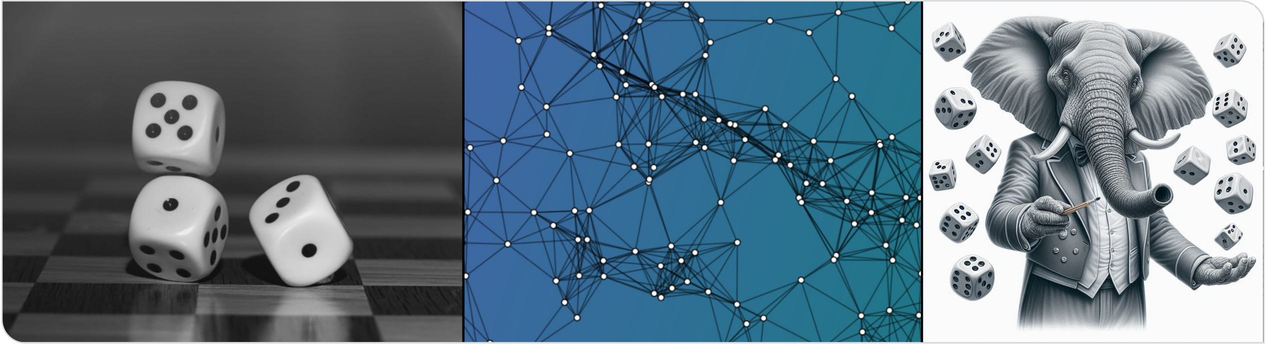
Probability and Computing –
Important Random Variables and How to Sample Them

Stefan Walzer | WS 2025/2026



Content

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"The probability that it is going to rain tomorrow is 33%*."*

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See `https://en.wikipedia.org/wiki/Probability_interpretations`.

In this lecture, we use a naive notion.

Bernoulli Distribution

Definition: $\mathrm{Ber}(p)$ for $p \in [0, 1]$

$B \sim \mathrm{Ber}(p)$ is a random variable with

$$\Pr[B = 1] = p \text{ and } \Pr[B = 0] = 1 - p.$$

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Standard Assumption: Access to Coin Flips

Algorithms have access to a sequence $B_1, B_2, \ldots \sim \mathrm{Ber}(1/2)$ in independent uniformly random bits.

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Exercise: $\mathrm{Ber}(1/3)$ from $\mathrm{Ber}(1/2)$

Design an algorithm that outputs B such that $B \sim \mathrm{Ber}(1/3)$.

Uniform Distribution

Definition: $\mathcal{U}(D)$ on finite D

If $|D| < \infty$, then $X \sim \mathcal{U}(D)$ is a random variable with

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Definition: $\mathcal{U}(D)$ on infinite D

If D is infinite but has finite measure[a] then $X \sim \mathcal{U}(D)$ is a random variable with uniform density function on D. Important example:

$$X \sim \mathcal{U}([0,1]) \Leftrightarrow \forall x \in [0,1] : \Pr[X < x] = x.$$

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[a] Formal details: Not in this lecture.

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In practice: Initialise the significand[a] of floating point number with random bits.

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Exercise: $\mathcal{U}(\{1,\ldots,n\})$ from $\mathcal{U}([0,1])$

Design an algorithm that outputs X such that $X \sim \mathcal{U}(\{1,\ldots,n\})$.

Uniform Distribution on a Disc

Task

Sample $P \sim \mathcal{U}(D)$ for $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

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Flawed Attempt

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sample $\Phi \sim \mathcal{U}([0, 2\pi])$
sample $R \sim \mathcal{U}([0, 1])$
**return** $(R \cdot \cos \Phi, R \cdot \sin \Phi)$
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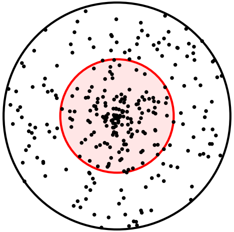

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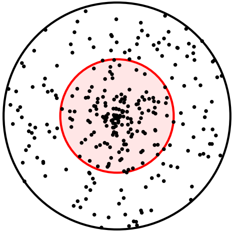

Issue

Disc of half the radius is hit 50% of the time but makes up only $1/4$ of the area!

