Karlsruhe Institute of Technology

Probability & Computing

Probability Amplification

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  - **One-sided error**: either *false-biased* or *true-biased*

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		✗	✓
Algo Output	✗	true neg	false neg
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Probability Amplification for true-biased algorithms

- Execute independently t times.
  - If ✓ at least once: Return ✓. (surely correct)
  - Otherwise: Return ✗. $\Pr[\text{“correct”}] \geq 1 - (1-p)^t \geq 1 - e^{-pt}$

$1 + x \leq e^x$ for $x \in \mathbb{R}$

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Probability Amplification for optimization algorithms

- Execute independently t times.
  - output best result

$$\Pr[\text{“optimal”}] \geq 1 - (1-p)^t \geq 1 - e^{-pt}$$

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Input

- Set P of points in a feature space (e.g., $\mathbb{R}^d$)
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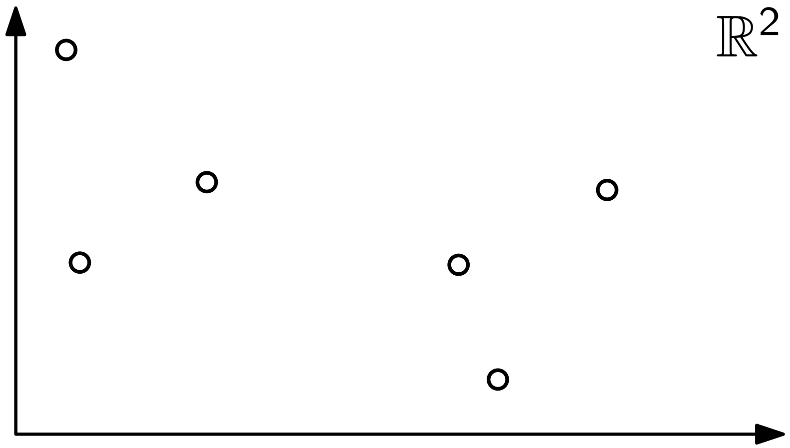

Example

- six points in $\mathbb{R}^2$
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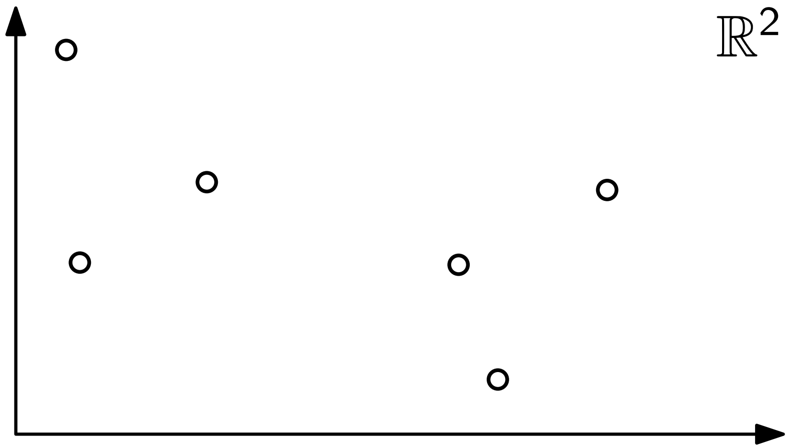

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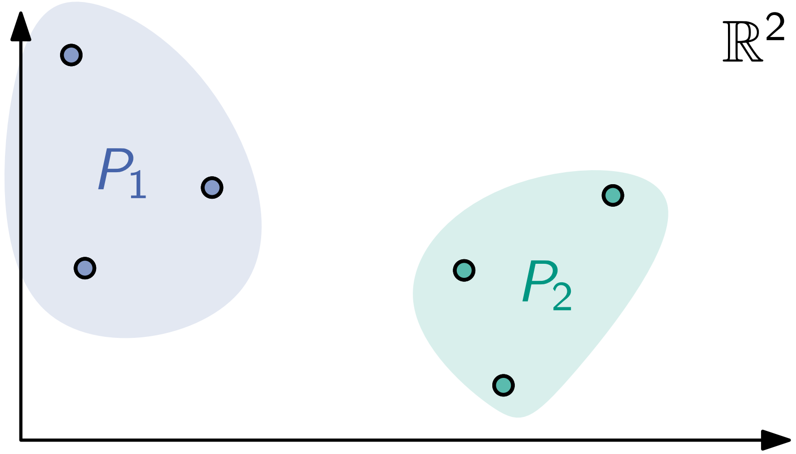

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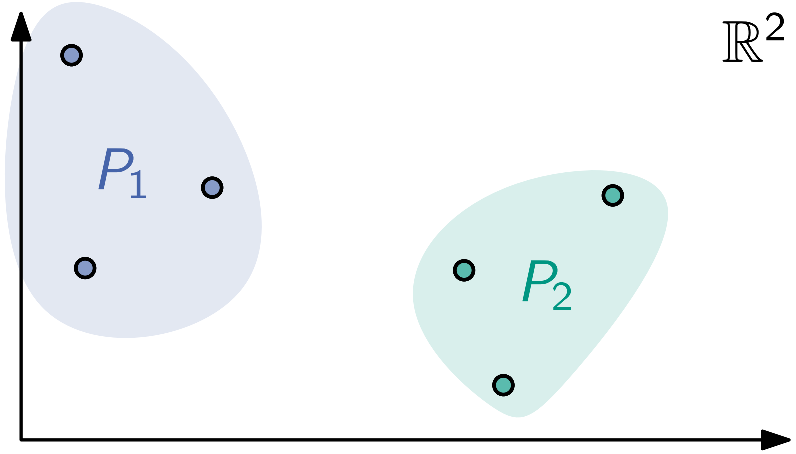

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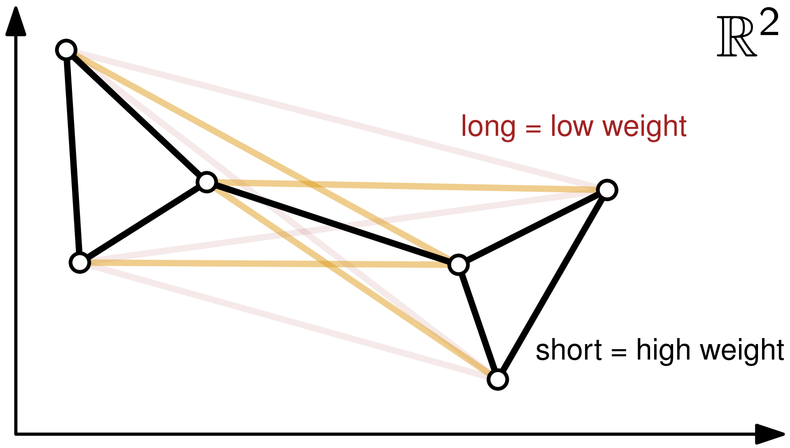
- Each point is a node
- Edges between all node pairs, with the weight given by the similarity of the two nodes



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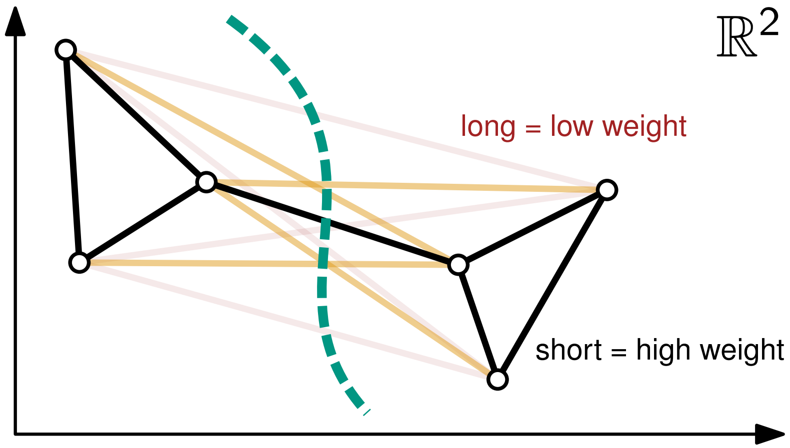

