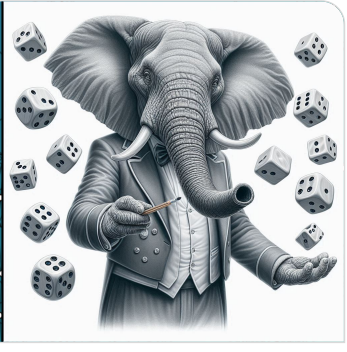
Karlsruhe Institute of Technology

Algorithm Engineering

Probability and Computing – Important Random Variables and How to Sample Them

Stefan Walzer | WS 2025/2026



Content

What is a Probability?

Physical Accounts

Probabilities are persistent rates of outcomes when observing the same (random) process over and over again.

It's about **objective stuff**:

"The probability that the coin comes up heads is 50%*."*

Evidential / Bayesian Accounts

Probabilities reflect how much a rational agent believes in a proposition and about how much they are willing to bet on it.

It's about **what I subjectively know**:

"The probability that it is going to rain tomorrow is 33%*."*

See `https://en.wikipedia.org/wiki/Probability_interpretations`.
In this lecture, we use a naive notion.

Bernoulli Distribution

Karlsruhe Institute of Technology

Definition: $\mathrm{Ber}(p)$ for $p \in [0,1]$

$B \sim \mathrm{Ber}(p)$ is a random variable with

$$\Pr[B = 1] = p \text{ and } \Pr[B = 0] = 1 - p.$$

Standard Assumption: Access to Coin Flips

Algorithms have access to a sequence $B_1, B_2, \ldots \sim \mathrm{Ber}(1/2)$ in independent uniformly random bits.

Exercise: $\mathrm{Ber}(1/3)$ from $\mathrm{Ber}(1/2)$

Design an algorithm that outputs B such that $B \sim \mathrm{Ber}(1/3)$.

Uniform Distribution

Definition: $\mathcal{U}(D)$ on finite D

If $|D| < \infty$, then $X \sim \mathcal{U}(D)$ is a random variable with

$$\Pr[X = x] = \frac{1}{|D|} \text{ for all } x \in D.$$

Definition: $\mathcal{U}(D)$ on infinite D

If D is infinite but has finite measure[a] then $X \sim \mathcal{U}(D)$ is a random variable with uniform density function on D. Important example:

$$X \sim \mathcal{U}([0,1]) \Leftrightarrow \forall x \in [0,1] : \Pr[X < x] = x.$$

[a] Formal details: Not in this lecture.

Standard Assumption

Algorithms have access to $X_1, X_2, \ldots \sim \mathcal{U}([0,1])$.
In practice: Initialise the significand[a] of floating point number with random bits.

[a] Deutsch: Mantisse.

Exercise: $\mathcal{U}(\{1,\ldots,n\})$ from $\mathcal{U}([0,1])$

Design an algorithm that outputs X such that $X \sim \mathcal{U}(\{1,\ldots,n\})$.

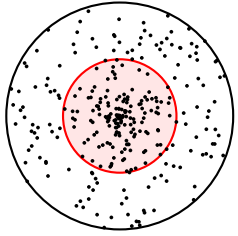
Uniform Distribution on a Disc

Task

Sample $P \sim \mathcal{U}(D)$ for $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Flawed Attempt

sample $\Phi \sim \mathcal{U}([0, 2\pi])$
sample $R \sim \mathcal{U}([0, 1])$
return $(R \cdot \cos \Phi, R \cdot \sin \Phi)$



Issue

Disc of half the radius is hit 50% of the time but makes up only $1/4$ of the area!

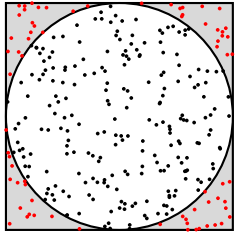
Uniform Distribution on a Disc with Rejection Sampling

Task

Sample $P \sim \mathcal{U}(D)$ for $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Solution with Rejection Sampling

repeat
  sample $X \sim \mathcal{U}([-1, 1])$
  sample $Y \sim \mathcal{U}([-1, 1])$
until $X^2 + Y^2 \leq 1$
return (X, Y)



- Idea: $P \sim \mathcal{U}([-1, 1]^2)$ *conditioned* on $P \in D$ is uniform on D.
- Each sample is accepted with probability $\pi/4$.
- Expected number of rounds is $1/(\pi/4) = \mathcal{O}(1)$.

Spoiler alert: We'll get worst-case constant time with inverse transform sampling later.

Rejection Sampling in General Discrete Distributions

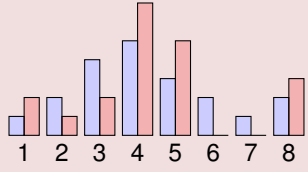
Exercise

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be distributions on a finite[a] set D. Assume

1. We can sample in constant time from $\mathcal{D}_1$.
2. There exists $C > 0$ such that for any $x \in D$ we have

$$\Pr_{X \sim \mathcal{D}_2}[X = x] \leq C \cdot \Pr_{X \sim \mathcal{D}_1}[X = x].$$

Design an algorithm that generates a sample from $\mathcal{D}_2$ in expected time $\mathcal{O}(C)$.



[a] This can be generalised.