Uniform Distribution on a Disc with Rejection Sampling

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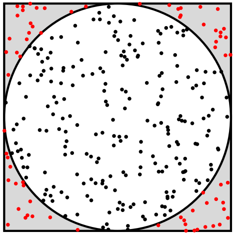

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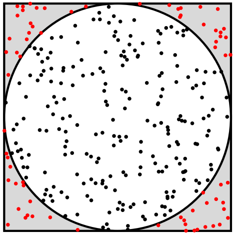

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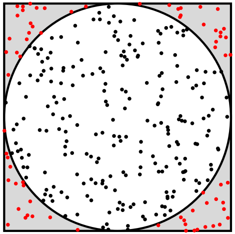

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- Expected number of rounds is $1/(\pi/4) = \mathcal{O}(1)$.

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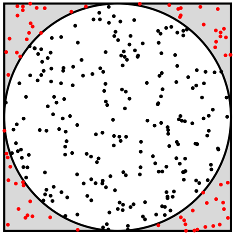

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Spoiler alert: We'll get worst-case constant time with inverse transform sampling later.

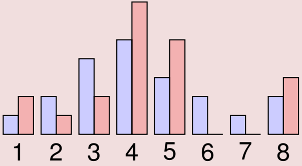
Rejection Sampling in General Discrete Distributions

Exercise

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be distributions on a finite[a] set D. Assume

1. We can sample in constant time from $\mathcal{D}_1$.
2. There exists $C > 0$ such that for any $x \in D$ we have

$$\Pr_{X \sim \mathcal{D}_2}[X = x] \leq C \cdot \Pr_{X \sim \mathcal{D}_1}[X = x].$$



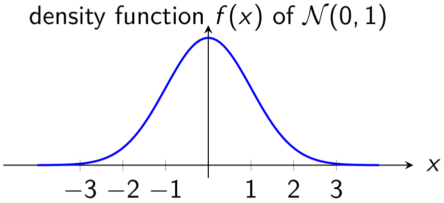
Design an algorithm that generates a sample from $\mathcal{D}_2$ in expected time $\mathcal{O}(C)$.

---

[a] This can be generalised.

Inverse Transform Sampling

- Let $\mathcal{D}$ be a distribution on $\mathbb{R}$.
  $\hookrightarrow$ e.g. $\mathcal{D} = \mathcal{N}(0, 1)$



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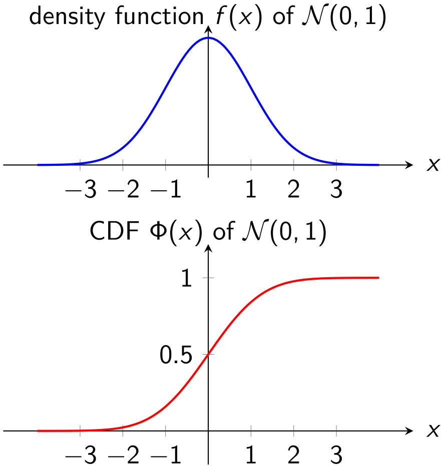
- Let $X \sim \mathcal{D}$ and $F_X(x) = \Pr[X \leq x]$.
  $\hookrightarrow$ F_X is the *cumulative distribution function* of X
  $\hookrightarrow$ the CDF of the normal distribution is called Φ



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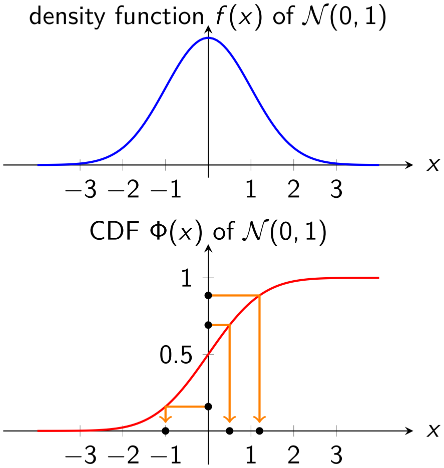
- Let $F_X^{-1}(u) := \inf\{x \in \mathbb{R} \mid F_X(x) \geq u\}$.
  ↪ ordinary inverse for strictly monotone F_X



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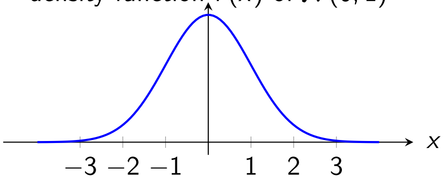
Theorem (Inverse Transform Sampling)