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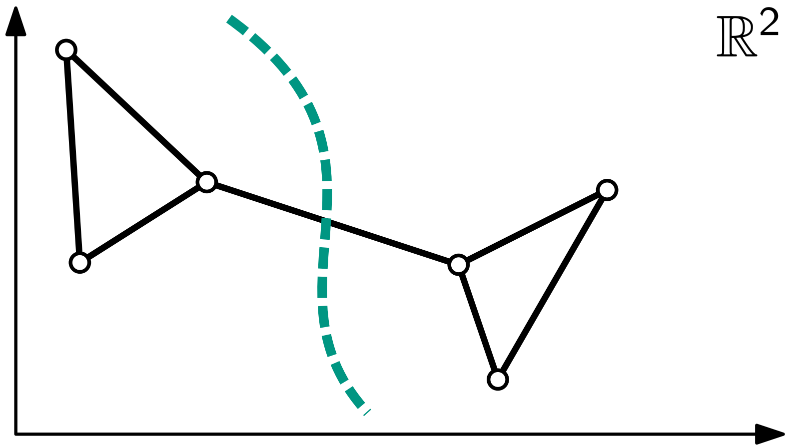

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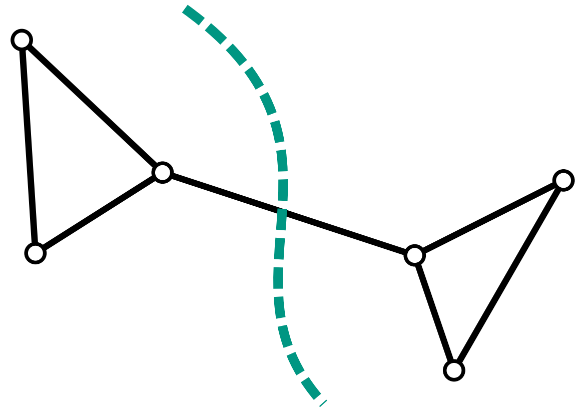
Today

$k = 2$ and $\sigma\colon P \times P \mapsto \{0, 1\}$

Computing Min Cuts

Cuts

- $G = (V, E)$ an unweighted, undirected, connected graph
- *Cut*: partition of V into non-empty parts V_1, V_2.



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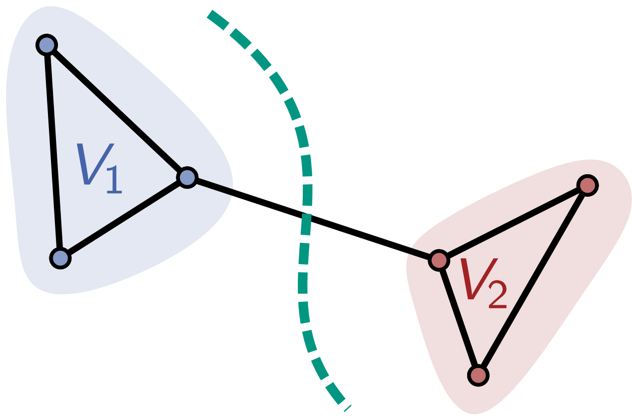

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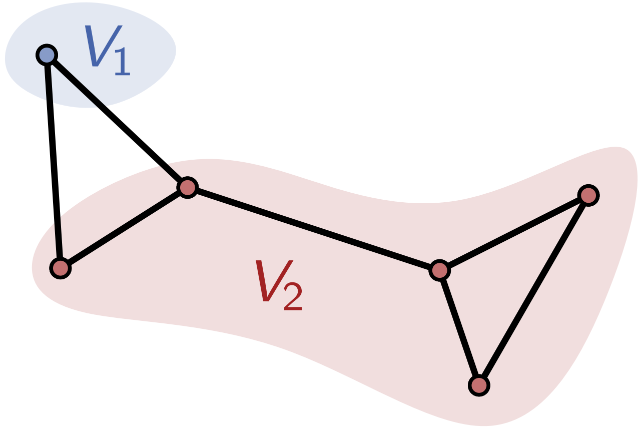

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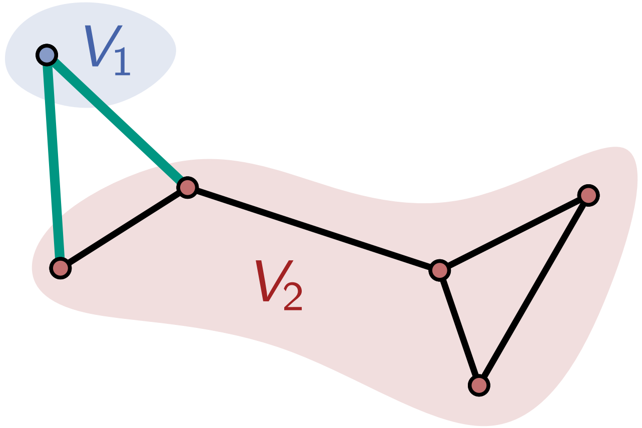

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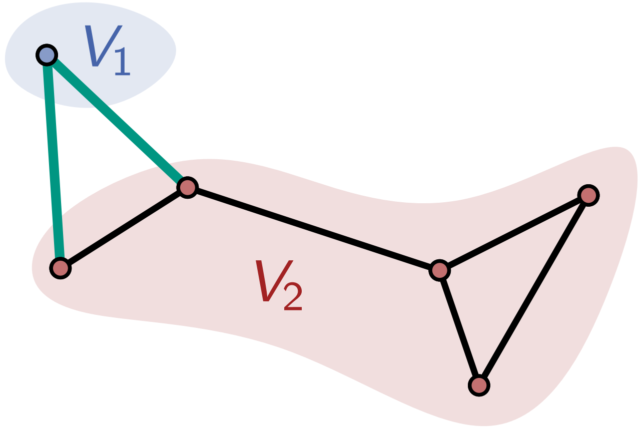

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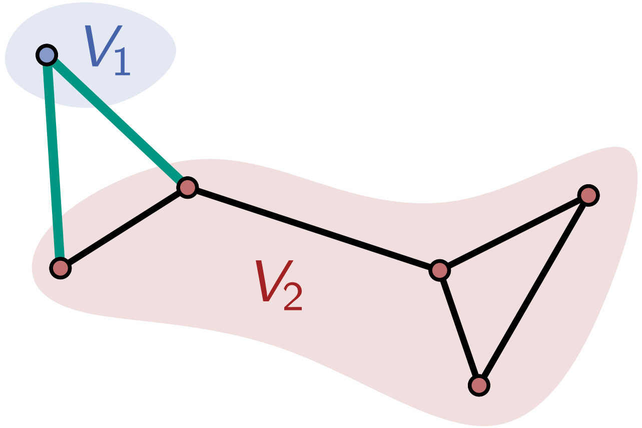

Today Goal: Compute a Min-Cut

i.e. a cut of minimum weight or cut-set of minimum size
the weight of the min-cut is known as the edge-connectivity of G

- Known deterministic strategies have worst case running time $\Omega(n^3)$.
- We'll see randomised algorithm with running time $O(n^2 \cdot \log^3(n))$.

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Observation: There are $2^{n-1} - 1$ cuts in a graph with n nodes.

