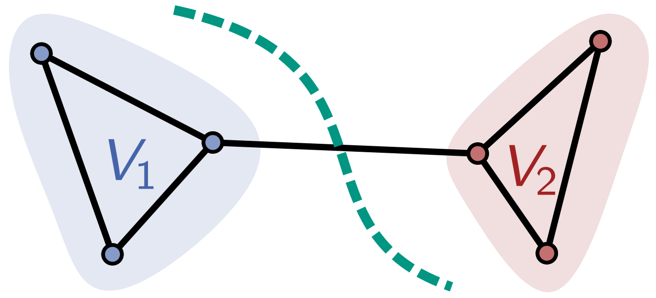
Success probability $\geq p$
Number of repetitions t
Amplified prob. $\geq 1 - e^{-pt}$

Corollary: On a graph with n nodes, $O(\log^2(n))$ repetitions of Karger-Stein run in $O(n^2 \log^3(n))$ total time and return a minimum cut with high probability.

- Compared to $O(n^4 \log(n))$ for Karger
- Compared to $\Omega(n^3)$ for deterministic approaches

Conclusion

Minimum Cut
- Fundamental graph problem
- Many deterministic flow-based algorithms ...
- ...with worst-case running times in $\Omega(n^3)$



Randomized Algorithms
- Karger's edge-contraction algorithm

Probability Amplification
- Monte Carlo algorithms with and without biases
- Repetitions amplify success probability
- Karger-Stein: Amplify before failure probability gets large

Algo Output \ Correct Answer	✗	✓
✗	true neg	false neg
✓	false pos	true pos

Outlook

"Minimum cuts in near-linear time", Karger, J.Acm. '00

Success w.h.p. in time $O(m \log^3(n))$

"Faster algorithms for edge connectivity via random 2-out contractions", Ghaffari & Nowicki & Thorup, SODA'20

Success w.h.p. in time $O(m \log(n))$ and $O(m + n \log^3(n))$

Possible Exam Questions

- What is a Monte Carlo algorithm?
  - Which variants exist?
- What is meant by probability amplification?
- How does probability amplification work...
  - ... in the case of one-sided error?
  - ... in the case of two-sided error?
  - ... for optimization problems?
  - How does the error probability relate to the number of repetitions?
- What is the Minimum Cut problem?
  - What do the best known deterministic algorithms achieve?
  - What are success probability and running time of the trivial random cut algorithm?
  - How does Karger's algorithm work?
    - What does $\Pr[\mathcal{E}_t]$ mean, and how did we estimate this probability?
    - What follows for the running time and success probability?
  - How is the algorithm by Karger and Stein obtained from Karger's algorithm?
    - How did we estimate the success probability and running time?
    - How do we achieve a success probability of $1 - \frac{1}{n}$?