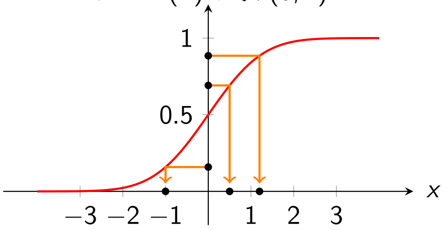
If $U \sim \mathcal{U}([0, 1])$ then $F_X^{-1}(U) \overset{\mathrm{d}}{=} X$, i.e. $F_X^{-1}(U) \sim \mathcal{D}$.

(“$\overset{\mathrm{d}}{=}$” means: “has the same distribution as”)

density function $f(x)$ of $\mathcal{N}(0, 1)$



CDF $\Phi(x)$ of $\mathcal{N}(0, 1)$



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Reason: $\Pr[F_X^{-1}(U) \leq x] = \Pr[U \leq F_X(x)] = F_X(x)$.

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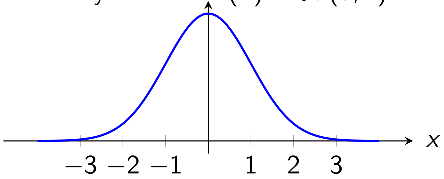

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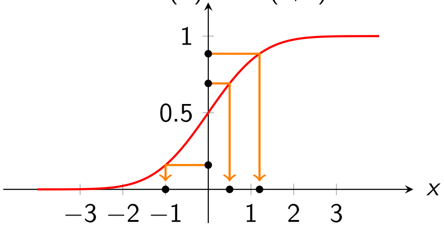

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Preparation

If $(x, y) \sim \mathcal{U}(D)$ then $R = \sqrt{x^2 + y^2}$ satisfies

$$F_R(r) = \Pr[R \leq r] = r^2\pi/\pi = r^2 \text{ hence } F_R^{-1}(u) = \sqrt{u}.$$

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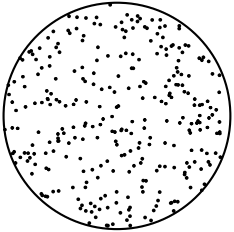

Solution with Inverse Transform Sampling

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Geometric Distribution

Karlsruhe Institute of Technology

Definition: $G \sim \mathrm{Geom}_1(p)$ and $G' \sim \mathrm{Geom}_0(p)$

Let $p \in (0,1]$ and $B_1, B_2, \ldots \sim \mathrm{Ber}(p)$.
Then we define the geometric random variables

$$G := \min\{i \in \mathbb{N} \mid B_i = 1\}$$

↪ number of $\mathrm{Ber}(p)$ trials until (and including) the first success

$$G' := G - 1$$

↪ number of $\mathrm{Ber}(p)$ failures before the first success

We write $G \sim \mathrm{Geom}_1(p)$ and $G' \sim \mathrm{Geom}_0(p)$.[a]

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Sampling $G \sim \mathrm{Geom}_1(p)$ in time $\mathcal{O}(G)$

```
$i \leftarrow 0$
repeat
    $i \leftarrow i + 1$
    sample $X \sim \mathrm{Ber}(p)$
until $X = 1$
return $i$
```

Quite bad: $\mathbb{E}[G] = 1/p$ might be large.

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Exercise

Use inverse transform sampling to sample $G \sim \text{Geom}_1(p)$ in time $\mathcal{O}(1)$.

Sampling Without Replacement

Karlsruhe Institute of Technology

Exercise

Design an algorithm that, given $k, n \in \mathbb{N}$ with $0 \leq k \leq n$ outputs a set $S \subseteq [n]$ of size $|S| = k$ uniformly at random.