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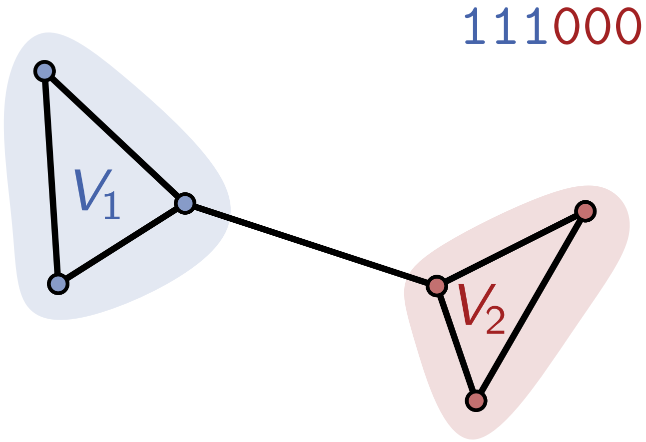

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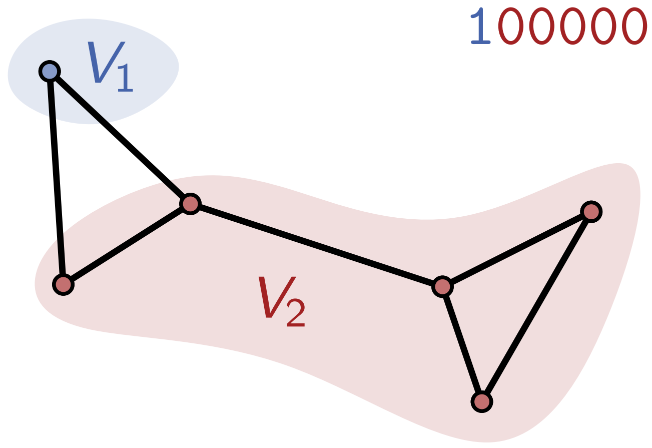

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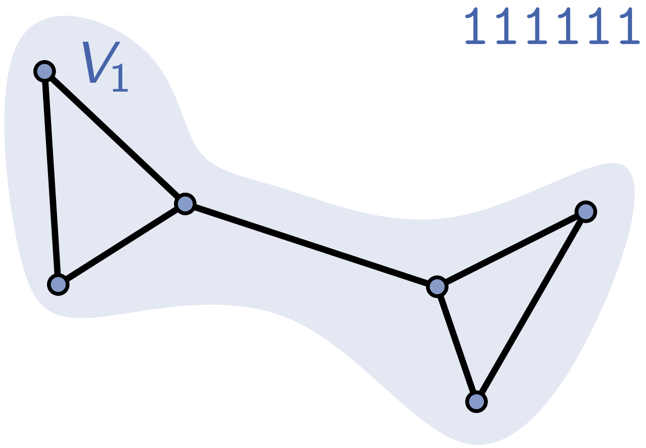

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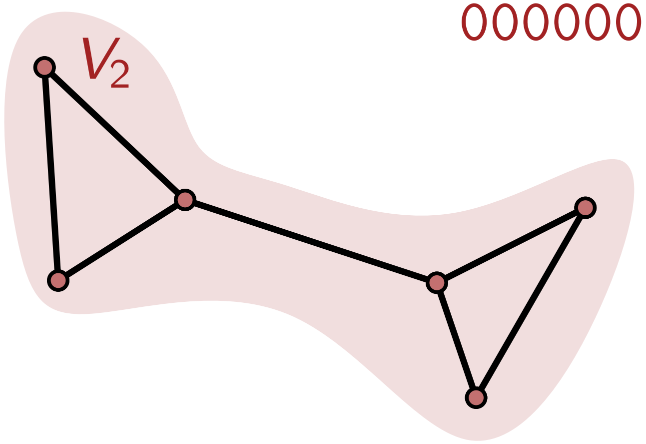

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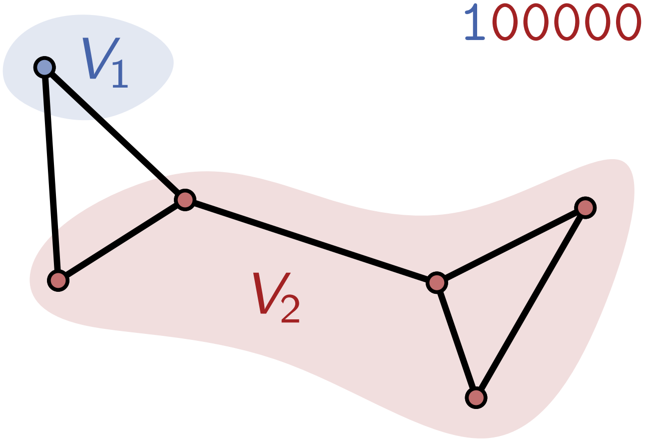

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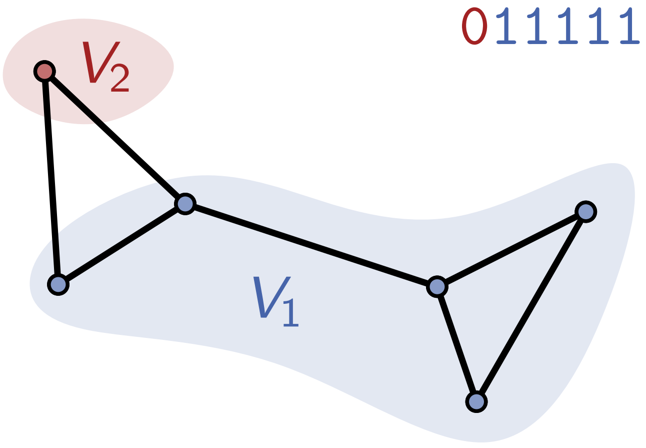

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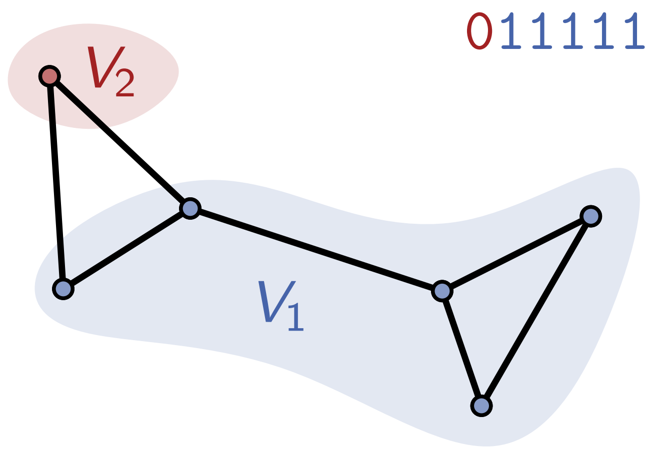

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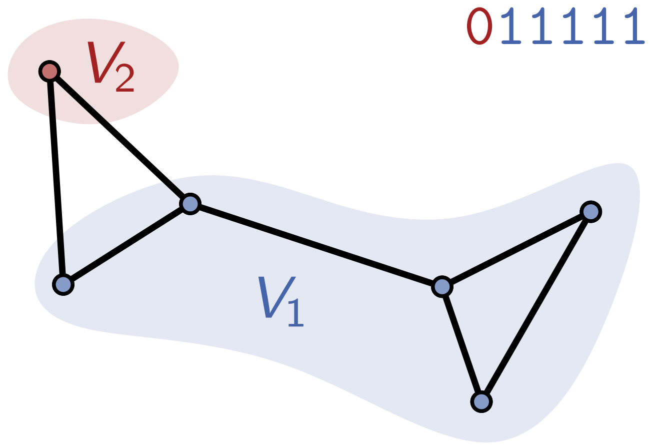