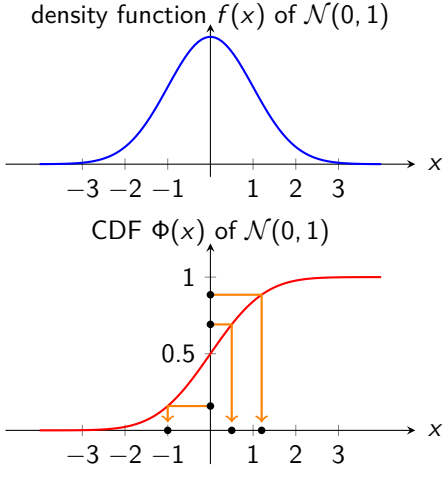
Inverse Transform Sampling

- Let $\mathcal{D}$ be a distribution on $\mathbb{R}$.
  $\hookrightarrow$ e.g. $\mathcal{D} = \mathcal{N}(0,1)$
- Let $X \sim \mathcal{D}$ and $F_X(x) = \Pr[X \leq x]$.
  $\hookrightarrow$ F_X is the *cumulative distribution function* of X
  $\hookrightarrow$ the CDF of the normal distribution is called Φ
- Let $F_X^{-1}(u) := \inf\{x \in \mathbb{R} \mid F_X(x) \geq u\}$.
  $\hookrightarrow$ ordinary inverse for strictly monotone F_X

Theorem (Inverse Transform Sampling)

If $U \sim \mathcal{U}([0,1])$ then $F_X^{-1}(U) \stackrel{\mathrm{d}}{=} X$, i.e. $F_X^{-1}(U) \sim \mathcal{D}$.
("$\stackrel{\mathrm{d}}{=}$" means: "has the same distribution as")

Reason: $\Pr[F_X^{-1}(U) \leq x] = \Pr[U \leq F_X(x)] = F_X(x)$.



Uniform Distribution on a Disc with Inverse Transform Sampling

Task

Sample $P \sim \mathcal{U}(D)$ for $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

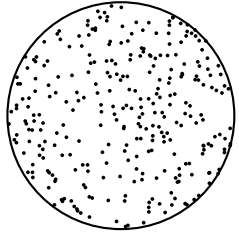
Preparation

If $(x, y) \sim \mathcal{U}(D)$ then $R = \sqrt{x^2 + y^2}$ satisfies

$$F_R(r) = \Pr[R \leq r] = r^2\pi/\pi = r^2 \text{ hence } F_R^{-1}(u) = \sqrt{u}.$$

Solution with Inverse Transform Sampling

sample $\Phi \sim \mathcal{U}([0, 2\pi])$
sample $U \sim \mathcal{U}([0, 1])$
$R \leftarrow \sqrt{U}$
return $(R \cdot \cos\Phi, R \cdot \sin\Phi)$



Geometric Distribution

Definition: $G \sim \text{Geom}_1(p)$ and $G' \sim \text{Geom}_0(p)$

Let $p \in (0, 1]$ and $B_1, B_2, \ldots \sim \text{Ber}(p)$.
Then we define the geometric random variables

$$G := \min\{i \in \mathbb{N} \mid B_i = 1\}$$

$\hookrightarrow$ number of $\text{Ber}(p)$ trials until (and including) the first success

$$G' := G - 1$$

$\hookrightarrow$ number of $\text{Ber}(p)$ failures before the first success

We write $G \sim \text{Geom}_1(p)$ and $G' \sim \text{Geom}_0(p)$.[a]

[a] In the literature Geom is used inconsistently.

Sampling $G \sim \text{Geom}_1(p)$ in time $\mathcal{O}(G)$

$i \leftarrow 0$
repeat
  $i \leftarrow i + 1$
  sample $X \sim \text{Ber}(p)$
until $X = 1$
return i

Quite bad: $\mathbb{E}[G] = 1/p$ might be large.

Exercise

Use inverse transform sampling to sample $G \sim \text{Geom}_1(p)$ in time $\mathcal{O}(1)$.

Sampling Without Replacement

Karlsruhe Institute of Technology

Exercise

Design an algorithm that, given $k, n \in \mathbb{N}$ with $0 \leq k \leq n$ outputs a set $S \subseteq [n]$ of size $|S| = k$ uniformly at random.

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

Algorithm init(k)**:**

allocate `reservoir`$[1..k]$
$n \leftarrow 0$

Algorithm observeItem(x)**:**

$n \leftarrow n + 1$
if $n \leq k$ **then**
  `reservoir`$[n] \leftarrow x$
else
  sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
  if $I \leq k$ **then**
    `reservoir`$[I] \leftarrow x$

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

Conclusion

Karlsruhe Institute of Technology

General Techniques

- rejection sampling
- inverse transform sampling

Distributions

- Bernoulli distribution
- uniform distribution
- geometric distribution

Other Stuff

- sampling from a set without replacement
- reservoir sampling

Appendix: Possible Exam Questions I

- How can one sample $B \sim \mathrm{Ber}(p)$? What about $X \sim \mathcal{U}(\{1, \ldots, n\})$? Under which assumptions?
- How does rejection sampling work in general? Under which conditions does rejection sampling lead to an efficient algorithm?
- How does inverse transform sampling work in general? Under which conditions does inverse transform sampling lead to an efficient algorithm?
- How can one sample a random point from a disk? Name two techniques and state their advantages and disadvantages.
- Given a set of size n. How can I determine a random subset of size $k \leq n$ and how long does that take?
- Explain reservoir sampling. Isn't that just a slower algorithm for "sampling without replacement"?