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Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

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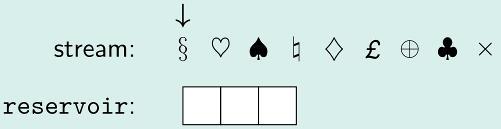

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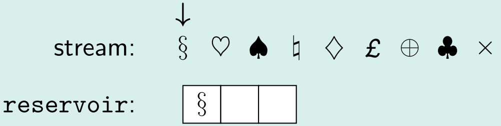
Example ($k = 3$)



stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: § ☐ ☐

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

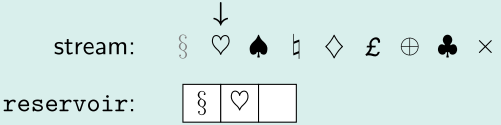
```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    **if** $n \leq k$ **then**
        reservoir[$n$] $\leftarrow x$
    **else**
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        **if** $I \leq k$ **then**
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



↓ (over ♡)

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`:

§	♡	

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

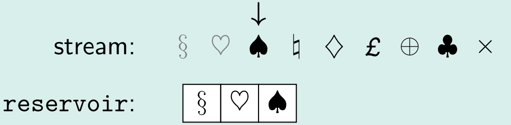
```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



↓

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | § | ♡ | ♠ |

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
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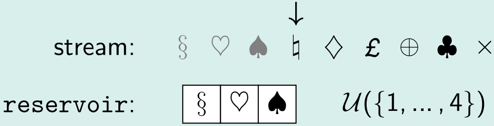
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir:` | § | ♡ | ♠ | $\mathcal{U}(\{1, \ldots, 4\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$

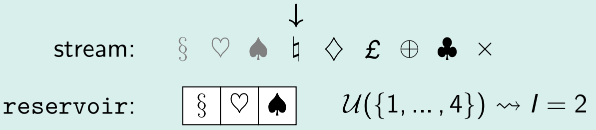
**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



stream: § ♡ ♠ ♮ ◇ £ ⊕ ♣ ×

`reservoir`: | § | ♡ | ♠ |

$\mathcal{U}(\{1, \ldots, 4\}) \rightsquigarrow I = 2$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

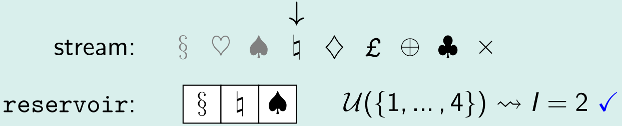
```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



stream: § ♡ ♠ $\overset{\downarrow}{\natural}$ ♢ £ ⊕ ♣ ×

`reservoir`: | § | ♮ | ♠ | $\mathcal{U}(\{1, \ldots, 4\}) \rightsquigarrow I = 2$ ✓

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
    $n \leftarrow 0$

Algorithm observeItem(x):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

$\downarrow$

stream: § ♡ ♠ ♮ ◇ £ ⊕ ♣ ×

`reservoir:` | § | ♮ | ♠ | $\mathcal{U}(\{1, \ldots, 5\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓ above ♢

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: § ♮ ♠ $\quad \mathcal{U}(\{1, \dots, 5\}) \rightsquigarrow I = 5$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

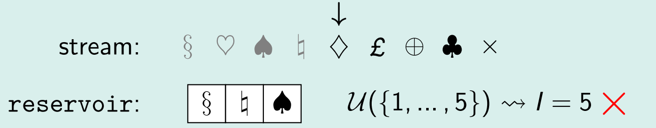
```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



stream: § ♡ ♠ ♮ ◇ £ ⊕ ♣ ×

`reservoir`: § ♮ ♠

$\mathcal{U}(\{1, \dots, 5\}) \rightsquigarrow I = 5$ ✗

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

$\downarrow$ (above £)

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | § | ♮ | ♠ | $\mathcal{U}(\{1, \ldots, 6\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
    $n \leftarrow 0$

Algorithm observeItem(x):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: $\boxed{§}\boxed{\natural}\boxed{\spadesuit}$ $\mathcal{U}(\{1, \dots, 6\}) \rightsquigarrow I = 1$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
    $n \leftarrow 0$
Algorithm observeItem(x):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[n] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[I] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | £ | ♮ | ♠ | $\mathcal{U}(\{1, \ldots, 6\}) \rightsquigarrow I = 1$ ✓

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
**Algorithm** init($k$):
    allocate reservoir[$1..k$]
    $n \leftarrow 0$

**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

$\downarrow$

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | £ | ♮ | ♠ | $\mathcal{U}(\{1, \dots, 7\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
    $n \leftarrow 0$

Algorithm observeItem(x):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[n] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[I] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: § ♡ ♠ ♮ ♢ £ $\overset{\downarrow}{\oplus}$ ♣ ×