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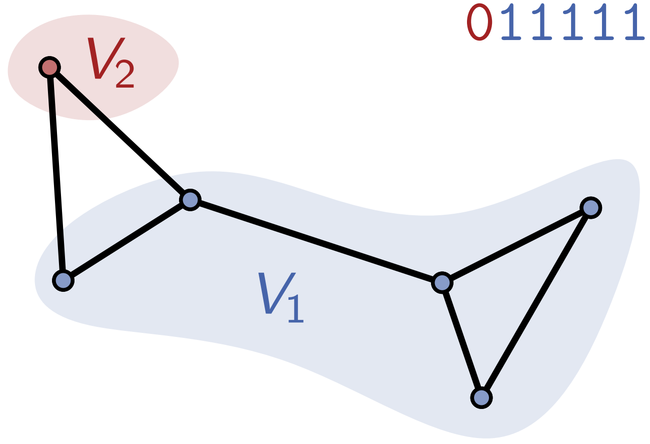
011111



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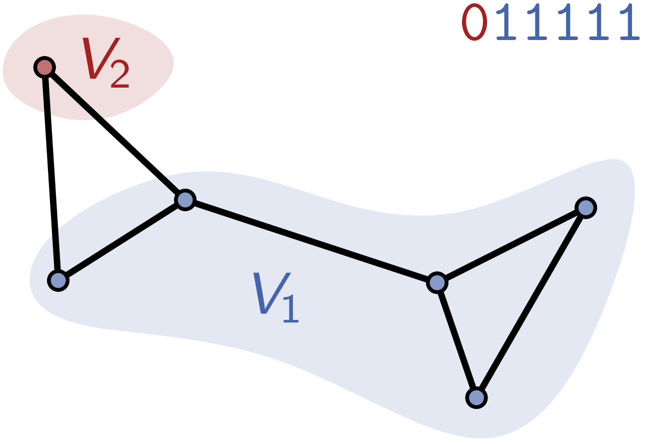

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- Swapping parts does not yield a new partition → $/2$

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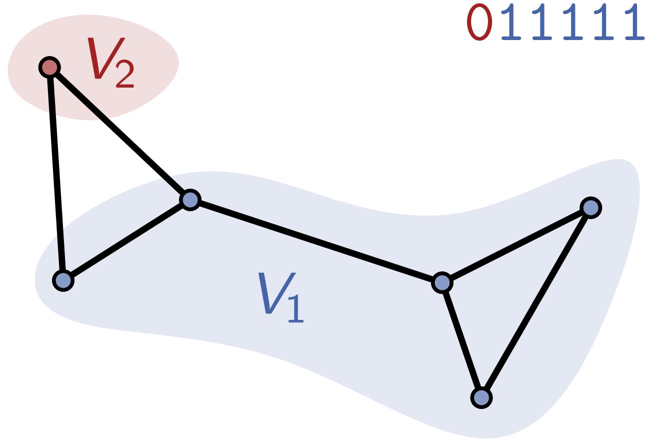

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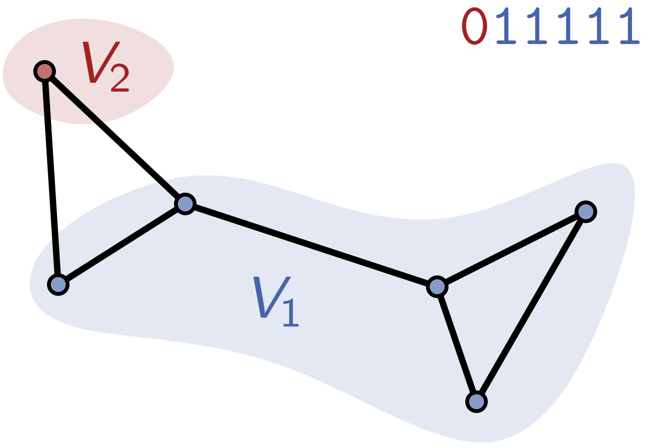

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    - actually for bounded running time: declare failure rather than sampling again
    - samples each cut with probability $1/2^{n-1}$

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Running time: $O(n)$ much better than the $\Omega(n^3)$ in the deterministic setting , but...

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$$\Pr[\text{“min cut found”}] \geq \underbrace{1 -}_{\text{not}} \left(1 - \underbrace{1/2^{n-1}}_{\text{minimum}}\right)^t$$

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Running time: $O(n)$ much better than the $\Omega(n^3)$ in the deterministic setting , but...

Success probability: $\geq 1/2^{n-1}$ "$=$" if there is only one min-cut.

$\rightarrow$ exponentially small!

Amplification

- Repeat the algorithm to obtain t independent random cuts, return the smallest

$$\Pr[\text{“min cut found”}] \geq \underbrace{1 -}_{\text{not}} \Big(1 - \underbrace{1/2^{n-1}}_{\text{minimum}}\Big)^{\overset{t}{\downarrow \; t \text{ times}}}$$

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- Repeat the algorithm to obtain t independent random cuts, return the smallest

$$\Pr[\text{“min cut found”}] \geq 1 - \left(1 - 1/2^{n-1}\right)^t \geq 1 - e^{-t/2^{n-1}}$$

$1 + x \leq e^x$ for $x \in \mathbb{R}$

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- For $t = 2^{n-1}$ min cut found with constant probability $1 - 1/e \approx 0.63$

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- For $t = 2^{n-1}$ min cut found with constant probability $1 - 1/e \approx 0.63$
- For $t = 2^{n-1} \cdot \ln(n)$ min cut found with high probability $1 - 1/n$

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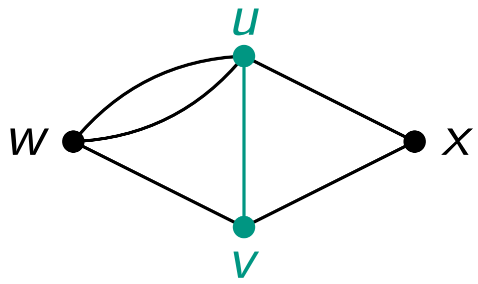
this is terrible so far...

Karger's Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops



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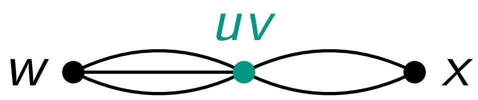

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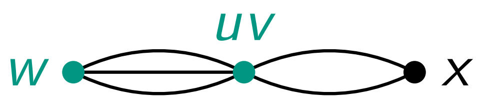
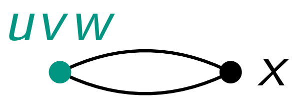


Karger's Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops
- A (multi) graph with two nodes has a unique cut-set



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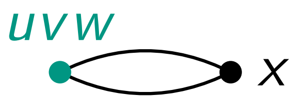

Contraction Algorithm

- Motivation: distinguish *non-essential* (not part of a min-cut)
  as well as *essential* edges (part of a min-cut)
  & hope there are few essential ones

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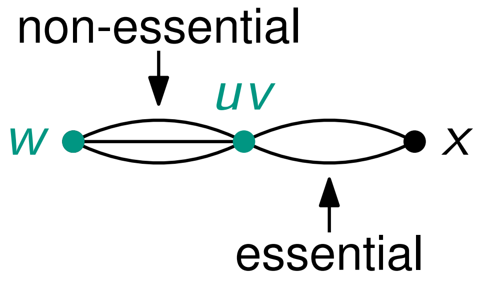

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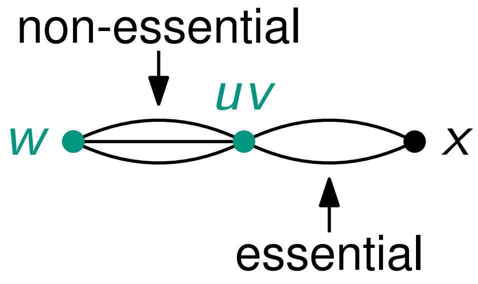

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Karger$(G_0 = (V_0, E_0))$

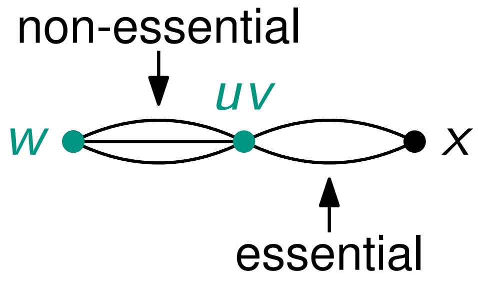
for $i = 1$ to $n - 2$ **do**

  sample $e \sim \mathcal{U}(E_{i-1})$

  $G_i \leftarrow G_{i-1}$.**contract**(e)

return unique cut-set in G_{n-2}

Karger's Algorithm



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for i = 1 to n − 2 do        // O(n)
    sample e ~ U(E_{i−1})    // O(1)
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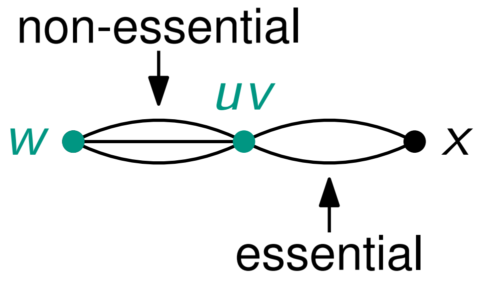
return unique cut-set in G_{n−2}
```

- Running time in $O(n^2)$
- Can be implemented to run in $O(m)$

Karger's Algorithm

Edge Contraction



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  for $i = 1$ to $n - 2$ **do** $\quad$ // $O(n)$
    sample $e \sim \mathcal{U}(E_{i-1})$ $\quad$ // $O(1)$
    $G_i \leftarrow G_{i-1}$.**contract**(e) $\quad$ // $O(n)$
  return unique cut-set in G_{n-2}

- Running time in $O(n^2)$
- Can be implemented to run in $O(m)$

Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

(the converse does not hold)

- Let C be a cut-set in G_i.
  - $G_i \setminus C$ is disconnected
- Assume C is not a cut-set in G_0.
  - $G_0 \setminus C$ is connected.
- $G_i \setminus C$ arises from $G_0 \setminus C$ by i edge contractions.
- $\lightning$contractions cannot disconnect a graph

Karger's Algorithm

Karlsruhe Institute of Technology

Edge Contraction

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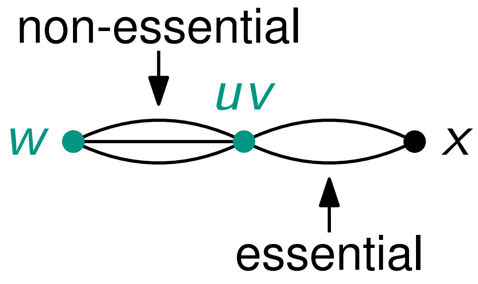

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- Running time in $O(n^2)$
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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

- Consider min-cut in G_0 with cut-set C and $|C| = k$

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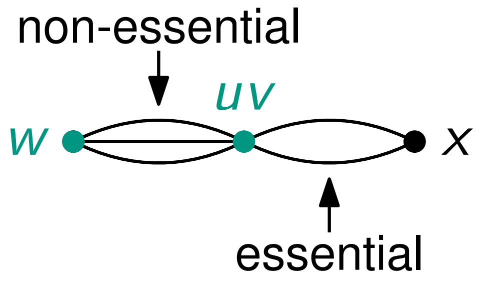

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- **return** unique cut-set in G_{n-2}

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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i =$ "C in G_i"

$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m}$

Karger's Algorithm

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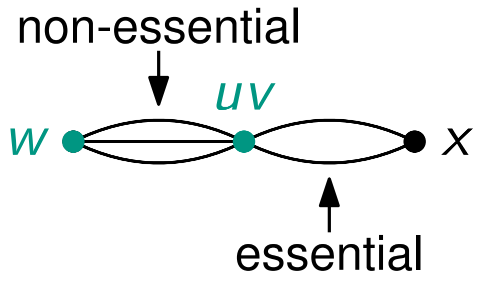

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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0. ↯

- Consider min-cut in G_0 with cut-set C and $|C| = k$
- $\mathcal{E}_i =$ "C in G_i"

Observation: min-degree $\geq k$

$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m}$

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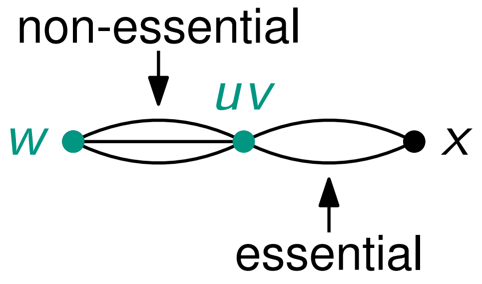

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(holds for all G_i due to 1st observation)

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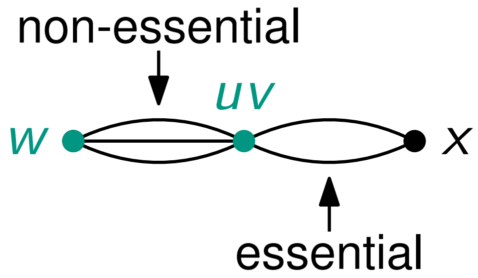