`reservoir`: £ ♮ ♠ $\quad \mathcal{U}(\{1, \ldots, 7\}) \rightsquigarrow I = 3$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[$1..k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: $\S \quad \heartsuit \quad \spadesuit \quad \natural \quad \diamondsuit \quad \pounds \quad \overset{\downarrow}{\oplus} \quad \clubsuit \quad \times$

`reservoir:` $\boxed{\pounds}\boxed{\natural}\boxed{\oplus}$ $\quad \mathcal{U}(\{1, \ldots, 7\}) \rightsquigarrow I = 3$ ✓

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    **if** $n \leq k$ **then**
        reservoir[$n$] $\leftarrow x$
    **else**
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        **if** $I \leq k$ **then**
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

$$\begin{array}{lccccccccc} & & & & & & & & \downarrow & \\ \text{stream:} & \S & \heartsuit & \spadesuit & \natural & \diamondsuit & \pounds & \oplus & \clubsuit & \times \end{array}$$

`reservoir:` $\boxed{\pounds}\boxed{\natural}\boxed{\oplus}$ $\mathcal{U}(\{1, \ldots, 8\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | £ | ♮ | ⊕ | $\mathcal{U}(\{1, \ldots, 8\}) \rightsquigarrow I = 3$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init(k):
    allocate reservoir[1..k]
    $n \leftarrow 0$
Algorithm observeItem(x):
    $n \leftarrow n + 1$
    if $n \le k$ then
        reservoir[n] ← x
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \le k$ then
            reservoir[I] ← x
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \ge k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

↓

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir:` | £ | ♮ | ♣ | $\mathcal{U}(\{1, \dots, 8\}) \rightsquigarrow I = 3$ ✓

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
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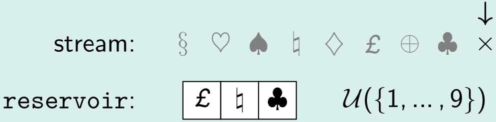
Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)



stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ↓×

`reservoir`: | £ | ♮ | ♣ | $\mathcal{U}(\{1, \dots, 9\})$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ $\overset{\downarrow}{\times}$

`reservoir`: | £ | ♮ | ♣ | $\mathcal{U}(\{1, \dots, 9\}) \rightsquigarrow I = 5$

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
Algorithm init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
Algorithm observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \ldots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \ldots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \ldots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

$\downarrow$

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | £ | ♮ | ♣ | $\mathcal{U}(\{1, \ldots, 9\}) \rightsquigarrow I = 5$ ✗

Reservoir Sampling

Task: Maintain a fair sample of k items while reading a (possibly infinite) stream.

```
**Algorithm** init($k$):
    allocate reservoir[1..$k$]
    $n \leftarrow 0$
**Algorithm** observeItem($x$):
    $n \leftarrow n + 1$
    if $n \leq k$ then
        reservoir[$n$] $\leftarrow x$
    else
        sample $I \sim \mathcal{U}(\{1, \dots, n\})$
        if $I \leq k$ then
            reservoir[$I$] $\leftarrow x$
```

Theorem

Assume we call init(k) and then observeItem(x) for $x \in \{x_1, \dots, x_n\}$ with $n \geq k$. Afterwards `reservoir` contains every subset of $\{x_1, \dots, x_n\}$ of size k with equal probability.

Proof by induction (not here).

Example ($k = 3$)

stream: § ♡ ♠ ♮ ♢ £ ⊕ ♣ ×

`reservoir`: | £ | ♮ | ♣ |

Conclusion

General Techniques

- rejection sampling
- inverse transform sampling

Distributions

- Bernoulli distribution
- uniform distribution
- geometric distribution

Other Stuff

- sampling from a set without replacement
- reservoir sampling

Appendix: Possible Exam Questions I

- How can one sample $B \sim \mathrm{Ber}(p)$? What about $X \sim \mathcal{U}(\{1, \dots, n\})$? Under which assumptions?
- How does rejection sampling work in general? Under which conditions does rejection sampling lead to an efficient algorithm?
- How does inverse transform sampling work in general? Under which conditions does inverse transform sampling lead to an efficient algorithm?
- How can one sample a random point from a disk? Name two techniques and state their advantages and disadvantages.
- Given a set of size n. How can I determine a random subset of size $k \leq n$ and how long does that take?
- Explain reservoir sampling. Isn't that just a slower algorithm for "sampling without replacement"?