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- $\mathcal{E}_i =$ "C in G_i"

Observation: min-degree $\geq k$

(holds for all G_i due to 1st observation)

$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m}$

$$m = \frac{1}{2}\sum_{v \in V} \deg(v) \geq \frac{1}{2}\sum_{v \in V} k \geq \frac{1}{2}nk$$

Karger's Algorithm

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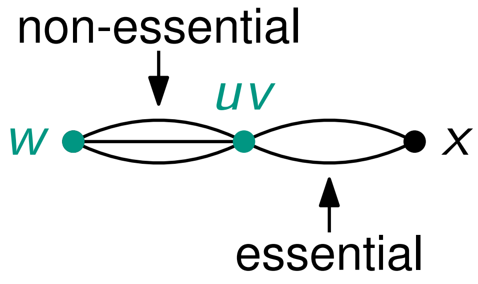

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Observation: min-degree $\geq k$

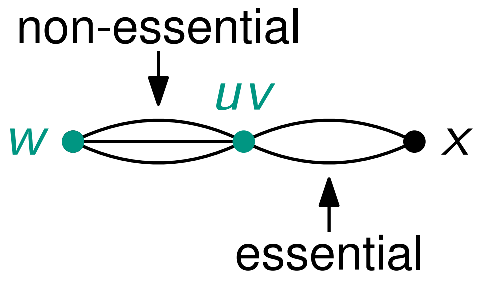
(holds for all G_i due to 1st observation)

$$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m} \geq 1 - \frac{k}{nk/2} = 1 - \frac{2}{n}$$

$$m = \frac{1}{2}\sum_{v \in V} \deg(v) \geq \frac{1}{2}\sum_{v \in V} k \geq \frac{1}{2}nk$$

Karger's Algorithm

Edge Contraction



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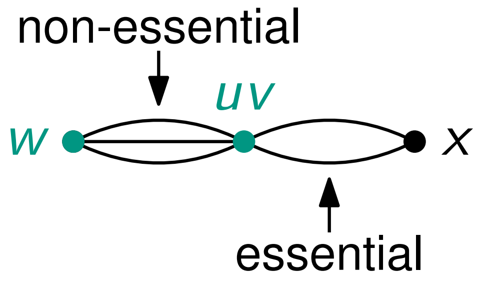

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$\Pr[\mathcal{E}_1] \geq 1 - \frac{2}{n}$

$\Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \geq 1 - \frac{2}{n-1}$

none of the k edges of C contracted

do not contract k edges in an $n - 1$-node graph

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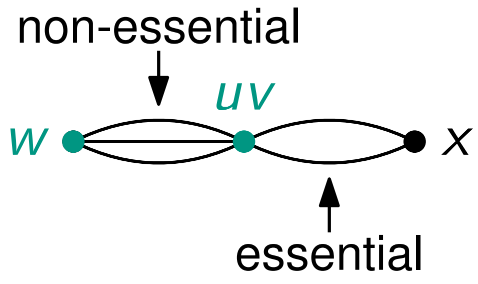

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$\Pr[\mathcal{E}_1] \geq 1 - \frac{2}{n}$

$\Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \geq 1 - \frac{2}{n-1} \longrightarrow \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}] \geq 1 - \frac{2}{n-i+1}$

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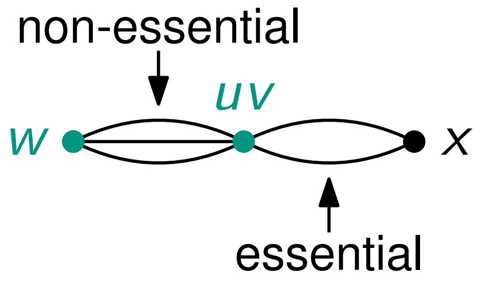

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$\Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \geq 1 - \frac{2}{n-1} \longrightarrow \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}] \geq 1 - \frac{2}{n-i+1}$

$$\underbrace{\Pr[\mathcal{E}_{n-2}] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_{n-2} \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{n-3}]}_{\text{chain rule of probability}}$$

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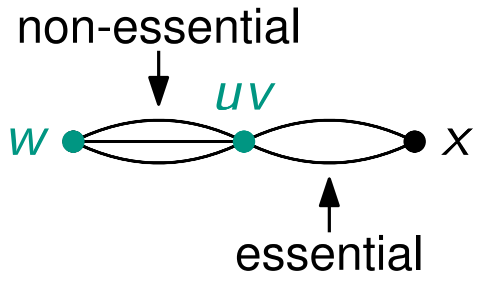

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```
for i = 1 to n − 2 do              // O(n)
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    G_i ← G_{i−1}.contract(e)      // O(n)
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```

- Running time in $O(n^2)$
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Success Probability

Observation: A cut-set in G_i is a cut-set in G_0.

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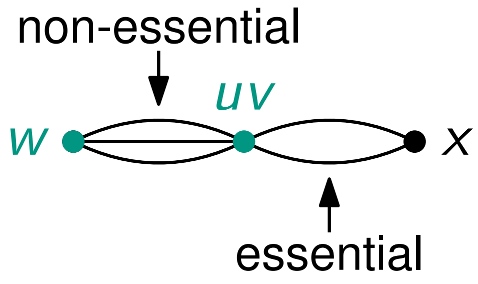
Karger's Algorithm



Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops
- A (multi) graph with two nodes has a unique cut-set

Contraction Algorithm

- Motivation: distinguish $\overbrace{\textit{non-essential}}^{\text{not part of a min-cut}}$ as well as *essential* edges (part of a min-cut) & hope there are few essential ones

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$1 = \frac{n-i}{n-i}$

Karger's Algorithm

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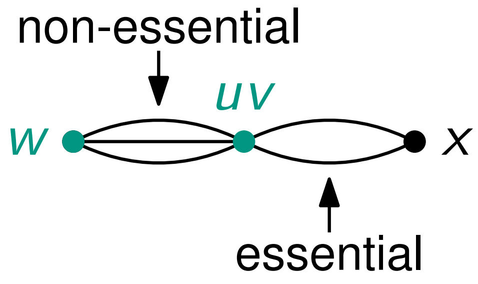

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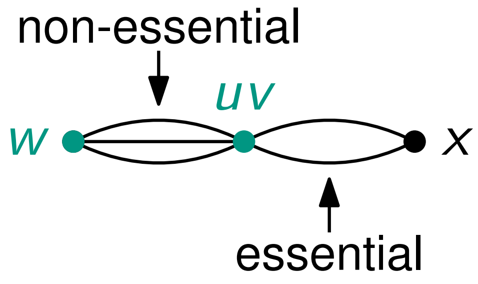

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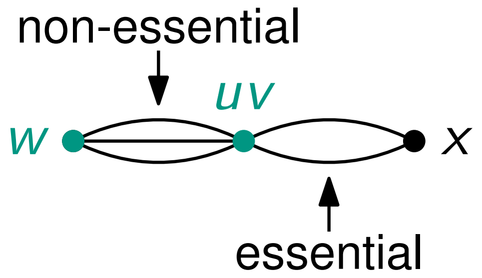

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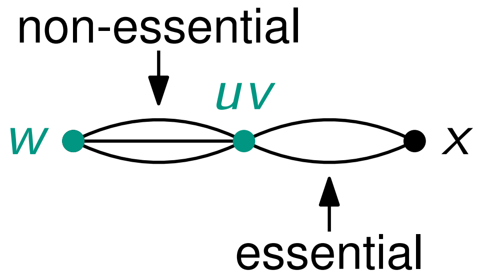

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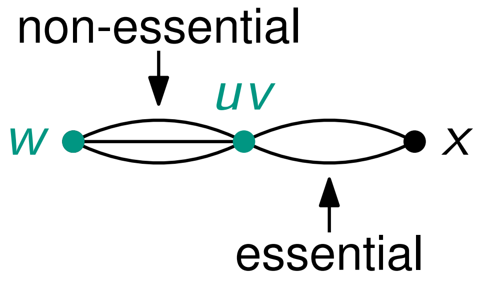

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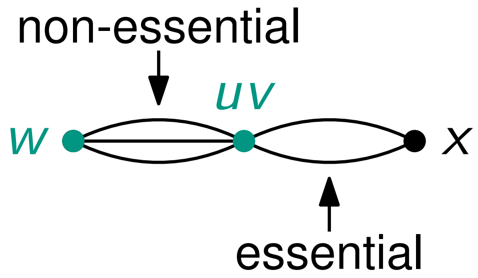

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$$= \frac{2}{n\,(n-1)} \geq \frac{2}{n^2}$$

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Theorem: On a graph with n nodes, Karger's algorithm runs in $O(n^2)$ time and returns a minimum cut with probability at least $\frac{2}{n^2}$.

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Karlsruhe Institute of Technology

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$$1 \geq \Pr\left[\bigcup_{i\in[\ell]} \mathcal{E}^i_{n-2}\right] = \sum_{i\in[\ell]} \Pr[\mathcal{E}^i_{n-2}] \geq \frac{2\cdot\ell}{n^2}$$

Observation: $\ell \leq \frac{n^2}{2}$.

More Amplification: Karger-Stein

Motivation

- Probability that a min-cut survives i contractions

$$\Pr[\mathcal{E}_i] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}]$$

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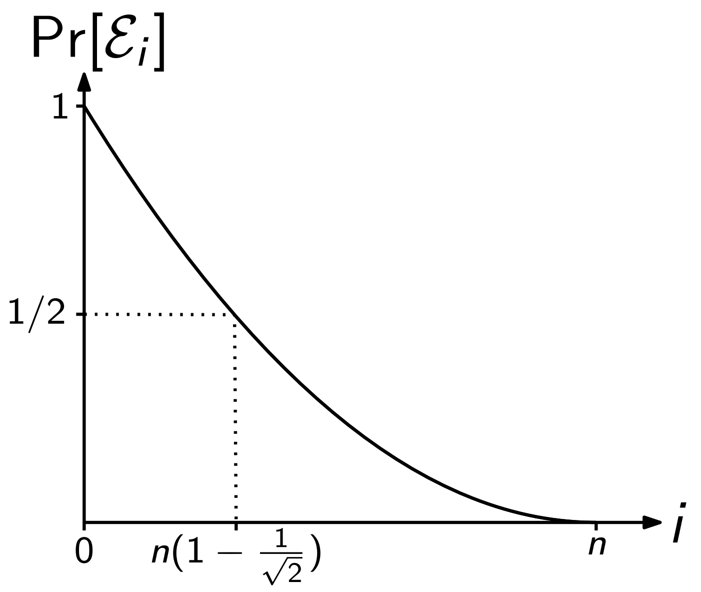

More Amplification: Karger-Stein

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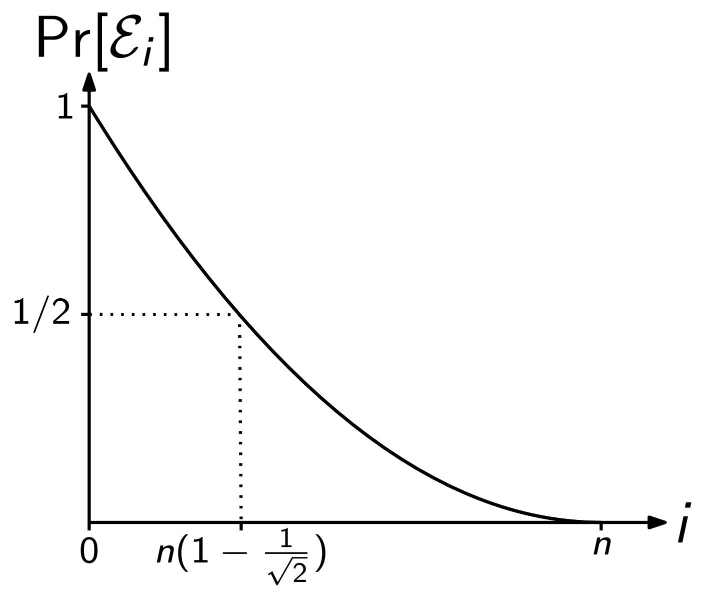

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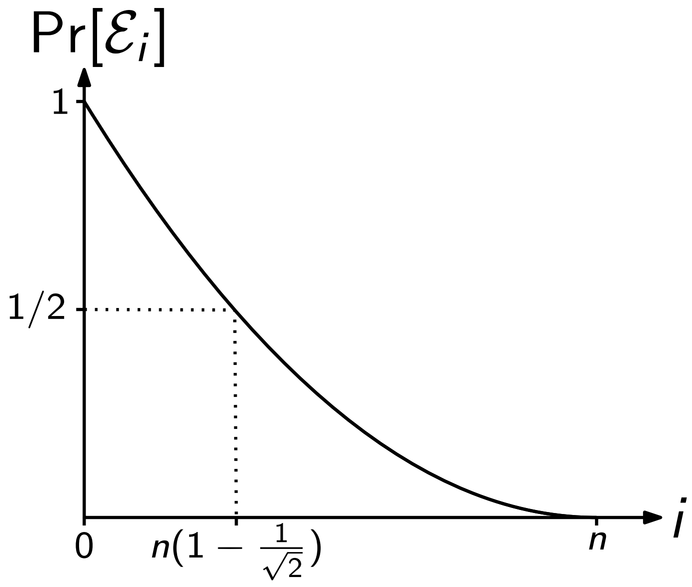

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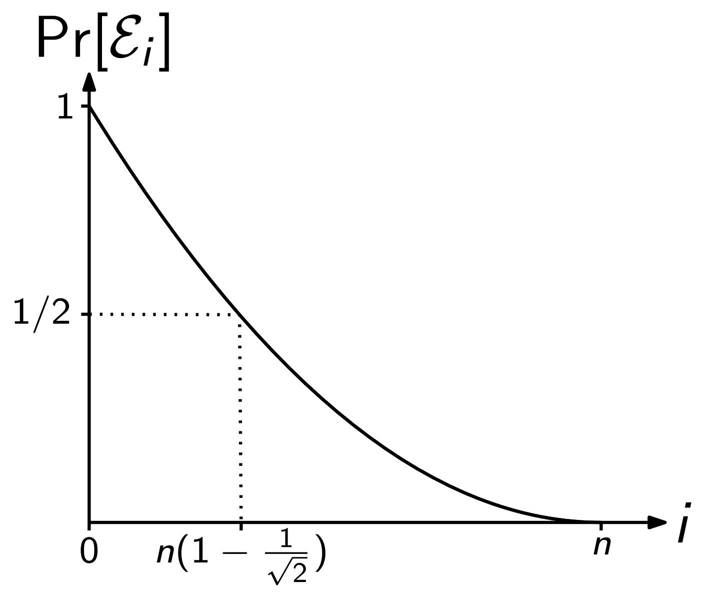

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Karlsruhe Institute of Technology

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KargerStein$(G_0 = (V_0, E_0))$
- **if** $|V_0| = 2$ **then** return unique cut-set
- **for** $i = 1$ to $s = |V_0| - \frac{|V_0|}{\sqrt{2}} - 1$ **do**
  - sample $e \sim \mathcal{U}(E_{i-1})$
  - $G_i \leftarrow G_{i-1}.$**contract**(e)
- $C_1 \leftarrow$ **KargerStein**(G_s) // inde-
- $C_2 \leftarrow$ **KargerStein**(G_s) // pendent // runs
- **return** smaller of C_1, C_2

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Karger-Stein: Running Time

Karlsruhe Institute of Technology

KargerStein$(G_0 = (V_0, E_0))$

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- After $t = n - n/\sqrt{2} - 1$ steps the number of nodes is $n/\sqrt{2} + 1$

$$T(n) = 2T\left(\frac{n}{\sqrt{2}} + 1\right) + O(n^2)$$

```
KargerStein(G_0 = (V_0, E_0))
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Solution (essentially by Master Theorem)

$$T(n) = O(n^2 \log n)$$

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Know: Each call to Karger-Stein breaks the min-cut with probability at most $\frac{1}{2}$.

before calling itself recursively

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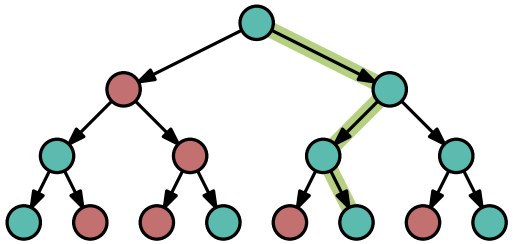
Given complete binary tree of height d where each node is randomly coloured red or green (with probability $\frac{1}{2}$ each).
What is the probability p_d that a green root-to-leaf path exists?



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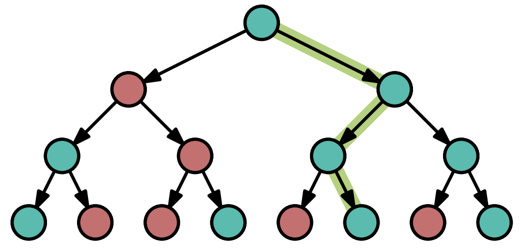

$p_0 = 1/2$ // root green $\quad$ $p_d = \frac{1}{2}(1 - (1 - p_{d-1})^2)$ // root green, **not** no path in both left and right subtree

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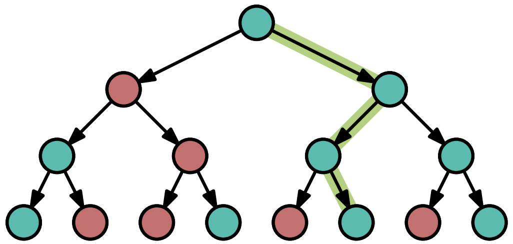

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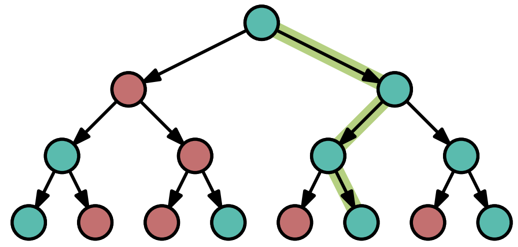

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$$p_0 = \frac{1}{2} = \frac{1}{0+2} \quad \checkmark$$

$$p_d = \frac{1}{2}\left(1 - (1 - p_{d-1})^2\right) \geq \frac{1}{2}\left(1 - \left(1 - \frac{1}{d+1}\right)^2\right) = \frac{1}{2}\left(\frac{2}{d+1} - \frac{1}{(d+1)^2}\right)$$

$$= \frac{1}{2} \cdot \frac{2d+2-1}{(d+1)^2} = \frac{1}{2} \cdot \frac{2d+1}{d^2+2d+1} \geq \frac{1}{2} \cdot \frac{2d}{d^2+2d} = \frac{1}{d+2}$$

// for $1 \leq a \leq b$ we have $\frac{a}{b} \geq \frac{a-1}{b-1}$

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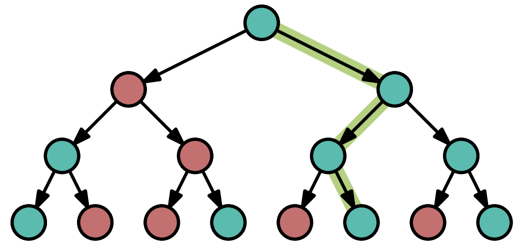

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Corollary: Karger-Stein succeeds with probability at least $p_{\log_{\sqrt{2}}(n)} = \frac{1}{O(\log n)}$.

Karger-Stein Amplified

Theorem: On a graph with n nodes, Karger-Stein runs in $O(n^2 \log(n))$ time and returns a minimum cut with probability at least $1/O(\log(n))$.

Karger-Stein Amplified

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Success probability $\geq p$
Number of repetitions t
Amplified prob. $\geq 1 - e^{-pt}$

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$$\Pr[\text{“min-cut found”}] \geq 1 - \exp\left(-\frac{t}{O(\log(n))}\right) = 1 - O\left(\frac{1}{n}\right)$$

for $t = \log^2(n)$

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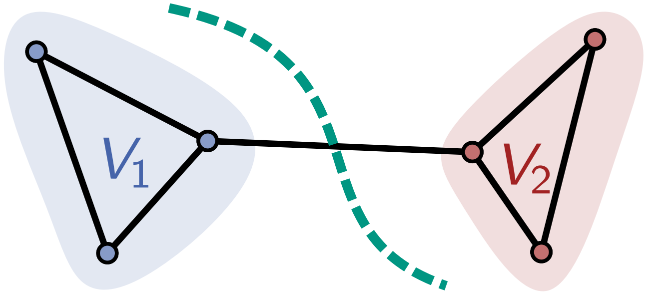
- Compared to $O(n^4 \log(n))$ for Karger
- Compared to $\Omega(n^3)$ for deterministic approaches

Conclusion

Minimum Cut

- Fundamental graph problem
- Many deterministic flow-based algorithms ...
- . . . with worst-case running times in $\Omega(n^3)$



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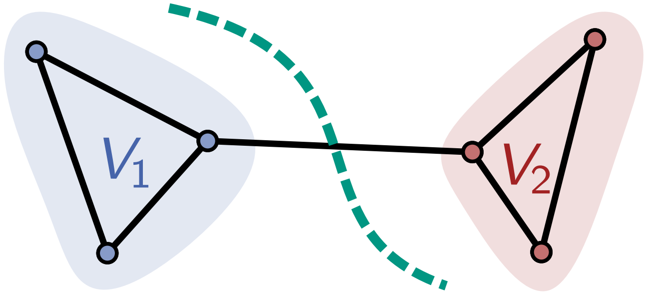

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- Karger's edge-contraction algorithm

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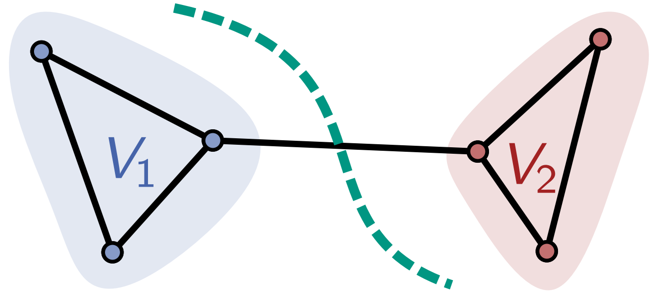

Randomized Algorithms

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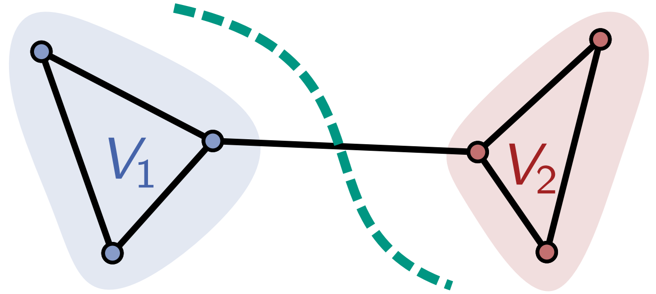
Probability Amplification

- Monte Carlo algorithms with and without biases
- Repetitions amplify success probability
- Karger-Stein: Amplify before failure probability gets large

		Correct Answer ✗	Correct Answer ✓
Algo Output	✗	true neg	false neg
Algo Output	✓	false pos	true pos

Conclusion



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Algo Output \ Correct Answer	✗	✓
✗	true neg	false neg
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Outlook

"Minimum cuts in near-linear time", Karger, J.Acm. '00

Success w.h.p. in time $O(m \log^3(n))$

"Faster algorithms for edge connectivity via random 2-out contractions", Ghaffari & Nowicki & Thorup, SODA'20

Success w.h.p. in time $O(m \log(n))$ and $O(m + n \log^3(n))$

Possible Exam Questions

- What is a Monte Carlo algorithm?
  - Which variants exist?
- What is meant by probability amplification?
- How does probability amplification work...
  - ... in the case of one-sided error?
  - ... in the case of two-sided error?
  - ... for optimization problems?
  - How does the error probability relate to the number of repetitions?
- What is the Minimum Cut problem?
  - What do the best known deterministic algorithms achieve?
  - What are success probability and running time of the trivial random cut algorithm?
  - How does Karger's algorithm work?
    - What does $\Pr[\mathcal{E}_t]$ mean, and how did we estimate this probability?
    - What follows for the running time and success probability?
  - How is the algorithm by Karger and Stein obtained from Karger's algorithm?
    - How did we estimate the success probability and running time?
    - How do we achieve a success probability of $1 - \frac{1}{n}$?